Symmetry Methods for Differential Equations and Conservation Laws

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Symmetry Groups of Differential Equations

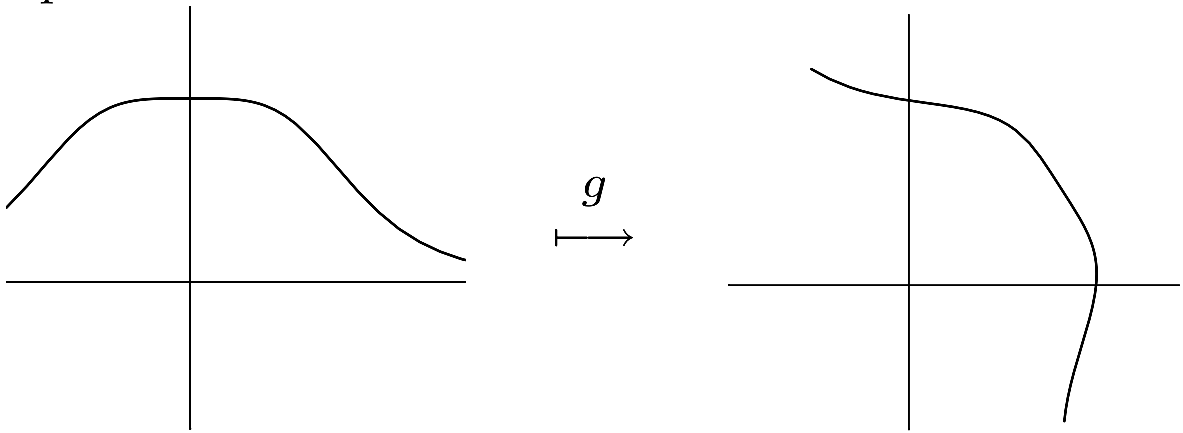
System of differential equations

$$\Delta(x, u^{(n)}) = 0$$

G — Lie group acting on the space of independent and dependent variables:

$$(\widetilde{x}, \widetilde{u}) = g \cdot (x, u) = (\Xi(x, u), \Phi(x, u))$$

G acts on functions $u = f(x)$ by transforming their graphs:



Definition. G is a symmetry group of the system $\Delta = 0$ if $\widetilde{f} = g \cdot f$ is a solution whenever f is.

Infinitesimal Generators

Vector field:

$$\mathbf{v}\big|_{(x,u)} = \frac{d}{d\varepsilon}\, g_\varepsilon \cdot (x,u)\big|_{\varepsilon=0}$$

In local coordinates:

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x,u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \varphi^\alpha(x,u)\frac{\partial}{\partial u^\alpha}$$

generates the one-parameter group

$$\frac{dx^i}{d\varepsilon} = \xi^i(x,u) \qquad \frac{du^\alpha}{d\varepsilon} = \varphi^\alpha(x,u)$$

Example. The vector field

$$\mathbf{v} = -u\,\frac{\partial}{\partial x} + x\,\frac{\partial}{\partial u}$$

generates the rotation group

$$\widetilde{x} = x\cos\varepsilon - u\sin\varepsilon \qquad \widetilde{u} = x\sin\varepsilon + u\cos\varepsilon$$

since

$$\frac{d\widetilde{x}}{d\varepsilon} = -\,\widetilde{u} \qquad \frac{d\widetilde{u}}{d\varepsilon} = \widetilde{x}$$

Jet Spaces

$x = (x^1, \ldots, x^p)$ — independent variables

$u = (u^1, \ldots, u^q)$ — dependent variables

$$u^\alpha_J = \frac{\partial^k u^\alpha}{\partial x^{j_1} \; \ldots \; \partial x^k}$$ — partial derivatives

$(x, u^{(n)}) = (\; \ldots \; x^i \; \ldots \; u^\alpha \; \ldots \; u^\alpha_J \; \ldots \;) \in \mathrm{J}^n$

— jet coordinates

$$\dim \mathrm{J}^n = p + q^{(n)} = p + q \binom{p+n}{n}$$

Prolongation to Jet Space

Since G acts on functions, it acts on their derivatives, leading to the prolonged group action on the jet space:

$$(\widetilde{x}, \widetilde{u}^{(n)}) = \mathrm{pr}^{(n)}\, g \cdot (x, u^{(n)})$$

$\Longrightarrow$ formulas provided by implicit differentiation

Prolonged vector field or infinitesimal generator:

$$\mathrm{pr}\, \mathbf{v} = \mathbf{v} + \sum_{\alpha, J} \varphi_J^\alpha(x, u^{(n)}) \frac{\partial}{\partial u_J^\alpha}$$

The coefficients of the prolonged vector field are given by the explicit prolongation formula:

$$\boxed{\varphi_J^\alpha = D_J\, Q^\alpha + \sum_{i=1}^{p} \xi^i\, u_{J,i}^\alpha}$$

$Q = (Q^1, \ldots, Q^q)$ — characteristic of $\mathbf{v}$

$$Q^\alpha(x, u^{(1)}) = \varphi^\alpha - \sum_{i=1}^{p} \xi^i \frac{\partial u^\alpha}{\partial x^i}$$

⋆ Invariant functions are solutions to

$$Q(x, u^{(1)}) = 0.$$

Symmetry Criterion

Theorem. (Lie) A connected group of transformations G is a symmetry group of a nondegenerate system of differential equations $\Delta = 0$ if and only if

$$\operatorname{pr} \mathbf{v}(\Delta) = 0 \qquad (*)$$

whenever u is a solution to $\Delta = 0$ for every infinitesimal generator $\mathbf{v}$ of G.

(*) are the determining equations of the symmetry group to $\Delta = 0$. For nondegenerate systems, this is equivalent to

$$\operatorname{pr} \mathbf{v}(\Delta) = A \cdot \Delta = \sum_{\nu} A_\nu \Delta_\nu$$

Nondegeneracy Conditions

Maximal Rank:

$$\operatorname{rank}\left(\cdots \frac{\partial \Delta_\nu}{\partial x^i} \cdots \frac{\partial \Delta_\nu}{\partial u_J^\alpha} \cdots \right) = \max$$

Local Solvability: Any each point $(x_0, u_0^{(n)})$ such that

$$\Delta(x_0, u_0^{(n)}) = 0$$

there exists a solution $u = f(x)$ with

$$u_0^{(n)} = \operatorname{pr}^{(n)} f(x_0)$$

Nondegenerate = maximal rank + locally solvable

Normal: There exists at least one non-characteristic direction at $(x_0, u_0^{(n)})$ or, equivalently, there is a change of variable making the system into Kovalevskaya form

$$\frac{\partial^n u^\alpha}{\partial t^n} = \Gamma^\alpha(x, \tilde{u}^{(n)})$$

Theorem. (Finzi) A system of q partial differential equations $\Delta = 0$ in q unknowns is not normal if and only if there is a nontrivial integrability condition:

$$\mathcal{D}\,\Delta = \sum_\nu \mathcal{D}_\nu \Delta_\nu = Q \qquad \text{order}\, Q < \text{order}\, \mathcal{D} + \text{order}\, \Delta$$

Under-determined: The integrability condition follows from lower order derivatives of the equation:

$$\widetilde{\mathcal{D}}\,\Delta \equiv 0$$

Example:

$$\Delta_1 = u_{xx} + v_{xy}, \quad \Delta_2 = u_{xy} + v_{yy}$$

$$D_x\Delta_2 - D_y\Delta_1 \equiv 0$$

Over-determined: The integrability condition is genuine.

Example:

$$\Delta_1 = u_{xx} + v_{xy} - v_y, \quad \Delta_2 = u_{xy} + v_{yy} + u_y$$

$$D_x\Delta_2 - D_y\Delta_1 = u_{xy} + v_{yy}$$

A Simple O.D.E.

$$\boxed{u_{xx} = 0}$$

Infinitesimal symmetry generator:

$$\mathbf{v} = \xi(x,u)\,\frac{\partial}{\partial x} + \varphi(x,u)\,\frac{\partial}{\partial u}$$

Second prolongation:

$$\mathbf{v}^{(2)} = \xi(x,u)\,\frac{\partial}{\partial x} + \varphi(x,u)\,\frac{\partial}{\partial u} + \\ + \;\varphi_1(x,u^{(1)})\,\frac{\partial}{\partial u_x} + \varphi_2(x,u^{(2)})\,\frac{\partial}{\partial u_{xx}}\,,$$

$$
\begin{aligned}
\varphi_1 &= \varphi_x + (\varphi_u - \xi_x)u_x - \xi_u u_x^2, \\
\varphi_2 &= \varphi_{xx} + (2\varphi_{xu} - \xi_{xx})u_x + (\varphi_{uu} - 2\xi_{xu})u_x^2 \; - \\
&\qquad\qquad - \xi_{uu}u_x^3 + (\varphi_u - 2\xi_x)u_{xx} - 3\xi_u u_x u_{xx}.
\end{aligned}
$$

Symmetry criterion:

$$
\varphi_2 = 0 \qquad \text{whenever} \qquad u_{xx} = 0.
$$

Symmetry criterion:

$$\varphi_{xx} + (2\varphi_{xu} - \xi_{xx})u_x + (\varphi_{uu} - 2\xi_{xu})u_x^2 - \xi_{uu}u_x^3 = 0.$$

Determining equations:

$$\varphi_{xx} = 0 \quad 2\varphi_{xu} = \xi_{xx} \quad \varphi_{uu} = 2\xi_{xu} \quad \xi_{uu} = 0$$

$$\Longrightarrow \quad \textit{Linear!}$$

General solution:

$$\xi(x,u) = c_1 x^2 + c_2 xu + c_3 x + c_4 u + c_5$$

$$\varphi(x,u) = c_1 xu + c_2 u^2 + c_6 x + c_7 u + c_8$$

Symmetry algebra:

$$\begin{aligned} \mathbf{v}_1 &= \partial_x & \mathbf{v}_5 &= u\partial_x \\ \mathbf{v}_2 &= \partial_u & \mathbf{v}_6 &= u\partial_u \\ \mathbf{v}_3 &= x\partial_x & \mathbf{v}_7 &= x^2\partial_x + xu\partial_u \\ \mathbf{v}_4 &= x\partial_u & \mathbf{v}_8 &= xu\partial_x + u^2\partial_u \end{aligned}$$

Symmetry Group:

$$(x, u) \longmapsto \left(\frac{ax + bu + c}{hx + ju + k}, \frac{dx + eu + f}{hx + ju + k} \right)$$

$$\Longrightarrow \quad \text{projective group}$$

Prolongation

Infinitesimal symmetry

$$\mathbf{v} = \xi(x,t,u)\,\frac{\partial}{\partial x} + \tau(x,t,u)\,\frac{\partial}{\partial t} + \varphi(x,t,u)\,\frac{\partial}{\partial u}$$

First prolongation

$$\mathrm{pr}^{(1)}\,\mathbf{v} = \xi\,\frac{\partial}{\partial x} + \tau\,\frac{\partial}{\partial t} + \varphi\,\frac{\partial}{\partial u} + \varphi^{x}\,\frac{\partial}{\partial u_x} + \varphi^{t}\,\frac{\partial}{\partial u_t}$$

Second prolongation

$$\mathrm{pr}^{(2)}\,\mathbf{v} = \mathrm{pr}^{(1)}\,\mathbf{v} + \varphi^{xx}\,\frac{\partial}{\partial u_{xx}} + \varphi^{xt}\,\frac{\partial}{\partial u_{xt}} + \varphi^{tt}\,\frac{\partial}{\partial u_{tt}}$$

where

$$\varphi^x = D_x Q + \xi u_{xx} + \tau u_{xt}$$

$$\varphi^t = D_t Q + \xi u_{xt} + \tau u_{tt}$$

$$\varphi^{xx} = D_x^2 Q + \xi u_{xxt} + \tau u_{xtt}$$

Characteristic

$$Q = \varphi - \xi u_x - \tau u_t$$

$$\begin{aligned}\varphi^x &= D_x Q + \xi u_{xx} + \tau u_{xt} \\ &= \varphi_x + (\varphi_u - \xi_x) u_x - \tau_x u_t - \xi_u u_x^2 - \tau_u u_x u_t\end{aligned}$$

$$\begin{aligned}\varphi^t &= D_t Q + \xi u_{xt} + \tau u_{tt} \\ &= \varphi_t - \xi_t u_x + (\varphi_u - \tau_t) u_t - \xi_u u_x u_t - \tau_u u_t^2\end{aligned}$$

$$\begin{aligned}\varphi^{xx} &= D_x^2 Q + \xi u_{xxt} + \tau u_{xtt} \\ &= \varphi_{xx} + (2\varphi_{xu} - \xi_{xx}) u_x - \tau_{xx} u_t \\ &\quad + (\varphi_{uu} - 2\xi_{xu}) u_x^2 - 2\tau_{xu} u_x u_t - \xi_{uu} u_x^3 - \\ &\quad - \tau_{uu} u_x^2 u_t + (\varphi_u - 2\xi_x) u_{xx} - 2\tau_x u_{xt} \\ &\quad - 3\xi_u u_x u_{xx} - \tau_u u_t u_{xx} - 2\tau_u u_x u_{xt}\end{aligned}$$

Heat Equation

$$u_t = u_{xx}$$

Infinitesimal symmetry criterion

$$\varphi^t = \varphi^{xx} \qquad \text{whenever} \qquad u_t = u_{xx}$$

Determining equations

Coefficient	*Monomial*
$0 = -2\tau_u$	$u_x u_{xt}$
$0 = -2\tau_x$	u_{xt}
$0 = -\tau_{uu}$	$u_x^2 u_{xx}$
$-\xi_u = -2\tau_{xu} - 3\xi_u$	$u_x u_{xx}$
$\varphi_u - \tau_t = -\tau_{xx} + \varphi_u - 2\xi_x$	u_{xx}
$0 = -\xi_{uu}$	u_x^3
$0 = \varphi_{uu} - 2\xi_{xu}$	u_x^2
$-\xi_t = 2\varphi_{xu} - \xi_{xx}$	u_x
$\varphi_t = \varphi_{xx}$	1

General solution

$$\xi = c_1 + c_4 x + 2c_5 t + 4c_6 xt$$

$$\tau = c_2 + 2c_4 t + 4c_6 t^2$$

$$\varphi = (c_3 - c_5 x - 2c_6 t - c_6 x^2)u + \alpha(x,t)$$

$$\alpha_t = \alpha_{xx}$$

Symmetry algebra

$\mathbf{v}_1 = \partial_x$	space transl.
$\mathbf{v}_2 = \partial_t$	time transl.
$\mathbf{v}_3 = u\partial_u$	scaling
$\mathbf{v}_4 = x\partial_x + 2t\partial_t$	scaling
$\mathbf{v}_5 = 2t\partial_x - xu\partial_u$	Galilean
$\mathbf{v}_6 = 4xt\partial_x + 4t^2\partial_t - (x^2 + 2t)u\partial_u$	inversions
$\mathbf{v}_\alpha = \alpha(x,t)\partial_u$	linearity

Potential Burgers' equation

$$u_t = u_{xx} + u_x^2$$

Infinitesimal symmetry criterion

$$\varphi^t = \varphi^{xx} + 2u_x \varphi^x$$

Determining equations

Coefficient	*Monomial*
$0 = -2\tau_u$	$u_x u_{xt}$
$0 = -2\tau_x$	u_{xt}
$-\tau_u = -\tau_u$	u_{xx}^2
$-2\tau_u = -\tau_{uu} - 3\tau_u$	$u_x^2 u_{xx}$
$-\xi_u = -2\tau_{xu} - 3\xi_u - 2\tau_x$	$u_x u_{xx}$
$\varphi_u - \tau_t = -\tau_{xx} + \varphi_u - 2\xi_x$	u_{xx}
$-\tau_u = -\tau_{uu} - 2\tau_u$	u_x^4
$-\xi_u = -2\tau_{xu} - \xi_{uu} - 2\tau_x - 2\xi_u$	u_x^3
$\varphi_u - \tau_t = -\tau_{xx} + \varphi_{uu} - 2\xi_{xu} + 2\varphi_u - 2\xi_x$	u_x^2
$-\xi_t = 2\varphi_{xu} - \xi_{xx} + 2\varphi_x$	u_x
$\varphi_t = \varphi_{xx}$	1

General solution

$$\xi = c_1 + c_4 x + 2c_5 t + 4c_6 xt$$

$$\tau = c_2 + 2c_4 t + 4c_6 t^2$$

$$\varphi = c_3 - c_5 x - 2c_6 t - c_6 x^2 + \alpha(x,t)e^{-u}$$

$$\alpha_t = \alpha_{xx}$$

Symmetry algebra

$$\mathbf{v}_1 = \partial_x$$

$$\mathbf{v}_2 = \partial_t$$

$$\mathbf{v}_3 = \partial_u$$

$$\mathbf{v}_4 = x\partial_x + 2t\partial_t$$

$$\mathbf{v}_5 = 2t\partial_x - x\partial_u$$

$$\mathbf{v}_6 = 4xt\partial_x + 4t^2\partial_t - (x^2 + 2t)\partial_u$$

$$\mathbf{v}_\alpha = \alpha(x,t)e^{-u}\partial_u$$

Hopf-Cole $w = e^u$ maps to heat equation.

Symmetry–Based Solution Methods

Ordinary Differential Equations

- Lie's method
- Solvable groups
- Variational and Hamiltonian systems
- Potential symmetries
- Exponential symmetries
- Generalized symmetries

Partial Differential Equations

- Group-invariant solutions
- Non-classical method
- Weak symmetry groups
- Clarkson-Kruskal method
- Partially invariant solutions
- Differential constraints
- Nonlocal Symmetries
- Separation of Variables

Integration of O.D.E.'s

First order ordinary differential equation

$$\frac{du}{dx} = F(x, u)$$

Symmetry Generator:

$$\mathbf{v} = \xi(x, u)\,\frac{\partial}{\partial x} + \varphi(x, u)\,\frac{\partial}{\partial u}$$

Determining equation

$$\varphi_x + (\varphi_u - \xi_x)F - \xi_u F^2 = \xi\,\frac{\partial F}{\partial x} + \varphi\,\frac{\partial F}{\partial u}$$

♠ Trivial symmetries

$$\frac{\varphi}{\xi} = F$$

Method 1: Rectify the vector field.

$$\mathbf{v}\big|_{(x_0,u_0)} \neq 0$$

Introduce new coordinates

$$y = \eta(x,u) \qquad w = \zeta(x,u)$$

near (x_0, u_0) so that

$$\mathbf{v} = \frac{\partial}{\partial w}$$

These satisfy first order p.d.e.'s

$$\xi\,\eta_x + \varphi\,\eta_u = 0 \qquad \xi\,\zeta_x + \varphi\,\zeta_u = 1$$

Solution by method of characteristics:

$$\frac{dx}{\xi(x,u)} = \frac{du}{\varphi(x,u)} = \frac{dt}{1}$$

The equation in the new coordinates will be invariant if and only if it has the form

$$\frac{dw}{dy} = h(y)$$

and so can clearly be integrated by quadrature.

Method 2: Integrating Factor

If $\mathbf{v} = \xi\,\partial_x + \varphi\,\partial_u$ is a symmetry for

$$P(x,u)\,dx + Q(x,u)\,du = 0$$

then

$$R(x,u) = \frac{1}{\xi\,P + \varphi\,Q}$$

is an integrating factor.

♠ If

$$\frac{\varphi}{\xi} = -\,\frac{P}{Q}$$

then the integrating factor is trivial. Also, rectification of the vector field is equivalent to solving the original ordinary differential equation.

Higher Order Ordinary Differential Equations

$$\Delta(x, u^{(n)}) = 0$$

If we know a one-parameter symmetry group

$$\mathbf{v} = \xi(x,u)\,\frac{\partial}{\partial x} + \varphi(x,u)\,\frac{\partial}{\partial u}$$

then we can reduce the order of the equation by 1.

Method 1: Rectify $\mathbf{v} = \partial_w$. Then the equation is invariant if and only if it does not depend on w:

$$\Delta(y, w', \ \ldots \ , w_n) = 0$$

Set $v = w'$ to reduce the order.

Method 2: Differential invariants.

$$h[\operatorname{pr}^{(n)} g \cdot (x, u^{(n)})] = h(x, u^{(n)}), \qquad g \in G$$

Infinitesimal criterion: $\operatorname{pr} \mathbf{v}(h) = 0.$

Proposition. If η, ζ are n^{th} order differential invariants, then

$$\frac{d\eta}{d\zeta} = \frac{D_x \eta}{D_x \zeta}$$

is an $(n+1)^{\text{st}}$ order differential invariant.

Corollary. Let

$$y = \eta(x, u), \qquad w = \zeta(x, u, u')$$

be the independent first order differential invariants for G. Any n^{th} order o.d.e. admitting G as a symmetry group can be written in terms of the differential invariants $y, w, dw/dy, \ \ldots \ , d^{n-1}w/dy^{n-1}$.

In terms of the differential invariants, the n^{th} order o.d.e. reduces to

$$\widetilde{\Delta}(y, w^{(n-1)}) = 0$$

For each solution $w = g(y)$ of the reduced equation, we must solve the auxiliary equation

$$\zeta(x, u, u') = g[\eta(x, u)]$$

to find $u = f(x)$. This first order equation admits G as a symmetry group and so can be integrated as before.

Multiparameter groups

- G - r-dimensional Lie group.

  Assume $\mathrm{pr}^{(r)}\, G$ acts regularly with r dimensional orbits.

Independent r^{th} order differential invariants:

$$y = \eta(x, u^{(r)}) \qquad w = \zeta(x, u^{(r)})$$

Independent n^{th} order differential invariants:

$$y, w, \frac{dw}{dy}\ ,\ \ldots\ ,\frac{d^{n-r} w}{dy^{n-r}}\ .$$

In terms of the differential invariants, the equation reduces in order by r:

$$\widetilde{\Delta}(y, w^{(n-r)}) = 0$$

For each solution $w = g(y)$ of the reduced equation, we must solve the auxiliary equation

$$\zeta(x, u^{(r)}) = g[\,\eta(x, u^{(r)})\,]$$

to find $u = f(x)$. In this case there is no guarantee that we can integrate this equation by quadrature.

Example. Projective group $G = \mathrm{SL}(2)$

$$(x, u) \quad \longmapsto \quad \left(x, \ \frac{a\,u + b}{c\,u + d} \right), \qquad a\,d - b\,c = 1.$$

Infinitesimal generators:

$$\partial_u, \qquad u\,\partial_u, \qquad u^2\,\partial_u$$

Differential invariants:

$$x, \qquad w = \frac{2\,u'\,u''' - 3\,u''^2}{u'^2}$$

$\Longrightarrow$ Schwarzian derivative

Reduced equation

$$\widetilde{\Delta}(y, w^{(n-3)}) = 0$$

Let $w = h(x)$ be a solution to reduced equation.

To recover $u = f(x)$ we must solve the auxiliary equation:

$$2\,u'\,u''' - 3\,u''^2 = u'^2\,h(x),$$

which still admits the full group SL(2).

Integrate using ∂_u:

$$u' = z \qquad 2\,z\,z'' - z'^2 = z^2\,h(x)$$

Integrate using $u\,\partial_u = z\,\partial_z$:

$$v = (\log z)' \qquad 2\,v' + v^2 = h(x)$$

No further symmetries, so we are stuck with a Riccati equation to effect the solution.

Solvable Groups

- Basis $\mathbf{v}_1, \ldots, \mathbf{v}_r$ of the symmetry algebra $\mathfrak{g}$ such that

$$[\,\mathbf{v}_i, \mathbf{v}_j\,] = \sum_{k<j} c_{ij}^k \mathbf{v}_k, \qquad i < j$$

If we reduce in the correct order, then we are guaranteed a symmetry at each stage. Reduced equation for subalgebra $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$:

$$\widetilde{\Delta}^{(k)}(y, w^{(n-k)}) = 0$$

admits a symmetry $\widetilde{\mathbf{v}}_{k+1}$ corresponding to $\mathbf{v}_{k+1}$.

Theorem. (Bianchi) If an n^{th} order o.d.e. has a (regular) r-parameter solvable symmetry group, then its solutions can be found by quadrature from those of the $(n-r)^{\text{th}}$ order reduced equation.

Example.

$$x^2\, u'' = f(x\, u' - u)$$

Symmetry group:

$$\mathbf{v} = x\,\partial_u, \qquad \mathbf{w} = x\,\partial_x,$$

$$[\,\mathbf{v}, \mathbf{w}\,] = -\,\mathbf{v}.$$

Reduction with respect to $\mathbf{v}$:

$$z = x\, u' - u$$

Reduced equation:

$$x\, z' = h(z)$$

still invariant under $\mathbf{w} = x\,\partial_x$, and hence can be solved by quadrature.

Wrong way reduction with respect to $\mathbf{w}$:

$$y = u, \qquad z = z(y) = x\,u'$$

Reduced equation:

$$z(z' - 1) = h(z - y)$$

- No remaining symmetry; not clear how to integrate directly.

Group Invariant Solutions

System of partial differential equations

$$\Delta(x, u^{(n)}) = 0$$

G — symmetry group

Assume G acts regularly on M with r-dimensional orbits

Definition. $u = f(x)$ is a G-invariant solution if

$$g \cdot f = f \qquad \text{for all} \qquad g \in G.$$

i.e. the graph $\Gamma_f = \{(x, f(x))\}$ is a (locally) G-invariant subset of M.

- Similarity solutions, travelling waves, ...

Proposition. Let G have infinitesimal generators $\mathbf{v}_1, \ldots, \mathbf{v}_r$ with associated characteristics $Q_1, \ldots, Q_r$. A function $u = f(x)$ is G-invariant if and only if it is a solution to the system of first order partial differential equations

$$Q_\nu(x, u^{(1)}) = 0, \qquad \nu = 1, \ \ldots \ , r.$$

Theorem. (Lie). If G has r-dimensional orbits, and acts transversally to the vertical fibers $\{x = \text{const.}\}$, then all the G-invariant solutions to $\Delta = 0$ can be found by solving a reduced system of differential equations $\Delta/G = 0$ in r fewer independent variables.

Method 1: Invariant Coordinates.

The new variables are given by a complete set of functionally independent invariants of G:

$$\eta_\alpha(g \cdot (x,u)) = \eta_\alpha(x,u) \qquad \text{for all} \qquad g \in G$$

Infinitesimal criterion:

$$\mathbf{v}_k[\eta_\alpha] = 0, \qquad k = 1, \ \ldots \ , r.$$

New independent and dependent variables:

$$y_1 = \eta_1(x,u), \ \ldots \ , y_{p-r} = \eta_{p-r}(x,u)$$

$$w_1 = \zeta_1(x,u), \ \ldots \ , w^q = \zeta^q(x,u)$$

Invariant functions:

$$w = \eta(y), \qquad \text{i.e.} \qquad \zeta(x,u) = h[\eta(x,u)]$$

Reduced equation:

$$\Delta/G(y, w^{(n)}) = 0$$

Every solution determines a G-invariant solution to the original p.d.e.

Example. The heat equation $\qquad u_t = u_{xx}$

Scaling symmetry: $\qquad x\,\partial_x + 2\,t\,\partial_t + a\,u\,\partial_u$

Invariants: $\qquad y = \dfrac{x}{\sqrt{t}}, \qquad w = t^{-a}u$

$$u = t^a w(y), \quad u_t = t^{a-1}(\,-\tfrac{1}{2}\,y\,w' + a\,w\,), \quad u_{xx} = t^a w''.$$

Reduced equation

$$w'' + 12yw' - aw = 0$$

Solution: $\qquad w = e^{-y^2/8}U(\,2\,a + \tfrac{1}{2}, y/\sqrt{2}\,)$

$\Longrightarrow$ parabolic cylinder function

Similarity solution:

$$u(x,t) = t^a e^{-x^2/8t}U(\,2\,a + \tfrac{1}{2}, x/\sqrt{2\,t}\,)$$

Example. The heat equation $u_t = u_{xx}$

Galilean symmetry: $2\,t\,\partial_x - x\,u\,\partial_u$

Invariants: $y = t \qquad w = e^{x^2/4t}u$

$$u = e^{-\,x^2/4t}w(y), \qquad u_t = e^{-\,x^2/4t}\left(w' + \frac{x^2}{4t^2}\,w \right),$$

$$u_{xx} = e^{-\,x^2/4t}\left(\frac{x^2}{4t^2} - \frac{1}{2\,t} \right) w.$$

Reduced equation: $2\,y\,w' + w = 0$

Source solution: $w = k\,y^{-1/2}, \qquad u = \dfrac{k}{\sqrt{t}}\,e^{x^2/4t}$

Method 2: Direct substitution:

Solve the combined system

$$\Delta(x, u^{(n)}) = 0 \qquad Q_k(x, u^{(1)}) = 0, \qquad k = 1, \ \ldots \ , r$$

as an overdetermined system of p.d.e.

For a one-parameter group, we solve

$$Q(x, u^{(1)}) = 0$$

for

$$\frac{\partial u^\alpha}{\partial x^p} = \frac{\varphi^\alpha}{\xi^n} - \sum_{i=1}^{p-1} \frac{\xi^i}{\xi^p} \frac{\partial u^\alpha}{\partial x^i}$$

Rewrite in terms of derivatives with respect to $x_1, \ldots, x_{p-1}$. The reduced equation has x^p as a parameter. Dependence on x^p can be found by by substituting back into the characteristic condition.

Classification of invariant solutions

Let G be the full symmetry group of the system $\Delta = 0$. Let $H \subset G$ be a subgroup. If $u = f(x)$ is an H-invariant solution, and $g \in G$ is another group element, then $\widetilde{f} = g \cdot f$ is an invariant solution for the conjugate subgroup $\widetilde{H} = g \cdot H \cdot g^{-1}$.

- Classification of subgroups of G under conjugation.
- Classification of subalgebras of $\mathfrak{g}$ under the adjoint action.
- Exploit symmetry of the reduced equation

Non-Classical Method

$\Longrightarrow$ Bluman and Cole

Here we require not invariance of the original partial differential equation, but rather invariance of the combined system

$$\Delta(x, u^{(n)}) = 0 \qquad Q_k(x, u^{(1)}) = 0, \qquad k = 1, \ \ldots \ , r$$

- Nonlinear determining equations.
- Most solutions derived using this approach come from ordinary group invariance anyway.

Weak (Partial) Symmetry Groups

Here we require invariance of

$$\Delta(x, u^{(n)}) = 0 \qquad Q_k(x, u^{(1)}) = 0, \qquad k = 1, \ \ldots \ , r$$

and all the associated integrability conditions

- Every group is a weak symmetry group.
- Every solution can be derived in this way.
- Compatibility of the combined system?
- Overdetermined systems of partial differential equations.

The Boussinesq Equation

$$u_{tt} + \tfrac{1}{2}(u^2)_{xx} + u_{xxxx} = 0$$

Classical symmetry group:

$$\mathbf{v}_1 = \partial_x \qquad \mathbf{v}_2 = \partial_t \qquad \mathbf{v}_3 = x\,\partial_x + 2\,t\,\partial_t - 2\,u\,\partial_u$$

For the scaling group

$$-\,Q = x\,u_x + 2\,t\,u_t + 2\,u = 0$$

Invariants:

$$y = \frac{x}{\sqrt{t}} \qquad w = t\,u \qquad u = \frac{1}{t}\,w\left(\frac{x}{\sqrt{t}}\right)$$

Reduced equation:

$$w'''' + \tfrac{1}{2}\,(w^2)'' + \tfrac{1}{4}\,y^2 w'' + \tfrac{7}{4}\,y\,w' + 2w = 0$$

$$u_{tt} + \tfrac{1}{2}(u^2)_{xx} + u_{xxxx} = 0$$

Group classification:

$$\mathbf{v}_1 = \partial_x \qquad \mathbf{v}_2 = \partial_t \qquad \mathbf{v}_3 = x\,\partial_x + 2\,t\,\partial_t - 2\,u\,\partial_u$$

Note:

$$\mathrm{Ad}(\varepsilon\,\mathbf{v}_3)\,\mathbf{v}_1 = e^{\varepsilon}\,\mathbf{v}_1 \qquad \mathrm{Ad}(\varepsilon\,\mathbf{v}_3)\mathbf{v}_2 = e^{2\,\varepsilon}\,\mathbf{v}_2$$

$$\mathrm{Ad}(\delta\,\mathbf{v}_1 + \varepsilon\,\mathbf{v}_2)\mathbf{v}_3 = \mathbf{v}_3 - \delta\,\mathbf{v}_1 - \varepsilon\,\mathbf{v}_2$$

so the one-dimensional subalgebras are classified by:

$$\{\,\mathbf{v}_3\,\} \qquad \{\,\mathbf{v}_1\,\} \qquad \{\,\mathbf{v}_2\,\} \qquad \{\,\mathbf{v}_1 + \mathbf{v}_2\,\} \qquad \{\,\mathbf{v}_1 - \mathbf{v}_2\,\}$$

and we only need to determine solutions invariant under these particular subgroups to find the most general group-invariant solution.

$$u_{tt} + \tfrac{1}{2}(u^2)_{xx} + u_{xxxx} = 0$$

Non-classical: Galilean group

$$\mathbf{v} = t\,\partial_x + \partial_t - 2\,t\,\partial_u$$

Not a symmetry, but the combined system

$$u_{tt} + \tfrac{1}{2}(u^2)_{xx} + u_{xxxx} = 0 \qquad -Q = t\,u_x + u_t + 2\,t = 0$$

does admit $\mathbf{v}$ as a symmetry. Invariants:

$$y = x - \tfrac{1}{2}\,t^2, \qquad w = u + t^2, \qquad u(x,t) = w(y) - t^2$$

Reduced equation:

$$w'''' + ww'' + (w')^2 - w' + 2 = 0$$

$$u_{tt} + \tfrac{1}{2}(u^2)_{xx} + u_{xxxx} = 0$$

Weak Symmetry: Scaling group: $\quad x\,\partial_x + t\,\partial_t$

Not a symmetry of the combined system

$$u_{tt} + \tfrac{1}{2}(u^2)_{xx} + u_{xxxx} = 0 \qquad Q = x\,u_x + t\,u_t = 0$$

Invariants: $\quad y = \dfrac{x}{t} \quad u \qquad$ Invariant solution: $\quad u(x,t) = w(y)$

The Boussinesq equation reduces to

$$t^{-4} w'''' + t^{-2}[(w+1-y)w'' + (w')^2 - y\,w'] = 0$$

so we obtain an overdetermined system

$$w'''' = 0 \qquad (w+1-y)w'' + (w')^2 - y\,w' = 0$$

Solutions: $\quad w(y) = \tfrac{2}{3}\,y^2 - 1, \qquad$ or $\qquad w = \text{constant}$

Similarity solution: $\quad u(x,t) = \dfrac{2\,x^2}{3\,t^2} - 1$

Symmetries and Conservation Laws

Variational problems

$$L[u] = \int_\Omega L(x, u^{(n)})\, dx$$

Euler-Lagrange equations

$$\Delta = E(L) = 0$$

Euler operator (variational derivative)

$$E^\alpha(L) = \frac{\delta L}{\delta u^\alpha} = \sum_J (-D)^J \frac{\partial L}{\partial u_J^\alpha}$$

Theorem. (Null Lagrangians)

$$E(L) \equiv 0 \qquad \text{if and only if} \qquad L = \operatorname{Div} P$$

Theorem. The system $\Delta = 0$ is the Euler-Lagrange equations for some variational problem if and only if the Fréchet derivative D_Δ is self-adjoint:

$$D_\Delta^* = D_\Delta.$$

$\Longrightarrow$ Helmholtz conditions

Fréchet derivative

Given $P(x, u^{(n)})$, its Fréchet derivative or formal linearization is the differential operator D_P defined by

$$D_P[w] = \frac{d}{d\varepsilon} P[u + \varepsilon w] \Big|_{\varepsilon = 0}$$

Example.

$$P = u_{xxx} + uu_x$$

$$D_P = D_x^3 + uD_x + u_x$$

Adjoint (formal)

$$\mathcal{D} = \sum_J A_J D^J \qquad \mathcal{D}^* = \sum_J (-D)^J \cdot A_J$$

Integration by parts formula:

$$P\,\mathcal{D} Q = Q\,\mathcal{D}^* P + \operatorname{Div} A$$

where A depends on P, Q.

Conservation Laws

Definition. A conservation law of a system of partial differential equations is a divergence expression

$$\mathrm{Div}\, P = 0$$

which vanishes on all solutions to the system.

$$P = (P_1(x, u^{(k)}), \ldots, P_p(x, u^{(k)}))$$

$\Longrightarrow$ The integral

$$\int P \cdot dS$$

is path (surface) independent.

If one of the coordinates is time, a conservation law takes the form

$$D_t T + \operatorname{Div} X = 0$$

T — conserved density $\qquad$ X — flux

By the divergence theorem,

$$\int_\Omega T(x,t,u^{(k)})\,dx\,\Big|_{t=a}^{b} = \int_a^b \int_\Omega X \cdot dS\,dt$$

depends only on the boundary behavior of the solution.

- If the flux X vanishes on $\partial\Omega$, then $\int_\Omega T\,dx$ is conserved (constant).

Trivial Conservation Laws

Type I If $P = 0$ for all solutions to $\Delta = 0$, then $\operatorname{Div} P = 0$ on solutions too

Type II (Null divergences) If $\operatorname{Div} P = 0$ for *all* functions $u = f(x)$, then it trivially vanishes on solutions.

Examples:

$$D_x(u_y) + D_y(-u_x) \equiv 0$$

$$D_x \frac{\partial(u,v)}{\partial(y,z)} + D_y \frac{\partial(u,v)}{\partial(z,x)} + D_z \frac{\partial(u,v)}{\partial(x,y)} \equiv 0$$

Theorem.

$$\operatorname{Div} P(x, u^{(k)}) \equiv 0$$

for all u if and only if

$$P = \operatorname{Curl} Q(x, u^{(k)})$$

i.e.

$$P_i = \sum_{j=1}^{p} D_j Q_{ij} \qquad Q_{ij} = -Q_{ji}$$

Two conservation laws P and $\widetilde{P}$ are equivalent if they differ by a sum of trivial conservation laws:

$$P = \widetilde{P} + P_I + P_{II}$$

where

$$P_I = 0 \;\text{ on solutions} \qquad \operatorname{Div} P_{II} \equiv 0.$$

Proposition. Every conservation law of a system of partial differential equations is equivalent to a conservation law in characteristic form

$$\operatorname{Div} P = Q \cdot \Delta = \sum_{\nu} Q_\nu \Delta_\nu$$

Proof:

$$\operatorname{Div} P = \sum_{\nu, J} Q_\nu^J D^J \Delta_\nu$$

Integrate by parts:

$$\operatorname{Div} \widetilde{P} = \sum_{\nu, J} (-D)^J Q_\nu^J \cdot \Delta_\nu \qquad Q_\nu = \sum_J (-D)^J Q_\nu^J$$

Q is called the characteristic of the conservation law.

Theorem. Q is the characteristic of a conservation law for $\Delta = 0$ if and only if

$$D_\Delta^* Q + D_Q^* \Delta = 0.$$

Proof:

$$0 = E(\operatorname{Div} P) = E(Q \cdot \Delta) = D_\Delta^* Q + D_Q^* \Delta$$

Normal Systems

A characteristic is trivial if it vanishes on solutions. Two characteristics are equivalent if they differ by a trivial one.

Theorem. Let $\Delta = 0$ be a normal system of partial differential equations. Then there is a one-to-one correspondence between (equivalence classes of) nontrivial conservation laws and (equivalence classes of) nontrivial characteristics.

Variational Symmetries

Definition. A (restricted) variational symmetry is a transformation $(\widetilde{x}, \widetilde{u}) = g \cdot (x, u)$ which leaves the variational problem invariant:

$$\int_{\widetilde{\Omega}} L(\widetilde{x}, \widetilde{u}^{(n)})\, d\widetilde{x} = \int_{\Omega} L(x, u^{(n)})\, dx$$

Infinitesimal criterion:

$$\operatorname{pr} \mathbf{v}(L) + L \operatorname{Div} \xi = 0$$

Theorem. If $\mathbf{v}$ is a variational symmetry, then it is a symmetry of the Euler-Lagrange equations.

⋆⋆ But not conversely!

Noether's Theorem (Weak version). If $\mathbf{v}$ generates a one-parameter group of variational symmetries of a variational problem, then the characteristic Q of $\mathbf{v}$ is the characteristic of a conservation law of the Euler-Lagrange equations:

$$\operatorname{Div} P = Q\, E(L)$$

Elastostatics

$$\int W(x, \nabla u)\, dx \qquad \text{— stored energy}$$

$$x, u \in \mathbb{R}^p, \quad p = 2, 3$$

Frame indifference

$$u \longmapsto R\,u + a, \quad R \in \mathrm{SO}(p)$$

Conservation laws = path independent integrals:

$$\operatorname{Div} P = 0.$$

1. Translation invariance

$$P_i = \frac{\partial W}{\partial u_i^\alpha}$$

$$\Longrightarrow \quad \text{Euler-Lagrange equations}$$

2. Rotational invariance

$$P_i = u_i^\alpha \frac{\partial W}{\partial u_j^\beta} - u_i^\beta \frac{\partial W}{\partial u_j^\alpha}$$

3. Homogeneity : $W = W(\nabla u) \quad x \longmapsto x + a$

$$P_i = \sum_{\alpha=1}^{p} u_j^\alpha \frac{\partial W}{\partial u_i^\alpha} - \delta_j^i W$$

$$\Longrightarrow \quad \text{Energy-momentum tensor}$$

4. Isotropy : $W(\nabla u \cdot Q) = W(\nabla u) \quad Q \in \mathrm{SO}(p)$

$$P_i = \sum_{\alpha=1}^{p} (x^j u_k^\alpha - x^k u_j^\alpha) \frac{\partial W}{\partial u_i^\alpha} + (\delta_j^i x^k - \delta_k^i x^j) W$$

5. Dilation invariance : $W(\lambda \nabla u) = \lambda^n W(\nabla u)$

$$P_i = \frac{n-p}{n} \sum_{\alpha,j=1}^{p} (u^\alpha \delta_j^i - x^j u_j^\alpha) \frac{\partial W}{\partial u_i^\alpha} + x^i W$$

5A. Divergence identity

$$\mathrm{Div}\, \widetilde{P} = p\, W$$

$$\widetilde{P}_i = \sum_{j=1}^{p} (u^\alpha \delta_j^i - x^j u_j^\alpha) \frac{\partial W}{\partial u_i^\alpha} + x^i W$$

$\Longrightarrow$ Knops/Stuart, Pohozaev, Pucci/Serrin

Generalized Vector Fields

Allow the coefficients of the infinitesimal generator to depend on derivatives of u:

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x, u^{(k)}) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \varphi^\alpha(x, u^{(k)}) \frac{\partial}{\partial u^\alpha}$$

Characteristic :

$$Q_\alpha(x, u^{(k)}) = \varphi^\alpha - \sum_{i=1}^{p} \xi^i u_i^\alpha$$

Evolutionary vector field:

$$\mathbf{v}_Q = \sum_{\alpha=1}^{q} Q_\alpha(x, u^{(k)}) \frac{\partial}{\partial u^\alpha}$$

Prolongation formula:

$$\mathrm{pr}\,\mathbf{v} = \mathrm{pr}\,\mathbf{v}_Q + \sum_{i=1}^{p} \xi^i D_i$$

$$\mathrm{pr}\,\mathbf{v}_Q = \sum_{\alpha,J} D^J Q_\alpha \frac{\partial}{\partial u^\alpha_J} \qquad D_i = \sum_{\alpha,J} u^\alpha_{J,i} \frac{\partial}{\partial u^\alpha_J}$$

$\Longrightarrow$ total derivative

Generalized Flows

- The one-parameter group generated by an evolutionary vector field is found by solving the Cauchy problem for an associated system of evolution equations

$$\frac{\partial u^\alpha}{\partial \varepsilon} = Q_\alpha(x, u^{(n)}) \qquad u\big|_{\varepsilon=0} = f(x)$$

Example. $\mathbf{v} = \dfrac{\partial}{\partial x}$ generates the one-parameter group of translations:

$$(x, y, u) \quad \longmapsto \quad (x + \varepsilon, y, u)$$

Evolutionary form:

$$\mathbf{v}_Q = -u_x \frac{\partial}{\partial x}$$

Corresponding group:

$$\frac{\partial u}{\partial \varepsilon} = -u_x$$

Solution

$$u = f(x, y) \quad \longmapsto \quad u = f(x - \varepsilon, y)$$

Generalized Symmetries of Differential Equations

Determining equations :

$$\mathrm{pr}\,\mathbf{v}(\Delta) = 0 \qquad \text{whenever} \qquad \Delta = 0$$

For totally nondegenerate systems, this is equivalent to

$$\mathrm{pr}\,\mathbf{v}(\Delta) = \mathcal{D}\Delta = \sum_{\nu} \mathcal{D}_{\nu}\Delta_{\nu}$$

★ $\mathbf{v}$ is a generalized symmetry if and only if its evolutionary form $\mathbf{v}_Q$ is.

- A generalized symmetry is trivial if its characteristic vanishes on solutions to Δ. Two symmetries are equivalent if their evolutionary forms differ by a trivial symmetry.

General Variational Symmetries

Definition. A generalized vector field is a variational symmetry if it leaves the variational problem invariant up to a divergence:

$$\mathrm{pr}\,\mathbf{v}(L) + L\,\mathrm{Div}\,\xi = \mathrm{Div}\,B$$

⋆ $\mathbf{v}$ is a variational symmetry if and only if its evolutionary form $\mathbf{v}_Q$ is.

$$\mathrm{pr}\,\mathbf{v}_Q(L) = \mathrm{Div}\,\widetilde{B}$$

Theorem. If $\mathbf{v}$ is a variational symmetry, then it is a symmetry of the Euler-Lagrange equations.

Proof:

First, $\mathbf{v}_Q$ is a variational symmetry if

$$\mathrm{pr}\,\mathbf{v}_Q(L) = \mathrm{Div}\,P.$$

Secondly, integration by parts shows

$$\mathrm{pr}\,\mathbf{v}_Q(L) = D_L(Q) = QD_L^*(1) + \mathrm{Div}\,A = QE(L) + \mathrm{Div}\,A$$

for some A depending on Q, L. Therefore

$$0 = E(\mathrm{pr}\,\mathbf{v}_Q(L)) = E(QE(L)) = E(Q\,\Delta) = D_\Delta^* Q + D_Q^* \Delta$$

$$= D_\Delta Q + D_Q^* \Delta = \mathrm{pr}\,\mathbf{v}_Q(\Delta) + D_Q^* \Delta$$

Noether's Theorem. Let $\Delta = 0$ be a normal system of Euler-Lagrange equations. Then there is a one-to-one correspondence between (equivalence classes of) nontrivial conservation laws and (equivalence classes of) nontrivial variational symmetries. The characteristic of the conservation law is the characteristic of the associated symmetry.

Proof: Nother's Identity:

$$QE(L) = \mathrm{pr}\,\mathbf{v}_Q(L) - \mathrm{Div}\,A = \mathrm{Div}(P - A)$$

The Kepler Problem

$$x_{tt} + \frac{\mu\, x}{r^3} = 0 \qquad L = \tfrac{1}{2}\, x_t^2 - \frac{\mu}{r}$$

Generalized symmetries:

$$\mathbf{v} = (x \cdot x_{tt})\, \partial_x + x_t\, (x \cdot \partial_x) - 2\, x\, (x_t \cdot \partial_x)$$

Conservation law

$$\mathrm{pr}\ \mathbf{v}(L) = D_t R$$

where

$$R = x_t \wedge (x \wedge x_t) - \frac{\mu x}{r}$$

$\Longrightarrow$ Runge-Lenz vector

Noether's Second Theorem. A system of Euler-Lagrange equations is under-determined if and only if it admits an infinite dimensional variational symmetry group depending on an arbitrary function. The associated conservation laws are trivial.

Proof: If $f(x)$ is any function,

$$f(x)\mathcal{D}(\Delta) = \Delta\,\mathcal{D}^*(f) + \operatorname{Div} P[f, \Delta].$$

Set

$$Q = D^*(f).$$

Example.

$$\int\!\!\int (u_x + v_y)^2 \, dx \, dy$$

Euler-Lagrange equations:

$$\Delta_1 = E^u(L) = u_{xx} + v_{xy} = 0$$

$$\Delta_2 = E^v(L) = u_{xy} + v_{yy} = 0$$

$$D_x \Delta_2 - D_y \Delta_2 \equiv 0$$

Symmetries

$$(u, v) \longmapsto (u + \varphi_y, v - \varphi_x)$$