Symmetry Methods for
Differential Equations
and Conservation Laws

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Symmetry Groups of Differential Equations

System of differential equations

\[ \Delta(x, u^{(n)}) = 0 \]

\( G \) — Lie group acting on the space of independent and dependent variables:

\[ (\tilde{x}, \tilde{u}) = g \cdot (x, u) = (\Xi(x, u), \Phi(x, u)) \]
$G$ acts on functions $u = f(x)$ by transforming their graphs:

\[ g \mapsto -g \]

**Definition.** $G$ is a symmetry group of the system $\Delta = 0$ if $\tilde{f} = g \cdot f$ is a solution whenever $f$ is.
**Infinitesimal Generators**

Vector field:

\[
v|_{(x,u)} = \frac{d}{d\varepsilon} \, g_\varepsilon \cdot (x, u)|_{\varepsilon=0}
\]

In local coordinates:

\[
v = \sum_{i=1}^{p} \xi^i(x, u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \varphi^\alpha(x, u) \frac{\partial}{\partial u^\alpha}
\]

generates the one-parameter group

\[
\frac{dx^i}{d\varepsilon} = \xi^i(x, u) \quad \frac{du^\alpha}{d\varepsilon} = \varphi^\alpha(x, u)
\]
Example. The vector field
\[ \mathbf{v} = -u \frac{\partial}{\partial x} + x \frac{\partial}{\partial u} \]
generates the rotation group
\[ \tilde{x} = x \cos \varepsilon - u \sin \varepsilon \quad \tilde{u} = x \sin \varepsilon + u \cos \varepsilon \]
since
\[ \frac{d\tilde{x}}{d\varepsilon} = -\tilde{u} \quad \frac{d\tilde{u}}{d\varepsilon} = \tilde{x} \]
Jet Spaces

\[ x = (x^1, \ldots, x^p) \quad \text{— independent variables} \]

\[ u = (u^1, \ldots, u^q) \quad \text{— dependent variables} \]

\[ u^\alpha_J = \frac{\partial^k u^\alpha}{\partial x^{j_1} \ldots \partial x^k} \quad \text{— partial derivatives} \]

\[ (x, u^{(n)}) = (\ldots x^i \ldots u^\alpha \ldots u_J^\alpha \ldots) \in J^n \quad \text{— jet coordinates} \]

\[ \dim J^n = p + q^{(n)} = p + q \binom{p + n}{n} \]
Prolongation to Jet Space

Since $G$ acts on functions, it acts on their derivatives, leading to the prolonged group action on the jet space:

$$(\tilde{x}, \tilde{u}^{(n)}) = \text{pr}^{(n)} g \cdot (x, u^{(n)})$$

$\implies$ formulas provided by implicit differentiation

Prolonged vector field or infinitesimal generator:

$$\text{pr} \ v = v + \sum_{\alpha, J} \varphi^\alpha_J(x, u^{(n)}) \frac{\partial}{\partial u^\alpha_J}$$
The coefficients of the prolonged vector field are given by the explicit prolongation formula:

$$\varphi^\alpha_J = D_J Q^\alpha + \sum_{i=1}^{p} \xi^i u^\alpha_{J,i}$$

$$Q = (Q^1, \ldots, Q^q) \quad \text{—— characteristic of } \mathbf{v}$$

$$Q^\alpha(x, u^{(1)}) = \varphi^\alpha - \sum_{i=1}^{p} \xi^i \frac{\partial u^\alpha}{\partial x^i}$$

★ Invariant functions are solutions to

$$Q(x, u^{(1)}) = 0.$$
Symmetry Criterion

Theorem. (Lie) A connected group of transformations $G$ is a symmetry group of a nondegenerate system of differential equations $\Delta = 0$ if and only if

$$\text{pr } v(\Delta) = 0 \quad (*)$$

whenever $u$ is a solution to $\Delta = 0$ for every infinitesimal generator $v$ of $G$.

$(*)$ are the determining equations of the symmetry group to $\Delta = 0$. For nondegenerate systems, this is equivalent to

$$\text{pr } v(\Delta) = A \cdot \Delta = \sum_{\nu} A_{\nu} \Delta_{\nu}$$
Nondegeneracy Conditions

Maximal Rank:
\[
\text{rank}
\begin{pmatrix}
\partial \Delta \nu & \cdots & \partial \Delta \nu \\
\partial x^i & \cdots & \partial u^\alpha_j
\end{pmatrix}
= \max
\]

Local Solvability: Any each point \((x_0, u_0^{(n)})\) such that
\[
\Delta(x_0, u_0^{(n)}) = 0
\]
there exists a solution \(u = f(x)\) with
\[
u_0^{(n)} = \text{pr}^{(n)} f(x_0)
\]
Nondegenerate = maximal rank + locally solvable
Normal: There exists at least one non-characteristic direction at \((x_0, u_0^{(n)})\) or, equivalently, there is a change of variable making the system into Kovalevskaya form

\[
\frac{\partial^n u^\alpha}{\partial t^n} = \Gamma^\alpha(x, \tilde{u}^{(n)})
\]

Theorem. (Finzi) A system of \(q\) partial differential equations \(\Delta = 0\) in \(q\) unknowns is not normal if and only if there is a nontrivial integrability condition:

\[
\mathcal{D} \Delta = \sum \mathcal{D}_\nu \Delta_\nu = Q \quad \text{order } Q < \text{order } \mathcal{D} + \text{order } \Delta
\]
Under-determined: The integrability condition follows from lower order derivatives of the equation:

\[ \tilde{D} \Delta \equiv 0 \]

Example:

\[ \Delta_1 = u_{xx} + v_{xy}, \quad \Delta_2 = u_{xy} + v_{yy} \]

\[ D_x \Delta_2 - D_y \Delta_1 \equiv 0 \]

Over-determined: The integrability condition is genuine.

Example:

\[ \Delta_1 = u_{xx} + v_{xy} - v_y, \quad \Delta_2 = u_{xy} + v_{yy} + u_y \]

\[ D_x \Delta_2 - D_y \Delta_1 = u_{xy} + v_{yy} \]
A Simple O.D.E.

\[
\begin{align*}
  u_{xx} &= 0 \\

  \text{Infinitesimal symmetry generator:} & \\
  v &= \xi(x, u) \frac{\partial}{\partial x} + \varphi(x, u) \frac{\partial}{\partial u} \\

  \text{Second prolongation:} & \\
  v^{(2)} &= \xi(x, u) \frac{\partial}{\partial x} + \varphi(x, u) \frac{\partial}{\partial u} + \\
  &+ \varphi_1(x, u^{(1)}) \frac{\partial}{\partial u_x} + \varphi_2(x, u^{(2)}) \frac{\partial}{\partial u_{xx}},
\end{align*}
\]
\[ \varphi_1 = \varphi_x + (\varphi_u - \xi_x)u_x - \xi_u u_x^2, \]

\[ \varphi_2 = \varphi_{xx} + (2\varphi_{xu} - \xi_{xx})u_x + (\varphi_{uu} - 2\xi_{xu})u_x^2 - \xi_{uu} u_x^3 + (\varphi_u - 2\xi_x)u_{xx} - 3\xi_u u_x u_{xx}. \]

Symmetry criterion:

\[ \varphi_2 = 0 \quad \text{whenever} \quad u_{xx} = 0. \]
Symmetry criterion:

$$\varphi_{xx} + (2\varphi_{xu} - \xi_{xx})u_x + (\varphi_{uu} - 2\xi_{xu})u^2_x - \xi_{uu}u^3_x = 0.$$ 

Determining equations:

$$\varphi_{xx} = 0 \quad 2\varphi_{xu} = \xi_{xx} \quad \varphi_{uu} = 2\xi_{xu} \quad \xi_{uu} = 0$$

$$\implies \text{Linear!}$$

General solution:

$$\xi(x, u) = c_1x^2 + c_2xu + c_3x + c_4u + c_5$$

$$\varphi(x, u) = c_1xu + c_2u^2 + c_6x + c_7u + c_8$$
Symmetry algebra:

\[ \begin{align*}
\mathbf{v}_1 &= \partial_x \\
\mathbf{v}_2 &= \partial_u \\
\mathbf{v}_3 &= x\partial_x \\
\mathbf{v}_4 &= x\partial_u \\
\mathbf{v}_5 &= u\partial_x \\
\mathbf{v}_6 &= u\partial_u \\
\mathbf{v}_7 &= x^2\partial_x + xu\partial_u \\
\mathbf{v}_8 &= xu\partial_x + u^2\partial_u
\end{align*} \]

Symmetry Group:

\[(x, u) \mapsto \left( \frac{ax + bu + c}{hx + ju + k}, \frac{dx + eu + f}{hx + ju + k} \right) \]

\[\implies \text{projective group}\]
Prolongation

Infinitesimal symmetry

\[ \mathbf{v} = \xi(x, t, u) \frac{\partial}{\partial x} + \tau(x, t, u) \frac{\partial}{\partial t} + \varphi(x, t, u) \frac{\partial}{\partial u} \]

First prolongation

\[ \text{pr}^{(1)} \mathbf{v} = \xi \frac{\partial}{\partial x} + \tau \frac{\partial}{\partial t} + \varphi \frac{\partial}{\partial u} + \varphi^x \frac{\partial}{\partial u_x} + \varphi^t \frac{\partial}{\partial u_t} \]

Second prolongation

\[ \text{pr}^{(2)} \mathbf{v} = \text{pr}^{(1)} \mathbf{v} + \varphi^{xx} \frac{\partial}{\partial u_{xx}} + \varphi^{xt} \frac{\partial}{\partial u_{xt}} + \varphi^{tt} \frac{\partial}{\partial u_{tt}} \]
where
\[
\begin{align*}
\varphi^x &= D_xQ + \xi u_{xx} + \tau u_{xt} \\
\varphi^t &= D_tQ + \xi u_{xt} + \tau u_{tt} \\
\varphi^{xx} &= D_x^2Q + \xi u_{xxt} + \tau u_{xtt}
\end{align*}
\]

Characteristic
\[
Q = \varphi - \xi u_x - \tau u_t
\]
\[ \varphi^x = D_x Q + \xi u_{xx} + \tau u_{xt} \]
\[ = \varphi_x + (\varphi_u - \xi_x)u_x - \tau_x u_t - \xi_u u_x^2 - \tau_u u_x u_t \]

\[ \varphi^t = D_t Q + \xi u_{xt} + \tau u_{tt} \]
\[ = \varphi_t - \xi_t u_x + (\varphi_u - \tau_t)u_t - \xi_u u_x u_t - \tau_u u_t^2 \]

\[ \varphi^{xx} = D_x^2 Q + \xi u_{xxt} + \tau u_{xtt} \]
\[ = \varphi_{xx} + (2\varphi_{xu} - \xi_{xx})u_x - \tau_{xx} u_t + (\varphi_{uu} - 2\xi_{xu})u_x^2 - 2\tau_{xu} u_x u_t - \xi_{uu} u_x^3 - \tau_{uu} u_x^2 u_t + (\varphi_u - 2\xi_x)u_{xx} - 2\tau_x u_x u_t - 3\xi_u u_x u_{xx} - \tau_u u_t u_{xx} - 2\tau_u u_x u_{xt} \]
Heat Equation

\[ u_t = u_{xx} \]

Infinitesimal symmetry criterion

\[ \varphi_t = \varphi_{xx} \quad \text{whenever} \quad u_t = u_{xx} \]
Determining equations

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>Monomial</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0 = -2\tau_u$</td>
<td>$u_x u_{xt}$</td>
</tr>
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<td>$u_x u_{xx}$</td>
</tr>
<tr>
<td>$\varphi_u - \tau_t = -\tau_{xx} + \varphi_u - 2\xi_x$</td>
<td>$u_{xx}$</td>
</tr>
<tr>
<td>$0 = -\xi_{uu}$</td>
<td>$u^3_x$</td>
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<td>$u_x$</td>
</tr>
<tr>
<td>$\varphi_t = \varphi_{xx}$</td>
<td>1</td>
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</tbody>
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General solution

\[ \xi = c_1 + c_4x + 2c_5t + 4c_6xt \]

\[ \tau = c_2 + 2c_4t + 4c_6t^2 \]

\[ \varphi = (c_3 - c_5x - 2c_6t - c_6x^2)u + \alpha(x, t) \]

\[ \alpha_t = \alpha_{xx} \]
Symmetry algebra

\[ \mathbf{v}_1 = \partial_x \quad \text{space transl.} \]
\[ \mathbf{v}_2 = \partial_t \quad \text{time transl.} \]
\[ \mathbf{v}_3 = u \partial_u \quad \text{scaling} \]
\[ \mathbf{v}_4 = x \partial_x + 2t \partial_t \quad \text{scaling} \]
\[ \mathbf{v}_5 = 2t \partial_x - xu \partial_u \quad \text{Galilean} \]
\[ \mathbf{v}_6 = 4xt \partial_x + 4t^2 \partial_t - (x^2 + 2t)u \partial_u \quad \text{inversions} \]
\[ \mathbf{v}_\alpha = \alpha(x, t) \partial_u \quad \text{linearity} \]
Potential Burgers’ equation

\[ u_t = u_{xx} + u_x^2 \]

Infinitesimal symmetry criterion

\[ \varphi^t = \varphi^{xx} + 2u_x\varphi^x \]
Determining equations

**Coefficient**

\[
\begin{align*}
0 &= -2\tau_u \\
0 &= -2\tau_x \\
-\tau_u &= -\tau_u \\
-2\tau_u &= -\tau_{uu} - 3\tau_u \\
-\xi_u &= -2\tau_{xu} - 3\xi_u - 2\tau_x \\
\varphi_u - \tau_t &= -\tau_{xx} + \varphi_u - 2\xi \\
-\tau_u &= -\tau_{uu} - 2\tau_u \\
-\xi_u &= -2\tau_{xu} - \xi_{uu} - 2\tau_x - 2\xi_u \\
\varphi_u - \tau_t &= -\tau_{xx} + \varphi_{uu} - 2\xi_{xu} + 2\varphi_u - 2\xi_x \\
-\xi_t &= 2\varphi_{xu} - \xi_{xx} + 2\varphi_x \\
\varphi_t &= \varphi_{xx}
\end{align*}
\]

**Monomial**

\[
\begin{align*}
&u_x u_{xt} \\
u_{xt} \\
&u^2 \\
u^2_{xx} \\
u u_{xx} \\
u_x u_{xx} \\
u_{xx} \\
u^4_x \\
u^3_x \\
u^2_x \\
u_x \\
u_x \\
1
\end{align*}
\]
General solution

\[ \dot{\xi} = c_1 + c_4 x + 2c_5 t + 4c_6 x t \]

\[ \tau = c_2 + 2c_4 t + 4c_6 t^2 \]

\[ \varphi = c_3 - c_5 x - 2c_6 t - c_6 x^2 + \alpha(x, t)e^{-u} \]

\[ \alpha_t = \alpha_{xx} \]
Symmetry algebra

\[ \mathbf{v}_1 = \partial_x \]

\[ \mathbf{v}_2 = \partial_t \]

\[ \mathbf{v}_3 = \partial_u \]

\[ \mathbf{v}_4 = x\partial_x + 2t\partial_t \]

\[ \mathbf{v}_5 = 2t\partial_x - x\partial_u \]

\[ \mathbf{v}_6 = 4xt\partial_x + 4t^2\partial_t - (x^2 + 2t)\partial_u \]

\[ \mathbf{v}_\alpha = \alpha(x, t)e^{-u}\partial_u \]

Hopf-Cole \( w = e^u \) maps to heat equation.
Symmetry–Based Solution Methods

Ordinary Differential Equations

- Lie’s method
- Solvable groups
- Variational and Hamiltonian systems
- Potential symmetries
- Exponential symmetries
- Generalized symmetries
Partial Differential Equations

- Group-invariant solutions
- Non-classical method
- Weak symmetry groups
- Clarkson-Kruskal method
- Partially invariant solutions
- Differential constraints
- Nonlocal Symmetries
- Separation of Variables
Integration of O.D.E.’s

First order ordinary differential equation

\[
\frac{du}{dx} = F(x, u)
\]

Symmetry Generator:

\[
v = \xi(x, u) \frac{\partial}{\partial x} + \varphi(x, u) \frac{\partial}{\partial u}
\]

Determining equation

\[
\varphi_x + (\varphi_u - \xi_x)F - \xi_u F^2 = \xi \frac{\partial F}{\partial x} + \varphi \frac{\partial F}{\partial u}
\]

♠ Trivial symmetries

\[
\frac{\varphi}{\xi} = F
\]
Method 1: Rectify the vector field.

\[ \mathbf{v}|_{(x_0,u_0)} \neq 0 \]

Introduce new coordinates

\[ y = \eta(x,u) \quad w = \zeta(x,u) \]

near \((x_0,u_0)\) so that

\[ \mathbf{v} = \frac{\partial}{\partial w} \]

These satisfy first order p.d.e.’s

\[ \xi \eta_x + \varphi \eta_u = 0 \quad \xi \zeta_x + \varphi \zeta_u = 1 \]

Solution by method of characteristics:

\[ \frac{dx}{\xi(x,u)} = \frac{du}{\varphi(x,u)} = \frac{dt}{1} \]
The equation in the new coordinates will be invariant if and only if it has the form

\[ \frac{dw}{dy} = h(y) \]

and so can clearly be integrated by quadrature.
Method 2: Integrating Factor

If $v = \xi \partial_x + \varphi \partial_u$ is a symmetry for

$$P(x, u) \ dx + Q(x, u) \ du = 0$$

then

$$R(x, u) = \frac{1}{\xi P + \varphi Q}$$

is an integrating factor.

If

$$\frac{\varphi}{\xi} = -\frac{P}{Q}$$

then the integrating factor is trivial. Also, rectification of the vector field is equivalent to solving the original ordinary differential equation.
Higher Order Ordinary Differential Equations

\[ \Delta(x, u^{(n)}) = 0 \]

If we know a one-parameter symmetry group

\[ v = \xi(x, u) \frac{\partial}{\partial x} + \varphi(x, u) \frac{\partial}{\partial u} \]

then we can reduce the order of the equation by 1.
Method 1: Rectify $\mathbf{v} = \partial_w$. Then the equation is invariant if and only if it does not depend on $w$:

$$\Delta(y, w', \ldots, w_n) = 0$$

Set $v = w'$ to reduce the order.
Method 2: Differential invariants.

\[ h[pr^{(n)}g \cdot (x, u^{(n)})] = h(x, u^{(n)}), \quad g \in G \]

Infinitesimal criterion: \( pr \, v(h) = 0. \)

**Proposition.** If \( \eta, \zeta \) are \( n^{th} \) order differential invariants, then

\[
\frac{d\eta}{d\zeta} = \frac{D_x \eta}{D_x \zeta}
\]

is an \((n+1)^{st}\) order differential invariant.

**Corollary.** Let

\[ y = \eta(x, u), \quad w = \zeta(x, u, u') \]

be the independent first order differential invariants
for $G$. Any $n^\text{th}$ order o.d.e. admitting $G$ as a symmetry group can be written in terms of the differential invariants $y, w, dw/dy, \ldots, d^{n-1}w/dy^{n-1}$.

In terms of the differential invariants, the $n^\text{th}$ order o.d.e. reduces to

$$\widetilde{\Delta}(y, w^{(n-1)}) = 0$$

For each solution $w = g(y)$ of the reduced equation, we must solve the auxiliary equation

$$\zeta(x, u, u') = g[\eta(x, u)]$$

to find $u = f(x)$. This first order equation admits $G$ as a symmetry group and so can be integrated as before.
Multiparameter groups

- $G$ - $r$-dimensional Lie group.

Assume $\text{pr}^{(r)} G$ acts regularly with $r$ dimensional orbits.

Independent $r^{\text{th}}$ order differential invariants:

$$y = \eta(x, u^{(r)}) \quad w = \zeta(x, u^{(r)})$$

Independent $n^{\text{th}}$ order differential invariants:

$$y, w, \frac{dw}{dy}, \cdots, \frac{d^{n-r}w}{dy^{n-r}}.$$
In terms of the differential invariants, the equation reduces in order by \( r \):

\[
\Delta(y, w^{(n-r)}) = 0
\]

For each solution \( w = g(y) \) of the reduced equation, we must solve the auxiliary equation

\[
\zeta(x, u^{(r)}) = g[\eta(x, u^{(r)})]
\]

to find \( u = f(x) \). In this case there is no guarantee that we can integrate this equation by quadrature.
Example. Projective group \( G = SL(2) \)

\[
(x, u) \mapsto \left( x, \frac{a u + b}{c u + d} \right), \quad a d - b c = 1.
\]

Infinitesimal generators:

\[ \partial_u, \quad u \partial_u, \quad u^2 \partial_u \]

Differential invariants:

\[
x, \quad w = \frac{2 u' u''' - 3 u''^2}{u'^2}
\]

\[ \Rightarrow \text{Schwarzian derivative} \]

Reduced equation

\[
\widetilde{\Delta}(y, w^{(n-3)}) = 0
\]
Let $w = h(x)$ be a solution to reduced equation.
To recover $u = f(x)$ we must solve the auxiliary equation:

$$2 u' u''' - 3 u''^2 = u'^2 h(x),$$

which still admits the full group $SL(2)$.
Integrate using $\partial_u$:

$$u' = z \quad 2 z z'' - z'^2 = z^2 h(x)$$

Integrate using $u \partial_u = z \partial_z$:

$$v = (\log z)' \quad 2 v' + v^2 = h(x)$$

No further symmetries, so we are stuck with a Riccati equation to effect the solution.
Solvable Groups

• Basis $v_1, \ldots, v_r$ of the symmetry algebra $g$ such that

$$[v_i, v_j] = \sum_{k<j} c_{ij}^k v_k, \quad i < j$$

If we reduce in the correct order, then we are guaranteed a symmetry at each stage. Reduced equation for subalgebra $\{v_1, \ldots, v_k\}$:

$$\widetilde{\Delta}^{(k)}(y, w^{(n-k)}) = 0$$

admits a symmetry $\tilde{v}_{k+1}$ corresponding to $v_{k+1}$. 
Theorem. (Bianchi) If an $n^{th}$ order o.d.e. has a (regular) $r$-parameter solvable symmetry group, then its solutions can be found by quadrature from those of the $(n-r)^{th}$ order reduced equation.
Example.

\[ x^2 u'' = f(x u' - u) \]

Symmetry group:

\[ v = x \partial_u, \quad w = x \partial_x, \]

\[ [v, w] = -v. \]

Reduction with respect to \( v \):

\[ z = x u' - u \]

Reduced equation:

\[ x z' = h(z) \]

still invariant under \( w = x \partial_x \), and hence can be solved by quadrature.
Wrong way reduction with respect to \( w \):

\[
y = u, \quad z = z(y) = xu' \]

Reduced equation:

\[
z(z' - 1) = h(z - y) \]

- No remaining symmetry; not clear how to integrate directly.
Group Invariant Solutions

System of partial differential equations
\[ \Delta(x, u^{(n)}) = 0 \]

\[ G \quad \text{ symmetry group} \]
Assume \( G \) acts regularly on \( M \) with \( r \)-dimensional orbits

**Definition.** \( u = f(x) \) is a \( G \)-invariant solution if
\[ g \cdot f = f \quad \text{for all} \quad g \in G. \]
i.e. the graph \( \Gamma_f = \{(x, f(x))\} \) is a (locally) \( G \)-invariant subset of \( M \).

- Similarity solutions, travelling waves, ...
Proposition. Let $G$ have infinitesimal generators $v_1, \ldots, v_r$ with associated characteristics $Q_1, \ldots, Q_r$. A function $u = f(x)$ is $G$-invariant if and only if it is a solution to the system of first order partial differential equations

$$Q_\nu(x, u^{(1)}) = 0, \quad \nu = 1, \ldots, r.$$ 

Theorem. (Lie). If $G$ has $r$-dimensional orbits, and acts transversally to the vertical fibers $\{x = \text{const.}\}$, then all the $G$-invariant solutions to $\Delta = 0$ can be found by solving a reduced system of differential equations $\Delta/G = 0$ in $r$ fewer independent variables.
**Method 1: Invariant Coordinates.**

The new variables are given by a complete set of functionally independent invariants of $G$:

$$\eta_\alpha(g \cdot (x, u)) = \eta_\alpha(x, u) \quad \text{for all} \quad g \in G$$

Infinitesimal criterion:

$$\mathbf{v}_k[\eta_\alpha] = 0, \quad k = 1, \ldots, r.$$  

New independent and dependent variables:

$$y_1 = \eta_1(x, u), \ldots, y_{p-r} = \eta_{p-r}(x, u)$$

$$w_1 = \zeta_1(x, u), \ldots, w^q = \zeta^q(x, u)$$
Invariant functions:

\[ w = \eta(y), \quad \text{i.e.} \quad \zeta(x, u) = h[\eta(x, u)] \]

Reduced equation:

\[ \Delta/G(y, w^{(n)}) = 0 \]

Every solution determines a \( G \)-invariant solution to the original p.d.e.
Example. The heat equation $u_t = u_{xx}$

Scaling symmetry: $x \partial_x + 2 t \partial_t + a u \partial_u$

Invariants: $y = \frac{x}{\sqrt{t}}$, $w = t^{-a}u$

$u = t^a w(y)$, $u_t = t^{a-1}(-\frac{1}{2} y w' + a w)$, $u_{xx} = t^a w''$.

Reduced equation

$$w'' + 12yw' - aw = 0$$

Solution: $w = e^{-y^2/8}U\left(2a + \frac{1}{2}, \frac{y}{\sqrt{2}}\right)$

$\Longrightarrow$ parabolic cylinder function

Similarity solution:

$$u(x, t) = t^a e^{-x^2/8t}U\left(2a + \frac{1}{2}, \frac{x}{\sqrt{2t}}\right)$$
Example. The heat equation \( u_t = u_{xx} \)

Galilean symmetry: \( 2t \partial_x - xu \partial_u \)

Invariants: \( y = t, \quad w = e^{x^2/4t} u \)

\[
\begin{align*}
u &= e^{-x^2/4t} w(y), \\
u_t &= e^{-x^2/4t} \left( w' + \frac{x^2}{4t^2} w \right), \\
u_{xx} &= e^{-x^2/4t} \left( \frac{x^2}{4t^2} - \frac{1}{2t} \right) w.
\end{align*}
\]

Reduced equation: \( 2yw' + w = 0 \)

Source solution: \( w = ky^{-1/2}, \quad u = \frac{k}{\sqrt{t}} e^{x^2/4t} \)
Method 2: Direct substitution:
Solve the combined system
\[ \Delta(x, u^{(n)}) = 0 \quad Q_k(x, u^{(1)}) = 0, \quad k = 1, \ldots, r \]
as an overdetermined system of p.d.e.
For a one-parameter group, we solve
\[ Q(x, u^{(1)}) = 0 \]
for
\[ \frac{\partial u^\alpha}{\partial x^p} = \frac{\varphi^\alpha}{\xi^n} - \sum_{i=1}^{p-1} \frac{\xi^i}{\xi^p} \frac{\partial u^\alpha}{\partial x^i} \]
Rewrite in terms of derivatives with respect to \( x_1, \ldots, x_{p-1} \).
The reduced equation has \( x^p \) as a parameter. Dependence on \( x^p \) can be found by by substituting back into the characteristic condition.
Classification of invariant solutions

Let $G$ be the full symmetry group of the system $\Delta = 0$. Let $H \subset G$ be a subgroup. If $u = f(x)$ is an $H$-invariant solution, and $g \in G$ is another group element, then $\tilde{f} = g \cdot f$ is an invariant solution for the conjugate subgroup $\tilde{H} = g \cdot H \cdot g^{-1}$.

- Classification of subgroups of $G$ under conjugation.
- Classification of subalgebras of $\mathfrak{g}$ under the adjoint action.
- Exploit symmetry of the reduced equation
Non-Classical Method

⇒ Bluman and Cole

Here we require not invariance of the original partial differential equation, but rather invariance of the combined system

\[ \Delta(x, u^{(n)}) = 0 \quad Q_k(x, u^{(1)}) = 0, \quad k = 1, \ldots, r \]

- Nonlinear determining equations.
- Most solutions derived using this approach come from ordinary group invariance anyway.
Weak (Partial) Symmetry Groups

Here we require invariance of

\[ \Delta(x, u^{(n)}) = 0 \quad Q_k(x, u^{(1)}) = 0, \quad k = 1, \ldots, r \]

and all the associated integrability conditions

- Every group is a weak symmetry group.
- Every solution can be derived in this way.
- Compatibility of the combined system?
- Overdetermined systems of partial differential equations.
The Boussinesq Equation

\[ u_{tt} + \frac{1}{2}(u^2)_{xx} + u_{xxxx} = 0 \]

Classical symmetry group:

\[ v_1 = \partial_x \quad v_2 = \partial_t \quad v_3 = x \partial_x + 2t \partial_t - 2u \partial_u \]

For the scaling group

\[ -Q = x u_x + 2t u_t + 2u = 0 \]

Invariants:

\[ y = \frac{x}{\sqrt{t}} \quad w = tu \quad u = \frac{1}{t} w \left( \frac{x}{\sqrt{t}} \right) \]

Reduced equation:

\[ w'''' + \frac{1}{2} (w^2)'' + \frac{1}{4} y^2 w'' + \frac{7}{4} y w' + 2w = 0 \]
\[ u_{tt} + \frac{1}{2}(u^2)_{xx} + u_{xxxx} = 0 \]

Group classification:

\[ \mathbf{v}_1 = \partial_x \quad \mathbf{v}_2 = \partial_t \quad \mathbf{v}_3 = x \partial_x + 2t \partial_t - 2u \partial_u \]

Note:

\[ \text{Ad}(\varepsilon \mathbf{v}_3) \mathbf{v}_1 = e^\varepsilon \mathbf{v}_1 \quad \text{Ad}(\varepsilon \mathbf{v}_3) \mathbf{v}_2 = e^{2\varepsilon} \mathbf{v}_2 \]

\[ \text{Ad}(\delta \mathbf{v}_1 + \varepsilon \mathbf{v}_2) \mathbf{v}_3 = \mathbf{v}_3 - \delta \mathbf{v}_1 - \varepsilon \mathbf{v}_2 \]

so the one-dimensional subalgebras are classified by:

\[ \{ \mathbf{v}_3 \} \quad \{ \mathbf{v}_1 \} \quad \{ \mathbf{v}_2 \} \quad \{ \mathbf{v}_1 + \mathbf{v}_2 \} \quad \{ \mathbf{v}_1 - \mathbf{v}_2 \} \]

and we only need to determine solutions invariant under these particular subgroups to find the most general group-invariant solution.
\[ u_{tt} + \frac{1}{2}(u^2)_{xx} + u_{xxxx} = 0 \]

**Non-classical**: Galilean group

\[ \mathbf{v} = t \partial_x + \partial_t - 2t \partial_u \]

Not a symmetry, but the combined system

\[ u_{tt} + \frac{1}{2}(u^2)_{xx} + u_{xxxx} = 0 \quad -Q = tu_x + u_t + 2t = 0 \]

does admit \( \mathbf{v} \) as a symmetry. Invariants:

\[ y = x - \frac{1}{2}t^2, \quad w = u + t^2, \quad u(x,t) = w(y) - t^2 \]

Reduced equation:

\[ w'''' + ww'' + (w')^2 - w' + 2 = 0 \]
\[ u_{tt} + \frac{1}{2}(u^2)_{xx} + u_{xxxx} = 0 \]

**Weak Symmetry:** Scaling group: \( x \partial_x + t \partial_t \)

Not a symmetry of the combined system

\[ u_{tt} + \frac{1}{2}(u^2)_{xx} + u_{xxxx} = 0 \quad Q = x u_x + t u_t = 0 \]

Invariants: \( y = \frac{x}{t} u \)  

Invariant solution: \( u(x, t) = w(y) \)

The Boussinesq equation reduces to

\[ t^{-4} w''' + t^{-2}[(w + 1 - y)w'' + (w')^2 - y w'] = 0 \]

so we obtain an overdetermined system

\[ w''' = 0 \quad (w + 1 - y)w'' + (w')^2 - y w' = 0 \]

Solutions: \( w(y) = \frac{2}{3} y^2 - 1 \), or \( w = \text{constant} \)

Similarity solution: \( u(x, t) = \frac{2}{3} \frac{x^2}{t^2} - 1 \)
Symmetries and Conservation Laws
Variational problems

\[ L[u] = \int_{\Omega} L(x, u^{(n)}) \, dx \]

Euler-Lagrange equations

\[ \Delta = E(L) = 0 \]

Euler operator (variational derivative)

\[ E^\alpha(L) = \frac{\delta L}{\delta u^\alpha} = \sum_J (-D)^J \frac{\partial L}{\partial u^\alpha_J} \]

Theorem. (Null Lagrangians)

\[ E(L) \equiv 0 \quad \text{if and only if} \quad L = \text{Div } P \]
Theorem. The system $\Delta = 0$ is the Euler-Lagrange equations for some variational problem if and only if the Fréchet derivative $D_\Delta$ is self-adjoint:

$$D^*_\Delta = D_\Delta.$$ 

$\implies$ Helmholtz conditions
Fréchet derivative

Given \( P(x, u^{(n)}) \), its Fréchet derivative or formal linearization is the differential operator \( D_P \) defined by

\[
D_P[w] = \frac{d}{d\varepsilon} P[u + \varepsilon w] \bigg|_{\varepsilon = 0}
\]

Example.

\[
P = u_{xxx} + uu_x
\]

\[
D_P = D_x^3 + uD_x + u_x
\]
Adjoint (formal)
\[ \mathcal{D} = \sum_J A_J D^J \quad \mathcal{D}^* = \sum_J (-D)^J \cdot A_J \]

Integration by parts formula:
\[ P \mathcal{D} Q = Q \mathcal{D}^* P + \text{Div} \ A \]
where \( A \) depends on \( P, Q \).
Conservation Laws

Definition. A conservation law of a system of partial differential equations is a divergence expression

$$\text{Div } P = 0$$

which vanishes on all solutions to the system.

$$P = (P_1(x, u^{(k)}), \ldots, P_p(x, u^{(k)}))$$

$$\implies$$ The integral

$$\int P \cdot dS$$

is path (surface) independent.
If one of the coordinates is time, a conservation law takes the form

\[ D_t T + \text{Div } X = 0 \]

\( T \) — conserved density \quad \( X \) — flux

By the divergence theorem,

\[ \int_\Omega T(x, t, u^{(k)}) \, dx \bigg|_{t=a}^{b} = \int_a^b \int_\Omega X \cdot dS \, dt \]

depends only on the boundary behavior of the solution.

- If the flux \( X \) vanishes on \( \partial\Omega \), then \( \int_\Omega T \, dx \) is conserved (constant).
Trivial Conservation Laws

Type I If $P = 0$ for all solutions to $\Delta = 0$, then $\text{Div } P = 0$ on solutions too.

Type II (Null divergences) If $\text{Div } P = 0$ for all functions $u = f(x)$, then it trivially vanishes on solutions.

Examples:

$$D_x(uy) + D_y(-ux) \equiv 0$$

$$D_x \frac{\partial(u, v)}{\partial(y, z)} + D_y \frac{\partial(u, v)}{\partial(z, x)} + D_z \frac{\partial(u, v)}{\partial(x, y)} \equiv 0$$
Theorem.

\[ \text{Div } P(x, u^{(k)}) \equiv 0 \]

for all \( u \) if and only if

\[ P = \text{Curl } Q(x, u^{(k)}) \]

i.e.

\[ P_i = \sum_{j=1}^{p} D_j Q_{ij} \quad Q_{ij} = -Q_{ji} \]
Two conservation laws $P$ and $\tilde{P}$ are equivalent if they differ by a sum of trivial conservation laws:

$$P = \tilde{P} + P_I + P_{II}$$

where

$$P_I = 0 \text{ on solutions} \quad \text{Div } P_{II} \equiv 0.$$
Proposition. Every conservation law of a system of partial differential equations is equivalent to a conservation law in characteristic form

\[ \text{Div } P = Q \cdot \Delta = \sum_{\nu} Q_{\nu} \Delta_{\nu} \]

Proof:

\[ \text{Div } P = \sum_{\nu, J} Q_{\nu}^J D^J \Delta_{\nu} \]

Integrate by parts:

\[ \text{Div } \tilde{P} = \sum_{\nu, J} (-D)^J Q_{\nu}^J \cdot \Delta_{\nu} \quad Q_{\nu} = \sum_{J} (-D)^J Q_{\nu}^J \]

\( Q \) is called the characteristic of the conservation law.
Theorem. $Q$ is the characteristic of a conservation law for $\Delta = 0$ if and only if

$$D^*_\Delta Q + D^*_Q \Delta = 0.$$ 

Proof:

$$0 = E(\text{Div } P) = E(Q \cdot \Delta) = D^*_\Delta Q + D^*_Q \Delta$$
Normal Systems

A characteristic is trivial if it vanishes on solutions. Two characteristics are equivalent if they differ by a trivial one.

**Theorem.** Let $\Delta = 0$ be a normal system of partial differential equations. Then there is a one-to-one correspondence between (equivalence classes of) nontrivial conservation laws and (equivalence classes of) nontrivial characteristics.
Variational Symmetries

**Definition.** A (restricted) variational symmetry is a transformation \((\tilde{x}, \tilde{u}) = g \cdot (x, u)\) which leaves the variational problem invariant:

\[
\int_{\tilde{\Omega}} L(\tilde{x}, \tilde{u}^{(n)}) \, d\tilde{x} = \int_{\Omega} L(x, u^{(n)}) \, dx
\]

Infinitesimal criterion:

\[
\text{pr } v(L) + L \text{ Div } \xi = 0
\]

**Theorem.** If \(v\) is a variational symmetry, then it is a symmetry of the Euler-Lagrange equations.

\[\star \star \text{ But not conversely!}\]
Noether’s Theorem (Weak version). If $v$ generates a one-parameter group of variational symmetries of a variational problem, then the characteristic $Q$ of $v$ is the characteristic of a conservation law of the Euler-Lagrange equations:

$$\text{Div } P = Q E(L)$$
Elastostatics

\[ \int W(x, \nabla u) \, dx \quad \text{— stored energy} \]

\[ x, u \in \mathbb{R}^p, \quad p = 2, 3 \]

Frame indifference

\[ u \mapsto R u + a, \quad R \in SO(p) \]

Conservation laws = path independent integrals:

\[ \text{Div } P = 0. \]
1. Translation invariance

\[ P_i = \frac{\partial W}{\partial u_i^\alpha} \]

\[ \Rightarrow \text{ Euler-Lagrange equations} \]

2. Rotational invariance

\[ P_i = u_i^\alpha \frac{\partial W}{\partial u_j^\beta} - u_i^\beta \frac{\partial W}{\partial u_j^\alpha} \]

3. Homogeneity: \( W = W(\nabla u) \) \( x \rightarrow x + a \)

\[ P_i = \sum_{\alpha=1}^{p} u_j^\alpha \frac{\partial W}{\partial u_i^\alpha} - \delta_j^i W \]

\[ \Rightarrow \text{ Energy-momentum tensor} \]
4. Isotropy: \( W(\nabla u \cdot Q) = W(\nabla u) \quad Q \in \text{SO}(p) \)

\[
P_i = \sum_{\alpha=1}^{p} \left( x^j u_k^\alpha - x^k u_j^\alpha \right) \frac{\partial W}{\partial u_i^\alpha} + (\delta_j^i x^k - \delta_k^i x^j)W
\]

5. Dilation invariance: \( W(\lambda \nabla u) = \lambda^n W(\nabla u) \)

\[
P_i = \frac{n - p}{n} \sum_{\alpha,j=1}^{p} \left( u^\alpha \delta_j^i - x^j u_j^\alpha \right) \frac{\partial W}{\partial u_i^\alpha} + x^i W
\]

5A. Divergence identity

\[
\text{Div} \tilde{P} = p W
\]

\[
\tilde{P}_i = \sum_{j=1}^{p} \left( u^\alpha \delta_j^i - x^j u_j^\alpha \right) \frac{\partial W}{\partial u_i^\alpha} + x^i W
\]

\[\implies\] Knops/Stuart, Pohozaev, Pucci/Serrin
Generalized Vector Fields

Allow the coefficients of the infinitesimal generator to depend on derivatives of $u$:

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x, u^{(k)}) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \varphi^\alpha(x, u^{(k)}) \frac{\partial}{\partial u^\alpha}$$

Characteristic:

$$Q^\alpha(x, u^{(k)}) = \varphi^\alpha - \sum_{i=1}^{p} \xi^i u^\alpha_i$$

Evolutionary vector field:

$$\mathbf{v}_Q = \sum_{\alpha=1}^{q} Q^\alpha(x, u^{(k)}) \frac{\partial}{\partial u^\alpha}$$
Prolongation formula:

\[ \text{pr } v = \text{pr } v_Q + \sum_{i=1}^{p} \xi^i D_i \]

\[ \text{pr } v_Q = \sum_{\alpha,J} D^J Q_\alpha \frac{\partial}{\partial u^\alpha_J} \]

\[ D_i = \sum_{\alpha,J} u_{J,i}^\alpha \frac{\partial}{\partial u^\alpha_J} \]

\[ \implies \text{ total derivative} \]
Generalized Flows

• The one-parameter group generated by an evolutionary vector field is found by solving the Cauchy problem for an associated system of evolution equations

\[
\frac{\partial u^\alpha}{\partial \varepsilon} = Q_\alpha(x, u^{(n)}) \quad u|_{\varepsilon=0} = f(x)
\]
Example. \( v = \frac{\partial}{\partial x} \) generates the one-parameter group of translations:

\[(x, y, u) \quad \mapsto \quad (x + \varepsilon, y, u)\]

Evolutionary form:

\[v_Q = -u_x \frac{\partial}{\partial x}\]

Corresponding group:

\[\frac{\partial u}{\partial \varepsilon} = -u_x\]

Solution

\[u = f(x, y) \quad \mapsto \quad u = f(x - \varepsilon, y)\]
Generalized Symmetries of Differential Equations

Determining equations:

$$\text{pr } \mathbf{v}(\Delta) = 0 \quad \text{whenever} \quad \Delta = 0$$

For totally nondegenerate systems, this is equivalent to

$$\text{pr } \mathbf{v}(\Delta) = \mathcal{D}\Delta = \sum_{\nu} \mathcal{D}_{\nu}\Delta_{\nu}$$

★ $\mathbf{v}$ is a generalized symmetry if and only if its evolutionary form $\mathbf{v}_{\mathcal{Q}}$ is.

• A generalized symmetry is trivial if its characteristic vanishes on solutions to $\Delta$. Two symmetries are equivalent if their evolutionary forms differ by a trivial symmetry.
General Variational Symmetries

**Definition.** A generalized vector field is a variational symmetry if it leaves the variational problem invariant up to a divergence:

\[ \text{pr } \mathbf{v}(L) + L \text{Div } \xi = \text{Div } B \]

\* \( \mathbf{v} \) is a variational symmetry if and only if its evolutionary form \( \mathbf{v}_Q \) is.

\[ \text{pr } \mathbf{v}_Q(L) = \text{Div } \tilde{B} \]
**Theorem.** If $v$ is a variational symmetry, then it is a symmetry of the Euler-Lagrange equations.

**Proof:**

First, $v_Q$ is a variational symmetry if

$$\text{pr } v_Q(L) = \text{Div } P.$$ 

Secondly, integration by parts shows

$$\text{pr } v_Q(L) = D_L(Q) = Q D^*_L(1) + \text{Div } A = Q E(L) + \text{Div } A$$

for some $A$ depending on $Q, L$. Therefore

$$0 = E(\text{pr } v_Q(L)) = E(QE(L)) = E(Q \Delta) = D^*_\Delta Q + D^*_Q \Delta$$

$$= D_\Delta Q + D^*_Q \Delta = \text{pr } v_Q(\Delta) + D^*_Q \Delta$$
**Noether’s Theorem.** Let $\Delta = 0$ be a normal system of Euler-Lagrange equations. Then there is a one-to-one correspondence between (equivalence classes of) nontrivial conservation laws and (equivalence classes of) nontrivial variational symmetries. The characteristic of the conservation law is the characteristic of the associated symmetry.

**Proof:** Noether’s Identity:

$$QE(L) = \text{pr} \, v_Q(L) - \text{Div} \, A = \text{Div}(P - A)$$
The Kepler Problem

\[ x_{tt} + \frac{\mu x}{r^3} = 0 \quad L = \frac{1}{2} x_t^2 - \frac{\mu}{r} \]

Generalized symmetries:

\[ \mathbf{v} = (x \cdot x_{tt}) \partial_x + x_t (x \cdot \partial_x) - 2x (x_t \cdot \partial_x) \]

Conservation law

\[ \text{pr } \mathbf{v}(L) = D_t R \]

where

\[ R = x_t \wedge (x \wedge x_t) - \frac{\mu x}{r} \]

\[ \implies \text{ Runge-Lenz vector} \]
Noether’s Second Theorem. A system of Euler-Lagrange equations is under-determined if and only if it admits an infinite dimensional variational symmetry group depending on an arbitrary function. The associated conservation laws are trivial.

Proof: If $f(x)$ is any function,
\[ f(x) \mathcal{D}(\Delta) = \Delta \mathcal{D}^*(f) + \text{Div} \, P[f, \Delta]. \]
Set
\[ Q = D^*(f). \]
Example.

\[ \int \int (u_x + v_y)^2 \, dx \, dy \]

Euler-Lagrange equations:

\[ \Delta_1 = E^u(L) = u_{xx} + v_{xy} = 0 \]

\[ \Delta_2 = E^v(L) = u_{xy} + v_{yy} = 0 \]

\[ D_x \Delta_2 - D_y \Delta_2 \equiv 0 \]

Symmetries

\[ (u, v) \mapsto (u + \varphi_y, v - \varphi_x) \]