Fractalization and Quantization in Dispersive Systems

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Dispersion of surface waves on a pond
Dispersion

**Definition.** A linear partial differential equation is called **dispersive** if the different Fourier modes travel unaltered but at different speeds.

Substituting

\[ u(t, x) = e^{i(kx - \omega t)} \]

produces the dispersion relation

\[ \omega = \omega(k), \quad \omega, k \in \mathbb{R} \]

relating frequency \( \omega \) and wave number \( k \).

---

Phase velocity: \( c_p = \frac{\omega(k)}{k} \)

Group velocity: \( c_g = \frac{d\omega}{dk} \) (stationary phase)
A Simple Linear Dispersive Wave Equation:

\[
\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3}
\]

\[\Rightarrow \text{linearized Korteweg–deVries equation}\]

Dispersion relation: \( \omega = k^3 \)

Phase velocity: \( c_p = \frac{\omega}{k} = k^2 \)

Group velocity: \( c_g = \frac{d\omega}{dk} = 3k^2 \)

Thus, wave packets (and energy) move faster (to the right) than the individual waves.
Linear Dispersion on the Line

\[
\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3} \quad u(0, x) = f(x)
\]

Fourier transform solution:

\[
u(t, x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k) e^{i(kx - k^3 t)} \, dk
\]

Fundamental solution \( u(0, x) = \delta(x) \)

\[
u(t, x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(kx - k^3 t)} \, dk = \frac{1}{\sqrt[3]{3} t} \text{Ai} \left( -\frac{x}{\sqrt[3]{3} t} \right)
\]
Fundamental solution to linearized KdV
Linear Dispersion on the Line

\[
\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3} \quad u(0, x) = f(x)
\]

Superposition solution formula:

\[
u(t, x) = \frac{1}{\sqrt[3]{3} t} \int_{-\infty}^{\infty} f(\xi) \ \text{Ai} \left( \frac{\xi - x}{\sqrt[3]{3} t} \right) d\xi
\]
Linear Dispersion on the Line

\[
\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3} \quad u(0, x) = f(x)
\]

Superposition solution formula:

\[
u(t, x) = \frac{1}{3\sqrt{3} t} \int_{-\infty}^{\infty} f(\xi) \text{Ai} \left( \frac{\xi - x}{\sqrt{3} t} \right) d\xi
\]

Step function initial data: \( u(0, x) = \sigma(x) = \left\{ \begin{array}{ll} 0, & x < 0, \\ 1, & x > 0. \end{array} \right. \)

\[
u(t, x) = \frac{1}{3} - H \left( -\frac{x}{3\sqrt{3} t} \right)
\]

\[
H(z) = \frac{z \Gamma\left(\frac{1}{3}\right)_{1}F_{2}\left(\frac{1}{3}; \frac{2}{3}, \frac{4}{3}; \frac{1}{9} z^3\right)}{3^{5/3} \Gamma\left(\frac{2}{3}\right) \Gamma\left(\frac{4}{3}\right)} - \frac{z^2 \Gamma\left(\frac{2}{3}\right)_{1}F_{2}\left(\frac{2}{3}; \frac{4}{3}, \frac{5}{3}; \frac{1}{9} z^3\right)}{3^{7/3} \Gamma\left(\frac{4}{3}\right) \Gamma\left(\frac{5}{3}\right)}
\]

\[\Rightarrow \text{MATHEMATICA} \quad \text{via Meijer G functions}\]
Step solution to linearized KdV
$t = 0.005$

$\begin{array}{lll}
\text{ } & t = 0.01 & \text{ } \\
\text{ } & t = 0.05 & \text{ } \\
\text{ } & t = 0.1 & \text{ } \\
\text{ } & t = 0.5 & \text{ } \\
\text{ } & t = 1.0 & \text{ }
\end{array}$

$\begin{array}{lll}
\text{ } & t = 0.005 & \text{ } \\
\text{ } & t = 0.01 & \text{ } \\
\text{ } & t = 0.05 & \text{ } \\
\text{ } & t = 0.1 & \text{ } \\
\text{ } & t = 0.5 & \text{ } \\
\text{ } & t = 1.0 & \text{ }
\end{array}$
Periodic Linear Dispersion

\[ \frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3} \]

\[ u(t, -\pi) = u(t, \pi) \quad \frac{\partial u}{\partial x} (t, -\pi) = \frac{\partial u}{\partial x} (t, \pi) \quad \frac{\partial^2 u}{\partial x^2} (t, -\pi) = \frac{\partial^2 u}{\partial x^2} (t, \pi) \]

Step function initial data:

\[ u(0, x) = \sigma(x) = \begin{cases} 
0, & x < 0, \\
1, & x > 0. 
\end{cases} \]
Periodic Linear Dispersion

\[ \frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3} \]

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Step function initial data:

\[ u(0, x) = \sigma(x) = \begin{cases} 0, & x < 0, \\ 1, & x > 0. \end{cases} \]

Fourier series solution formula:

\[ u^*(t, x) \sim \frac{1}{2} + \frac{2}{\pi} \sum_{j=0}^{\infty} \frac{\sin((2j + 1)x - (2j + 1)^3 t)}{2j + 1} \].
Periodic linearized KdV with $\Delta t = .01$
Periodic linearized KdV with $\Delta t = \pi/300$
Periodic linearized KdV — irrational times

$t = 0$

$t = 0.1$

$t = 0.2$

$t = 0.3$

$t = 0.4$

$t = 0.5$
Periodic linearized KdV — rational times

t = \frac{1}{30} \pi

t = \frac{1}{15} \pi

t = \frac{1}{10} \pi

t = \frac{2}{15} \pi

t = \frac{1}{6} \pi

t = \frac{1}{5} \pi
\[
t = \pi
\]
\[
t = \frac{1}{2} \pi
\]
\[
t = \frac{1}{3} \pi
\]
\[
t = \frac{1}{4} \pi
\]
\[
t = \frac{1}{5} \pi
\]
\[
t = \frac{1}{6} \pi
\]
\[
t = \frac{1}{7} \pi
\]
\[
t = \frac{1}{8} \pi
\]
\[
t = \frac{1}{9} \pi
\]
Periodic linearized KdV with $\Delta t = .0001$
Theorem. At rational time $t = 2\pi p/q$, the solution $u^*(t, x)$ is constant on every subinterval $2\pi j/q < x < 2\pi (j + 1)/q$. At irrational time $u^*(t, x)$ is a non-differentiable continuous fractal function.
Lemma.

\[ f(x) \sim \sum_{k=-\infty}^{\infty} c_k e^{i k x} \]

is piecewise constant on intervals \(2 \pi j/q < x < 2 \pi (j + 1)/q\) if and only if

\( \hat{c}_k = \hat{c}_l, \; k \equiv l \not\equiv 0 \mod q, \; \hat{c}_k = 0, \; 0 \not\equiv k \equiv 0 \mod q. \)

where

\( \hat{c}_k = \frac{2 \pi k c_k}{i q (e^{-2i \pi k/q} - 1)} \quad k \not\equiv 0 \mod q. \)

\[ \implies \text{DFT} \]
The Fourier coefficients of the solution $u^*(t, x)$ at rational time $t = 2\pi p/q$ are

$$c_k = b_k e^{-2\pi i k^3 p/q}$$

where, for the step function initial data,

$$b_k = \begin{cases} 
-\frac{i}{\pi k}, & k \text{ odd,} \\
1/2, & k = 0, \\
0, & 0 \neq k \text{ even.}
\end{cases}$$

Crucial observation: if $k \equiv l \mod q$ then $k^3 \equiv l^3 \mod q$ which implies

$$e^{-2\pi i k^3 p/q} = e^{-2\pi i l^3 p/q}$$

and hence the Fourier coefficients (* ) satisfy the condition in the Lemma.

Q.E.D.
Revival

Fundamental Solution: \( F(0, x) = \delta(x) \).

**Theorem.** At rational time \( t = 2\pi p/q \), the fundamental solution \( F(t, x) \) is a linear combination of finitely many periodically extended delta functions, based at \( 2\pi j/q \) for integers \(-\frac{1}{2} q < j \leq \frac{1}{2} q\).
Revival

Fundamental Solution: \( F(0, x) = \delta(x). \)

**Theorem.** At rational time \( t = 2\pi p/q, \) the fundamental solution \( F(t, x) \) is a linear combination of finitely many periodically extended delta functions, based at \( 2\pi j/q \) for integers \(-\frac{1}{2}q < j \leq \frac{1}{2}q. \)

**Corollary.** At rational time, any solution profile \( u(2\pi p/q, x) \) to the periodic initial-boundary value problem is a linear combination of \( \leq q \) translates of the initial data, namely \( f(x + 2\pi j/q) \), and hence its value depends on only finitely many values of the initial data.
The same quantization/fractalization phenomenon appears in any linearly dispersive equation with “integral polynomial” dispersion relation:

\[ \omega(k) = \sum_{m=0}^{n} c_m k^m \]

where

\[ c_m = \alpha n_m \quad n_m \in \mathbb{Z} \]
Linear Free-Space Schrödinger Equation

\[ i \frac{\partial u}{\partial t} = - \frac{\partial^2 u}{\partial x^2} \]

Dispersion relation: \( \omega = k^2 \)

Phase velocity: \( c_p = \frac{\omega}{k} = k \)

Group velocity: \( c_g = \frac{d\omega}{dk} = 2k \)
The Talbot Effect

\[ i \frac{\partial u}{\partial t} = -\frac{\partial^2 u}{\partial x^2} \]

\[ u(t, -\pi) = u(t, \pi) \quad \frac{\partial u}{\partial x}(t, -\pi) = \frac{\partial u}{\partial x}(t, \pi) \]

- Michael Berry et. al.
- Oskolkov
- Kapitanski, Rodnianski
  
  "Does a quantum particle know the time?"

- Michael Taylor
- Bernd Thaller, Visual Quantum Mechanics
William Henry Fox Talbot  (1800–1877)
Talbot’s 1835 image of a latticed window in Lacock Abbey

⇒ oldest photographic negative in existence.
A Talbot Experiment

Fresnel diffraction by periodic gratings (1836):

“It was very curious to observe that though the grating was greatly out of the focus of the lens ... the appearance of the bands was perfectly distinct and well defined ... the experiments are communicated in the hope that they may prove interesting to the cultivators of optical science.”

— Fox Talbot
A Talbot Experiment

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“It was very curious to observe that though the grating was greatly out of the focus of the lens ... the appearance of the bands was perfectly distinct and well defined ... the experiments are communicated in the hope that they may prove interesting to the cultivators of optical science.”

— Fox Talbot

⇒ Lord Rayleigh calculates the Talbot distance (1881)
The Quantized/Fractal Talbot Effect

- Optical experiments — Berry & Klein
- Diffraction of matter waves (helium atoms) — Nowak et. al.
Quantum Revival

- Electrons in potassium ions — Yeazell & Stroud
- Vibrations of bromine molecules — Vrakking, Villeneuve, Stolow
Periodic Linear Schrödinger Equation

\[ i \frac{\partial u}{\partial t} = -\frac{\partial^2 u}{\partial x^2} \]

\[ u(t, -\pi) = u(t, \pi) \quad \frac{\partial u}{\partial x} (t, -\pi) = \frac{\partial u}{\partial x} (t, \pi) \]

Integrated fundamental solution:

\[ u(t, x) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} e^{i(kx-k^2t)} \]

For \( x/t \in \mathbb{Q} \), this is known as a Gauss sum (or, more generally, \( k^n \), a Weyl sum), of great importance in number theory

\[ \Rightarrow \text{Hardy, Littlewood, Weil, I. Vinogradov, etc.} \]
Periodic Linear Schrödinger Equation

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For \( x/t \in \mathbb{Q} \), this is known as a Gauss sum (or, more generally, \( k^n \), a Weyl sum), of great importance in number theory

\[ \Rightarrow \text{Hardy, Littlewood, Weil, I. Vinogradov, etc.} \]

\[ \star \star \text{The Riemann Hypothesis!} \]
Integrated fundamental solution:

\[ u(t, x) = \frac{1}{2\pi} \sum_{0 \neq k = -\infty}^{\infty} e^{i(kx - k^2 t)} \frac{1}{k}. \]

**Theorem.**

- The fundamental solution \( \partial u / \partial x \) is a Jacobi theta function. At rational times \( t = 2\pi p/q \), it linear combination of delta functions concentrated at rational nodes \( x_j = 2\pi j/q \).
- At irrational times \( t \), the integrated fundamental solution is a continuous but nowhere differentiable function.
Dispersive Carpet

Schrödinger Carpet
Periodic Linear Dispersion

\[
\frac{\partial u}{\partial t} = L(D_x) u, \quad u(t, x + 2\pi) = u(t, x)
\]

Dispersion relation:

\[u(t, x) = e^{i(k x - \omega t)} \implies \omega(k) = -i L(-i k) \quad \text{assumed real}\]

Riemann problem: step function initial data

\[u(0, x) = \sigma(x) = \begin{cases} 0, & x < 0, \\ 1, & x > 0. \end{cases}\]

Solution:

\[u(t, x) \sim \frac{1}{2} + \frac{2}{\pi} \sum_{j=0}^{\infty} \frac{\sin[(2j + 1)x - \omega(2j + 1)t]}{2j + 1}.\]

\[\star \star \ \omega(-k) = -\omega(k) \quad \text{odd}\]

Polynomial dispersion, rational \(t\) \implies Weyl exponential sums
Water Waves
2D Water Waves

\[ y = h + \eta(t, x) \quad \text{surface elevation} \]

\[ \phi(t, x, y) \quad \text{velocity potential} \]
2D Water Waves

- Incompressible, irrotational fluid.
- No surface tension

\[
\begin{align*}
\phi_t + \frac{1}{2} \phi_x^2 + \frac{1}{2} \phi_y^2 + g \eta &= 0 \\
\eta_t &= \phi_y - \eta_x \phi_x \\
\phi_{xx} + \phi_{yy} &= 0 \\
\phi_y &= 0
\end{align*}
\]

\[
y = h + \eta(t, x)
\]

\[
0 < y < h + \eta(t, x)
\]

\[
y = 0
\]

- Wave speed (maximum group velocity): \( c = \sqrt{gh} \)
- Dispersion relation: \( \sqrt{gk \tanh(hk)} = ck - \frac{1}{6} ch^2 k^3 + \cdots \)
Small parameters — long waves in shallow water (KdV regime)

\[ \alpha = \frac{a}{h} \quad \beta = \frac{h^2}{\ell^2} = O(\alpha) \]
Rescale:

\[ x \rightarrow \ell x \quad y \rightarrow h y \quad t \rightarrow \frac{\ell t}{c} \]

\[ \eta \rightarrow a \eta \quad \phi \rightarrow \frac{g a \ell \phi}{c} \quad c = \sqrt{gh} \]

Rescaled water wave system:

\[
\begin{align*}
\phi_t + \frac{\alpha}{2} \phi_x^2 + \frac{\alpha}{2 \beta} \phi_y^2 + \eta &= 0 \\
\eta_t &= \frac{1}{\beta} \phi_y - \alpha \eta_x \phi_x \\
\beta \phi_{xx} + \phi_{yy} &= 0 \\
\phi_y &= 0
\end{align*}
\]

\[ 0 < y < 1 + \alpha \eta \]

\[ y = 0 \]
Boussinesq expansion

Set

\[ \psi(t, x) = \phi(t, x, 0) \quad u(t, x) = \phi_x(t, x, \theta) \quad 0 \leq \theta \leq 1 \]

Solve Laplace equation:

\[ \phi(t, x, y) = \psi(t, x) - \frac{1}{2} \beta^2 y^2 \psi_{xx} + \frac{1}{4!} \beta^4 y^4 \psi_{xxxx} + \cdots \]

Plug expansion into free surface conditions: To first order

\[ \psi_t + \eta + \frac{1}{2} \alpha \psi^2_x - \frac{1}{2} \beta \psi_{xxt} = 0 \]
\[ \eta_t + \psi_x + \alpha (\eta \psi_x)_x - \frac{1}{6} \beta \psi_{xxxx} = 0 \]
Bidirectional Boussinesq systems:

\[
\begin{align*}
    u_t + \eta_x + \alpha u u_x - \frac{1}{2} \beta (\theta^2 - 1) u_{xxt} &= 0 \\
    \eta_t + u_x + \alpha (\eta u)_x - \frac{1}{6} \beta (3 \theta^2 - 1) u_{xxx} &= 0
\end{align*}
\]

\[\star \star\] at \( \theta = 1 \) this system is \textit{integrable} (tri-Hamiltonian)
but \textit{ill-posed} (!?!)

\[
\begin{align*}
    u_{tt} &= u_{xx} + \frac{1}{2} \alpha (u^2)_{xx} - \frac{1}{6} \beta u_{xxxx} \\
    \eta_{tt} + u_{xt} + \alpha (\eta u)_x - \frac{1}{6} \beta (3 \theta^2 - 1) u_{xxtt} &= 0
\end{align*}
\]

\[
\text{DNA dynamics (Scott)}
\]
Bidirectional Boussinesq systems:

\[
\begin{align*}
    u_t + \eta_x + \alpha uu_x - \frac{1}{2} \beta (\theta^2 - 1) u_{xxx} &= 0 \\
    \eta_t + u_x + \alpha (\eta u)_x - \frac{1}{6} \beta (3 \theta^2 - 1) u_{xxx} &= 0
\end{align*}
\]

★★ at θ = 1 this system is integrable (tri-Hamiltonian) but ill-posed (!?!)

Boussinesq equation

\[
u_{tt} = u_{xx} + \frac{1}{2} \alpha (u^2)_{xx} - \frac{1}{6} \beta u_{xxxx}
\]

Regularized Boussinesq equation

\[
u_{tt} = u_{xx} + \frac{1}{2} \alpha (u^2)_{xx} + \frac{1}{6} \beta u_{xxtt}
\]

⇒ DNA dynamics (Scott)
Unidirectional waves:

\[ u = \eta - \frac{1}{4} \alpha \eta^2 + \left( \frac{1}{3} - \frac{1}{2} \theta^2 \right) \beta \eta_{xx} + \cdots \]

Korteweg-deVries (1895) equation:

\[ \eta_t + \eta_x + \frac{3}{2} \alpha \eta \eta_x + \frac{1}{6} \beta \eta_{xxx} = 0 \]

\[ \Rightarrow \text{ Due to Boussinesq in 1877!} \]

Benjamin–Bona–Mahony (BBM) equation:

\[ \eta_t + \eta_x + \frac{3}{2} \alpha \eta \eta_x - \frac{1}{6} \beta \eta_{xxt} = 0 \]
# Shallow Water Dispersion Relations

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<td>$\frac{k}{1 + \frac{1}{6} k^2}$</td>
</tr>
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</table>
Water waves

\[ \omega = \sqrt{k \tanh k \text{ sign } k} \]
Water waves: \( t > 1000 \)

\[ \omega = \sqrt{k \tanh k \operatorname{sign} k} \]
Water waves

\[ \omega = \sqrt{k \tanh k \text{ sign } k} \]
Square root dispersion

\[ \omega = \sqrt{|k|} \ \text{sign} \ k \]
BBM equation

\[ \omega = \frac{k}{\sqrt{1 + \frac{1}{3}k^2}} \]
BBM equation:  $t > 1000$

\[ \omega = \frac{k}{\sqrt{1 + \frac{1}{3} k^2}} \]
BBM equation: \( t > 10,000 \)

\[
\omega = \frac{k}{\sqrt{1 + \frac{1}{3}k^2}}
\]
Figure 2. RLW/BBM Dispersion:
\[ \omega = k \sqrt{1 + \frac{1}{3} k^2} \]

Figure 3. Regularized Boussinesq Dispersion:
\[ \omega = k \sqrt{\tanh(k \omega)} \]
\[ \omega = k \sqrt{1 + \frac{1}{3} k^2} \]
\[ \omega = k \sqrt{\frac{1}{1 + \frac{1}{6} k^2}} \]
\[ \omega = k \sqrt{\frac{1}{1 + \frac{1}{6} k^2}} \]
Boussinesq equation

\[ \omega = k \sqrt{1 + \frac{1}{3} k^2} \]
Boussinesq equation

\[ \omega = k \sqrt{1 + \frac{1}{3} k^2} \]
Regularized Boussinesq equation

\[ \omega = \frac{k}{1 + \frac{1}{6}k^2} \]
Regularized Boussinesq equation $t > 1000$

$$\omega = \frac{k}{1 + \frac{1}{6} k^2}$$
Regularized Boussinesq equation $t > 10,000$

$$
\omega = \frac{k}{1 + \frac{1}{6} k^2}
$$
Regularized Boussinesq equation

\[ \omega = \frac{k}{1 + \frac{1}{6}k^2} \]

Figure 2. RLW/BBM Dispersion:
\[ \omega = k_1 + \frac{1}{6}k_2 \]

Figure 3. Regularized Boussinesq Dispersion:
\[ \omega = \sqrt{k \tanh k} \]
\[ \omega = \sqrt{k} \]
\[ \omega = k \sqrt{1 + \frac{1}{3}k^2} \]
\[ \omega = k \left| 1 + \frac{1}{6}k^2 \right| \]
Dispersion Asymptotics

★ The qualitative behavior of the solution to the periodic problem depends crucially on the asymptotic behavior of the dispersion relation $\omega(k)$ for large wave number $k \rightarrow \pm \infty$.

$$\omega(k) \sim k^{\alpha}$$

- $\alpha = 0$ — large scale oscillations
- $0 < \alpha < 1$ — dispersive oscillations
- $\alpha = 1$ — traveling waves
- $1 < \alpha < 2$ — oscillatory becoming fractal
- $\alpha \geq 2$ — fractal/quantized
Linearized Benjamin Ono equation

\[ u_t = \mathcal{H}[u_{xx}], \quad \omega_{BO}(k) = k^2 \text{ sign } k. \]

Hilbert transform

\[ \mathcal{H}[f](x) = H * f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(y)}{x-y} \, dy \]

periodic Hilbert transform

\[ \mathcal{H}[f](x) = \frac{1}{\pi} \sum_{k=-\infty}^{\infty} \int_{-\pi}^{\pi} \frac{f(y)}{x-y+2\pi k} \, dy = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cot \left[ \frac{1}{2}(x-y) \right] f(y) \, dy \]
Benjamin–Ono equation: irrational times 

\[ \omega = k^2 \ \text{sign} \ k \]
Benjamin–Ono equation: rational times

\[ \omega = k^2 \text{ sign } k \]
Benjamin–Ono equation

\[ \omega = |k|^2 \text{ sign } k \]
Generalized Revival

**Theorem** At a rational time \( t = \pi p/q \), the solution to the periodic initial-boundary value problem for the linearized Benjamin–Ono equation on the interval \(-\pi < x < \pi\) is a linear combination of

- translates \( f(x + \pi j/q) \) of the initial condition \( u(0, x) = f(x) \), and
- translates \( g(x + \pi j/q) \) of its periodic Hilbert transform: \( g(x) = \mathcal{H}[f](x) \),

for \( j = 0, \ldots, 2q - 1 \).
Trigonometric hypergeometric functions

\[ S_j^k(x) = S_{j,1}^k(x) = \sum_{n=0}^{\infty} \frac{\sin(nk+j)x}{nk+j}. \]

\[ S_j^k(x) = \frac{1}{k} \sum_{l=1}^{k} \left[ \sin \left( \frac{2\pi j l}{k} \right) \log \left| 2 \sin \left( \frac{x}{2} + \frac{\pi l}{k} \right) \right| \right. \\
+ \cos \left( \frac{2\pi j l}{k} \right) \frac{\text{sign} (x + 2\pi l/k) \pi - (x + 2\pi l/k)}{2} \left. \right]. \]
For $r = 2$, at the nodes $x = 2 \pi l/k$, for $l \in \mathbb{Z}$, $S_k^j(x)$ has a point of infinite gradient, unless

$$\cos \frac{\pi 2 j l}{k} = 0 \quad \text{or} \quad \sin \frac{\pi 2 j l}{k} = 0.$$ 

in which case there is a corner. The most important case of trigonometric polylogarithmic functions for the present work is the case $r = 1$, and we will at times refer to these as trigonometric hypergeometric functions; see the Appendix. In this case we use the notation

$$S_k^j(x) = \sum_{n=0}^{\infty} \sin(nkx + j)nkx + j.$$ 

(3.5)

In the Appendix, we provide the alternative formula

$$S_k^j(x) = \sum_{l=1}^{k} \sin \left( \frac{\pi 2 j l}{k} \right) \log \left( \frac{2 \sin \left( \frac{\pi x}{l} \right) \pm \cos \left( \frac{\pi 2 j l}{k} \right) \text{sign} \left( x + 2 \frac{\pi l}{k} \right) \right).$$ 

(3.6)

The discontinuities of these functions are described in Proposition 1 and the low order ones are displayed in Figure 1 and 2. In the graphs, which were plotted in Mathematica, the infinite logarithmic cusps have been drawn in by hand, as the plotting routines made them misleadingly appear to be finite in height.

Figure 1: Graphs of trigonometric hypergeometric functions $S_k^j(x)$, $k = 1, 2, 3$. Both the horizontal and vertical axes are from $-\pi$ to $\pi$. 
Figure 2: Graphs of trigonometric hypergeometric functions $S_k^j(x)$, $k = 4, 5$.

Both the horizontal and vertical axes are from $\pi$ to $\pi$.

These functions have distributional derivatives

$$
\frac{dS_k^j(x)}{dx} = \pi^k 1 \sum_{l=0}^{k} \cos \left( \frac{2 \pi l}{k} \right) \left[ \pi, \pi \right] x + \frac{2 \pi l}{k} \sum_{l=1}^{k} \sin \left( \frac{2 \pi l}{k} \right) \cot \left( \pi \right) x.
$$

(3.7)

The first summation in expression (3.7) is a linear combination of Dirac deltas at the nodes and the second is a linear combination of pole singularities at the nodes that are not integer multiples of $\pi$.

Furthermore, for $j = 1, \ldots, k - 1$, the sum

$$
S_k^j(x) + S_k^{k-j}(x) = \frac{1}{2i} \sum_{n=1}^{k-j} \left( e^{i(nk+j)x} + e^{i(nk-j)x} \right) nk + j + \frac{1}{2k} \sum_{l=0}^{k-1} \left( e^{i(2\pi l/k)x} \right).
$$

(3.8)

is a piecewise constant function on the intervals $m\pi/k < x < (m+1)\pi/k$; the proof relies on the method used in [25] to characterize Fourier series representing piecewise constant functions.

Note that the pole terms cancel out in the sum

$$
\frac{dS_k^j(x)}{dx} + \frac{dS_k^{k-j}(x)}{dx} = \pi^k 1 \sum_{l=0}^{k} \cos \left( \frac{2 \pi l}{k} \right) \left[ \pi, \pi \right] x + \frac{2 \pi l}{k} \sum_{l=1}^{k} \sin \left( \frac{2 \pi l}{k} \right) \cot \left( \pi \right) x.
$$

(3.9)
Benjamin–Ono equation

$$\omega = |k|^2 \text{ sign } k$$
Trigonometric hypergeometric functions

\[ S_j^k(x) = S_{j,1}^k(x) = \sum_{n=0}^{\infty} \frac{\sin(nk+j)x}{nk+j}. \]

\[ S_j^k(x) = \frac{1}{k} \sum_{l=1}^{k} \left[ \sin \left( \frac{2\pi jl}{k} \right) \log \left| 2 \sin \left( \frac{x}{2} + \frac{\pi l}{k} \right) \right| \\
+ \cos \left( \frac{2\pi jl}{k} \right) \text{sign} \left( x + 2\pi l/k \right) \pi - \left( x + 2\pi l/k \right) \right]. \]

\[ \frac{dS_j^k}{dx} = \frac{\pi}{k} \sum_{l=0}^{k-1} \cos \left( \frac{2\pi jl}{k} \right) \delta[-\pi,\pi] \left( x + \frac{2\pi l}{k} \right) + \frac{1}{2k} \sum_{l=1}^{k-1} \sin \left( \frac{2\pi jl}{k} \right) \cot \left( \frac{1}{2} x + \frac{\pi l}{k} \right) \]

- Produces the periodic fundamental solution
- The cotangent is the Hilbert transform of the delta function
Linearized Intermediate Long Wave Equation

\[ \mathcal{L}[u] = \mathcal{I}[u_{xx}] - \frac{1}{\delta} u_x \]
\[ \omega_\delta(k) = k^2 \coth(\delta k) - \frac{k}{\delta}. \]
\[ \mathcal{I}[f](x) = -\frac{1}{2 \delta} \int_{-\infty}^{\infty} \coth \left( \frac{\pi}{2 \delta} (x - y) \right) f(y) \, dy \]

Periodic kernel:
\[ \mathcal{I}[f](x) = -\frac{1}{2 \delta} \int_{-\pi}^{\pi} \left[ \sum_{n = -\infty}^{\infty} \coth \left( \frac{\pi}{2 \delta} (x - y) + \frac{\pi^2 n}{\delta} \right) \right] f(y) \, dy \]
\[ = \frac{1}{\pi} \int_{-\pi}^{\pi} \left[ i \frac{\zeta(-i \delta)}{\delta} (x - y) - \zeta(x - y) + \frac{\pi^2 n}{\delta} \right] f(y) \, dy \]

Weierstrass zeta function
\[ \zeta(z) = \frac{\eta_1}{\omega_1} z + \frac{\pi}{2 \omega_1} \sum_{n = -\infty}^{\infty} \cot \left( \frac{\pi}{2 \omega_1} z + \frac{\pi \omega_3 n}{\omega_1} \right), \quad \eta_1 = \zeta(\omega_1), \quad \omega_1 = -i \delta, \quad \omega_3 = \pi \]
Linearized Smith Equation

\[ u_t = S_\delta[u_x] \]

\[ \omega_S(k) = k \sqrt{\frac{1}{\delta} + k^2}. \]

\[ S_\delta[f] = -\frac{i}{\pi \sqrt{\delta}} \int_{-\infty}^{\infty} \frac{K_1(|x-y|/\sqrt{\delta})}{|x-y|} f(y) \, dy. \]

\( K_1(x) \) denotes the modified Bessel function of the second kind

Periodic kernel:

\[ S_\delta[f] = -\frac{i}{\pi \sqrt{\delta}} \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{K_1(|x-y+2n\pi|/\sqrt{\delta})}{|x-y+2n\pi|} f(y) \, dy. \]
What about nonlinear equations?
Periodic Korteweg–deVries equation

\[
\frac{\partial u}{\partial t} = \alpha \frac{\partial^3 u}{\partial x^3} + \beta u \frac{\partial u}{\partial x} \quad u(t, x + 2\ell) = u(t, x)
\]

Zabusky–Kruskal (1965)

\[\alpha = 1, \quad \beta = .000484, \quad \ell = 1, \quad u(0, x) = \cos \pi x.\]

Lax–Levermore (1983) — small dispersion

\[\alpha \to 0, \quad \beta = 1.\]

Gong Chen (2011)

\[\alpha = 1, \quad \beta = .000484, \quad \ell = 1, \quad u(0, x) = \sigma(x).\]
Zabusky & Kruskal — birth of the soliton
Periodic KdV — dispersive quantization
Figure 13. Korteweg–deVries Equation: Irrational Times.

Figure 14. Korteweg–deVries Equation: Rational Times.
Figure 13. Korteweg–deVries Equation: Irrational Times.

Figure 14. Korteweg–deVries Equation: Rational Times.
Figure 15. Quartic Korteweg–deVries Equation: Irrational Times.
Figure 16. Quartic Korteweg–deVries Equation: Rational Times.
Periodic Korteweg–deVries Equation

Analysis of non-smooth initial data:

Estimates, existence, well-posedness, stability, ... 

• Kato
• Bourgain
• Kenig, Ponce, Vega
• Colliander, Keel, Staffilani, Takaoka, Tao
• Oskolkov
• D. Russell, B–Y Zhang
• Erdoğan, Tzirakis
Operator Splitting

\[ u_t = \alpha u_{xxx} + \beta uu_x = L[u] + N[u] \]

Flow operators: \( \Phi_L(t), \Phi_N(t) \)

Godunov scheme:
\[ u_G^\Delta(t_n) \simeq (\Phi_L(\Delta t) \Phi_N(\Delta t))^n u_0 \]

Strang scheme:
\[ u_S^\Delta(t_n) \simeq (\Phi_N(\frac{1}{2} \Delta t) \Phi_L(\Delta t) \Phi_N(\frac{1}{2} \Delta t))^n u_0 \]

Numerical implementation:
- FFT for \( \Phi_L \) — linearized KdV
- FFT + convolution for \( \Phi_N \) — conservative version of inviscid Burgers’, using Backward Euler + fixed point iteration to overcome mild stiffness. Shock dynamics doesn’t complicate due to small time stepping.
Periodic Linear Dispersive Equations

⇒ Chousionis, Erdoğan, Tzirakis

**Theorem.** Suppose $3 \leq k \in \mathbb{Z}$ and

$$iu_t + (-i \partial_x)^k u = 0, \quad x \in \mathbb{R}/\mathbb{Z}, \quad u(0, x) = g(x) \in \text{BV}$$

(i) $u(t, \cdot)$ is continuous for almost all $t$

(ii) When $g \notin \bigcup_{\epsilon > 0} H^{1/2+\epsilon}$, then, at almost all $t$, the real and imaginary parts of the graph of $u(t, \cdot)$ has fractal dimension $1 + 2^{1-k} \leq D \leq 2 - 2^{1-k}$.

---

**Theorem.** For the periodic Korteweg–deVries equation

$$u_t + u_{xxx} + uu_x = 0, \quad x \in \mathbb{R}/\mathbb{Z}, \quad u(0, x) = g(x) \in \text{BV}$$

(i) $u(t, \cdot)$ is continuous for almost all $t$

(ii) When $g \notin \bigcup_{\epsilon > 0} H^{1/2+\epsilon}$, then, at almost all $t$, the real and imaginary parts of the graph of $u(t, \cdot)$ has fractal dimension $\frac{5}{4} \leq D \leq \frac{7}{4}$. 
Theorem. (Erdoğan, Shakan)

Suppose $c_k$ are the complex Fourier coefficients of a function of bounded variation. Let $\omega(k) \sim |k|^{1/2}$ as $k \to \infty$, then, for any $t \neq 0$, the “dispersive” Fourier series

$$v(t, x) \sim \sum_{k = -\infty}^{\infty} c_k e^{i(kx - \omega(k)t)}$$

converges to a function whose real and imaginary parts have graphs whose maximal fractal dimension $D_t$ satisfies the following estimate:

$$\frac{5}{4} \leq D_t \leq \frac{7}{4}.$$
The Fermi–Pasta–Ulam–Tsingou Problem

⇒ Los Alamos Report, 1955

—Stanislaw Ulam, Adventures of a Mathematician, pp. 226–7

⇒ PJO + Ari Stern
The Fermi–Pasta–Ulam–Tsingou Problem

⇒ Los Alamos Report, 1955

Our problem turned out to have been felicitously chosen. The results were entirely different qualitatively from what even Fermi, with his great knowledge of wave motions, had expected. ... To our surprise, the string started playing a game of musical chairs, only between several low notes, and perhaps even more amazingly, after what would have been several hundred ordinary up and down vibrations, it came back almost exactly to its original sinusoidal shape.

— Stanislaw Ulam, Adventures of a Mathematician, pp. 226–7
The Fermi–Pasta–Ulam–Tsingou System

\[ \mu^{-2} \frac{d^2 u_n}{dt^2} = F(u_{n+1} - u_n) - F(u_n - u_{n-1}) \]

\[ = u_{n+1} - 2u_n + u_{n-1} + N(u_{n+1} - u_n) - N(u_n - u_{n-1}). \]

Forcing function and potential

\[ F(y) = y + N(y) = V'(y), \quad \text{where} \quad V(y) = \frac{1}{2} y^2 + W(y) \]

Classical potentials: \( N(y) = \alpha y^\beta, \quad \beta = 2, 3 \)

Toda lattice: \( N(y) = \alpha e^{\beta y} \)
Continuum Limit

Periodic problem: \( m \) masses on a circle of unit radius with intermass spacing \( h = \frac{2\pi}{m} \). We suppose \( m \to \infty \).

Rescale time: \( t \mapsto ht \)

\[
\frac{d^2 u_n}{dt^2} = \frac{c^2}{h^2} \left[ F(u_{n+1} - u_n) - F(u_n - u_{n-1}) \right],
\]

\( c = \mu h \) — wave speed

Assume the displacements are obtained by sampling a function \( u(t, x) \) at the nodes:

\( u_n(t) = u(t, x_n), \quad \text{where} \quad x_n = nh = \frac{2\pi n}{m}. \)

Taylor expansion:

\[
u_{n\pm 1}(t) = u(t, x_n \pm h) = u \pm h u_x + \frac{1}{2} h^2 u_{xx} \pm \frac{1}{6} h^3 u_{xxx} + \cdots,
\]
Continuum Models

\[ u_{tt} = c^2 (K[u] + M[u]) \]

Linear component

\[ K[u] = u_{xx} + \frac{1}{12} h^2 u_{xxxx} + O(h^4) \]

Quadratic nonlinear component:

\[ M[u] = 2 \alpha h u_x u_{xx} + \frac{1}{6} \alpha h^3 u_x u_{xxx} + \frac{1}{3} \alpha h^3 u_{xx} u_{xx} + O(h^5) \]

Bidirectional continuum model = potential Boussinesq equation

\[ u_{tt} = c^2 (u_{xx} + 2 \alpha h u_x u_{xx} + \frac{1}{12} h^2 u_{xxxx} ) \]

Unidirectional model = Korteweg–deVries equation:

\[ u_t = c (u_x + \alpha h u u_x + \frac{1}{24} h^2 u_{xxx} ) \]
Linear FPU

Discrete wave equation:

\[
\frac{d^2 u_n}{dt^2} = \frac{c^2}{h^2}(u_{n+1} - 2u_n + u_{n-1}),
\]

Bidirectional continuum model

\[
u_{tt} = c^2 u_{xx} + \frac{1}{12} c^2 h^2 u_{xxxx},
\]

★ linearized “bad Boussinesq equation” — ill-posed.

Dispersion relation:

\[
\omega^2 = p_4(k) = c^2 k^2 (1 - \frac{1}{12} h^2 k^2) < 0 \quad \text{for} \quad k \gg 0
\]
Regularized Bidirectional Models

Sixth order linearized model:

\[ u_{tt} = c^2(u_{xx} + \frac{1}{12} h^2 u_{xxxx} + \frac{1}{360} h^4 u_{xxxxxx}), \]

Dispersion relation:

\[ \omega^2 = p_6(k) = c^2 k^2 \left( 1 - \frac{1}{12} h^2 k^2 + \frac{1}{360} h^4 k^4 \right) > 0 \quad \text{for all} \quad k \neq 0 \]

Alternatively, replacing

\[ u_{xx} = c^{-2} u_{tt} + O(h^2) \]

leads to the linear Boussinesq equation

\[ u_{tt} = c^2 u_{xx} + \frac{1}{12} h^2 u_{xxtt} \]

Dispersion relation:

\[ \omega^2 = q(k) = \frac{c^2 k^2}{1 + \frac{1}{12} h^2 k^2} > 0 \quad \text{for all} \quad k \neq 0 \]
FPU Lattice Dispersion Relation

Substituting $u(t, x) = e^{i(kx - \omega t)}$ evaluated at $x = x_n = nh$ into the linearized FPU system

$$\frac{d^2 u_n}{dt^2} = \frac{c^2}{h^2}(u_{n+1} - 2u_n + u_{n-1}),$$

produces

$$-\omega^2 e^{i(kx_n - \omega t)} = \frac{c^2}{h^2} \left( e^{i(kx_n + kh - \omega t)} - 2e^{i(kx_n - \omega t)} + e^{i(kx_n - kh - \omega t)} \right)$$

$$= -\frac{2c^2}{h^2}(1 - \cos kh) e^{i(kx_n - \omega t)}.$$

Discrete FPU dispersion relation:

$$\omega^2 = \frac{2c^2}{h^2}(1 - \cos kh) = \frac{4c^2}{h^2} \sin^2 \frac{1}{2} kh = \frac{c^2m^2}{\pi^2} \sin^2 \frac{k\pi}{m}$$
The Continuum Riemann Problem

Step function initial data:

\[ u(0, x) = \sigma(x) = \frac{1}{2} + \frac{2}{\pi} \sum_{j=0}^{\infty} \frac{\sin (2j + 1)x}{2j + 1} \]

\[ u_t(0, x) = 0 \]

Bidirectional solution

\[ u(t, x) = \frac{1}{2} + \frac{2}{\pi} \sum_{j=0}^{\infty} \frac{\cos \omega(2j + 1)t \sin (2j + 1)x}{2j + 1} \]

Unidirectional right-moving constituent:

\[ u_R(t, x) = \frac{1}{2} + \frac{2}{\pi} \sum_{j=0}^{\infty} \frac{\sin[(2j + 1)x - \omega(2j + 1)t]}{2j + 1} \]
The Discrete Riemann Problem

\[ u_n(0) = \begin{cases} 
1, & 0 < n < m, \\
0, & -m < n < 0, \\
\frac{1}{2}, & n = -m, 0, m.
\end{cases} \]

Discrete Fourier Transform:

\[ u(0, x) \sim \frac{1}{2} + \frac{1}{m} \left[ \frac{m}{2} \right] \sum_{j=0}^{[m/2]} \cot \left( \frac{2j+1}{2m} \right) \sin(2j+1)x. \]

Linear FPU solution:

\[ u(t, x) \sim \frac{1}{2} + \frac{1}{m} \sum_{j=0}^{[m/2]} \cot \left( \frac{2j+1}{2m} \right) \cos \left( \frac{cm t}{\pi} \sin \left( \frac{2j+1}{m} \right) \right) \sin(2j+1)x, \]

Right-moving constituent:

\[ u_R(t, x) \sim \frac{1}{2} + \frac{1}{2m} \sum_{j=0}^{[m/2]} \cot \left( \frac{2j+1}{2m} \right) \sin \left( (2j+1)x - \frac{cm t}{\pi} \sin \left( \frac{2j+1}{m} \right) \right). \]
scales: what we will call short times, where $t = O(1)$, medium times, where $t = O(h^{-1})$, and long times, where $t = O(h^{-2})$.

First, on short time scales, the solutions to all four models exhibit little appreciable difference. For example, consider the profiles at $t = \frac{15}{5} \pi$ graphed below—the top row being the full bidirectional solution and the bottom row its right-moving unidirectional constituent. The only noticeable difference is that, on closer inspection, the oscillatory (or perhaps fractal) perturbation that is superimposed upon the interval so consistently is more concentrated near the discontinuities in the regularized Boussinesq (and sixth order) model, while in the FPU and KdV cases, the oscillations are more spread out, particularly in the unidirectional profiles.

![Figure 1. Bi- and uni-directional solution profiles at $t = \frac{1}{5} \pi$.](image)
scales: what we will call short times, where $t = O(1)$, medium times, where $t = O(h^{-1})$, and long times, where $t = O(h^{-2})$.

First, on short time scales, the solutions to all four models exhibit little appreciable difference. For example, consider the profiles at $t = \frac{15}{2}\pi$ graphed below— the top row being the full bidirectional solution and the bottom row its right-moving unidirectional constituent. The only noticeable difference is that, on closer inspection, the oscillatory (or perhaps fractal) perturbation that is superimposed upon interval so constant is more concentrated near the discontinuities in the regularized Boussinesq (and sixth order) model, while in the FPU and KdV cases, the oscillations are more spread out, particularly in the unidirectional profiles.

Figure 1. Bi- and uni-directional solution profiles at $t = 1$.

Figure 2. Bi- and unidirectional FPU solution profiles.
Since all profiles remain rather similar at short times, in Figure 2, we just graph the FPU solution profiles. What we observe is that, on these short timescale, the solution is an oscillatory perturbation of the solution to the bi- and unidirectional traveling wave solution to the limiting wave equations $u_{tt} = u_{xx}$ and $u_t + u_x = 0$, respectively. In particular, at $t = 1/2\pi$ the right- and left-moving frames have cancelled each other out, leaving only a constant solution profile for the traveling wave solution, with a superimposed fractal residue in the FPU system, as well as its continuum models, all three of which are very similar.

Figure 3. Bi- and unidirectional FPU solution profiles.
Since all profiles remain rather similar at short times, in Figure 2, we just graph the FPU solution profiles. What we observe is that, on these short timescale, the solution is an oscillatory perturbation of the solution to the - and unidirectional traveling wave solution to the limiting wave equations $u_{tt} = u_{xx}$ and $u_t + u_x = 0$, respectively.

In particular, at $t = 1/2$, the right- and left-moving frames have cancelled each other out, leaving only a constant solution profile for the traveling wave solution, with a superimposed fractal residue in the FPU system, as well as its continuum models, all three of which are very similar.

Figure 3. Bi- and unidirectional FPU solution profiles.

At medium times, of order $O(h^{-1})$, the fractal nature of the oscillations superimposed upon the traveling wave solution profile has become more pronounced. Again, both the KdV Boussinesq FPU

Figure 4. Bi- and unidirectional solution profiles at $t = 1/h^2$. 
FPU system and its continuum models exhibit very similar behavior; Figure 3 shows graphs of the former at some representative medium times.

Once we transition to the long time scale, of order $O(h^{-2})$, there is a dramatic difference in the observed behaviors. First consider the solution profiles at the irrational times $t = \frac{1}{h^2} \approx 26560$ and $t = 400000$.

All three solution profiles have a similar fractal nature. The unidirectional constituents are more "pure" fractals, while the bidirectional solution exhibits some semi-coherent regions, perhaps indicating a remnant of regions of constancy of a nearby rational profile.

Figure 5. Bi- and unidirectional solution profiles at $t = 400,000$. 
FPU system and its continuum models exhibit very similar behavior; Figure 3 shows graphs of the former at some representative medium times. Once we transition to the long time scale, of order $O(h^{-2})$, there is a dramatic difference in the observed behaviors. First consider the solution profiles at the irrational times $t = 24\pi/(5 \times h^2) \approx 400,527$.
The KdV solution has quantized to an essentially piecewise constant profile, while the FPU system and the Boussinesq models retain a similar fractal form. On the other hand, the latter profiles retain a noticeable adherence to the underlying piecewise constant KdV solution albeit with a superimposed fractal perturbation.

Figure 7. Bi- and unidirectional solution profiles at $t = 24\pi/h^2$. 
The KdV solution has quantized to an essentially piecewise constant profile, while the FPU system and the Boussinesq models retain a similar fractal form. On the other hand, the latter profiles retain a noticeable adherence to the underlying piecewise constant KdV solution albeit with a superimposed fractal perturbation.

Figure 7. Bi- and unidirectional solution profiles at $t = 24 \pi / (5h^2) \approx 400,527$.

Figure 8. Truncated unidirectional solution profiles at $t = 24 \pi / (5h^2) \approx 400,527$. 
as always, are obtained by explicitly summing over the first \( m \) modes. The first column plots the solutions to the bidirectional KdV model; the disc rete oscillatory peaks indicate the appearance of a revival. The second column plots the corresponding FPU solution; here, there is no appreciable signs of a concentration of the solution and hence no apparent revival. Similar behavior is observed at other (long) times, with varying number of masses.

The KdV profiles are fractal at rational times and concentrated in accordance with a revival at rational times, whereas the FPU profiles are more or less uniformly oscillatory at all times.

\[
t = 24\pi/(5h^2)
\]

\[
t = 24\pi/h^2
\]

**Figure 11.** Revival and lack thereof.
Future Directions

- General dispersion behavior explanation/justification
- Stability analysis
- Improved numerical solution techniques
- Other boundary conditions
- Nonlinearly dispersive models: Camassa–Holm, …
- Discrete systems: Fermi–Pasta–Ulam, spin chains, …
- Higher space dimensions and other domains: tori, spheres, …
- Experimental verification in dispersive media?