Ghost Symmetries

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Ghost symmetries,
Jet Notation

\((x^1, \ldots, x^p)\) — independent variables
\((u^1, \ldots, u^q)\) — dependent variables

Total derivative

\[
D_i(x^j) = \delta_i^j, \quad D_i(u^\alpha) = u^\alpha_{J+e_i},
\]

\[
D^J = (D_1)^{j_1} \cdots (D_p)^{j_p}
\]

\(u^\alpha_{J} = D^J(u^\alpha)\) — derivative coordinates

Differential polynomial:

\[
Q[u] = \sum_{K,J} c_K(x) u^{\alpha_1}_{J_1} \cdots u^{\alpha_k}_{J_k}
\]

Local algebra

\(J = (j_1, \ldots, j_p) \geq 0\)

Nonlocal algebra

\(J \in \mathbb{Z}^p\) i.e. \(j_\nu < 0\) allowed

or, more generally

\[
D^J Q \quad \text{is defined for all} \quad J \in \mathbb{Z}^p
\]
The Kadomtsev–Petviashvili equation

\[ u_{xt} = u_{yy} - 6u u_{xx} - 6u_x^2 - u_{xxxx} \]

Evolutionary form

\[ u_t = D_x^{-1}u_{yy} - 6u u_x - u_{xx} \]

\[ = u_{-1,2} - 6u_{0,0} u_{1,0} - u_{3,0} \]

Symmetries and loop algebras

- Chen, Lee, Lin: 1982
Nonlocal Symmetries

- Vinogradov, Krasil’shchik: 1980, 1984
  \[\Rightarrow\] “coverings”
- Kapcov: 1982
- Vladimirov, Volovich: 1985
- Fushchych, Nikitin: 1987
- Bluman, Kumei, Reid: 1988
- Akhatov, Gazizov, Ibragimov: 1991
- Galas: 1992
- Guthrie, Hickman: 1993
- Dodd: 1994
- Anco, Bluman: 1995
- Sanders, Wang: 2000
- Leo, Leo, Soliani, Tempesta: 2002
- Mikhailov, V. Novikov: 2002
Generalized vector field

\[ \mathbf{v} = \mathbf{v}_Q = \sum_{\alpha, J} D^J Q^\alpha \frac{\partial}{\partial u_j^\alpha} \]

Characteristic

\[ Q^\alpha = \mathbf{v}_Q(u^\alpha) \quad \alpha = 1, \ldots, q \]

\[ \mathbf{v}_Q(u_j^\alpha) = D_J(Q^\alpha) \]

\[ [ \mathbf{v}_Q, D_i ] = 0 \]

\[ \mathbf{v}_Q(P) = D_P(Q) = \sum_{\alpha=1}^{q} D_P^\alpha(Q^\alpha) \]

Fréchet derivative

\[ D_P^\alpha = \sum_J \frac{\partial P}{\partial u_j^\alpha} D^J \]
Commutator

*Lie bracket or commutator*

\[
[ \mathbf{v}_P, \mathbf{v}_Q ] = \mathbf{v}_{[P,Q]}
\]

where

\[
[ P, Q ] = \mathbf{v}_P(Q) - \mathbf{v}_Q(P) = \mathbf{D}_Q(P) - \mathbf{D}_P(Q)
\]

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**Theorem.** The generalized vector field \( \mathbf{v}_Q \) is a symmetry of the evolution equation

\[
\frac{\partial Q}{\partial t} + [ P, Q ] = 0.
\]

if and only if
A Jacobi Paradox?

\[
\begin{align*}
[1, [u_x, D_x^{-1}u]] + [u_x, [D_x^{-1}u, 1]] \\
+ [D_x^{-1}u, [1, u_x]] &= 1
\end{align*}
\]

\[
[1, u_x] = D_{u_x}(1) - D_1(u_x) = 0 \\
\implies [\partial_u, \partial_x] = 0
\]

\[
\begin{align*}
[u_x, D_x^{-1}u] &= D_{D_x^{-1}u}(u_x) - D_{u_x}(D_x^{-1}u) \\
&= D_x^{-1}u_x - D_x(D_x^{-1}u) \\
&= (u + c) - u = c
\end{align*}
\]

\[
[1, [u_x, D_x^{-1}u]] = [1, c] = 0
\]

\[
\begin{align*}
[D_x^{-1}u, 1] &= -D_x^{-1}(1) = -x + d \\
[u_x, [D_x^{-1}u, 1]] &= [u_x, -x + d] \\
&= -D_x(-x + d) = 1
\end{align*}
\]
General Framework

\[ \mathcal{A} \quad \text{— algebra of functions } f(x) \]

\[ \mathcal{U}_0 \quad \text{— } u\text{–dependent differential functions} \]

\[ \mathcal{U} = \mathcal{A} \oplus \mathcal{U}_0 \quad \text{— full differential algebra} \]

\[ F(x, \mathcal{u}^{(n)}) = f(x) + P(x, \mathcal{u}^{(n)}) \]

\[ \implies P(x, 0) = 0 \]

\[ D_i \quad \text{— total derivatives on } \mathcal{U} \]

\[ \ker D_i = \{ f(x^1, \ldots, x^{i-1}, x^{i+1}, \ldots, x^p) \} \subset \mathcal{A} \]
**Definition.** A *evolutionary vector field* \( \mathbf{v} \) is a derivation \( \mathbf{v}: \mathcal{U} \to \mathcal{U} \), with

\[
\mathbf{v}(P + Q) = \mathbf{v}(P) + \mathbf{v}(Q)
\]

\[
\mathbf{v}(P \cdot Q) = \mathbf{v}(P) \cdot Q + P \cdot \mathbf{v}(Q)
\]

\[
\mathbf{v}(x^i) = 0 \quad \implies \quad \mathcal{A} \subseteq \ker \mathbf{v}
\]

\[
[ \mathbf{v}, D_i ] = 0
\]

Commutator:

\[
[ \mathbf{v}, \mathbf{w} ](P) = \mathbf{v}(\mathbf{w}(P)) - \mathbf{w}(\mathbf{v}(P))
\]

**Theorem.** The Jacobi identity holds!

\[
[ \mathbf{u}, [ \mathbf{v}, \mathbf{w} ] ] + [ \mathbf{v}, [ \mathbf{w}, \mathbf{u} ] ] + [ \mathbf{w}, [ \mathbf{u}, \mathbf{v} ] ] = 0
\]
Characteristic:
\[ \mathbf{v}(u^\alpha) = Q^\alpha \quad \alpha = 1, \ldots, q. \]

Note
\[ \mathbf{v}(u_j^\alpha) = \mathbf{v}(D_j u^\alpha) = D_j \mathbf{v}(u^\alpha) = D_j Q^\alpha \]

**Key observation:**
For local differential algebras, an evolutionary vector field is uniquely determined by its characteristic. This is *not* true in nonlocal differential algebras. There are nonzero evolutionary vector fields with zero characteristic — *ghosts.*
Ghosts

**Definition.** An evolutionary vector field $\gamma$ is called a $K$-ghost if

$$\gamma(u^\alpha_L) = 0$$

for all $L \geq K$ and $\alpha = 1, \ldots, q$.

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**Example.** $p = 1 \quad u_n = D^n_x u$

The vector field $v_1$ with characteristic $Q = 1$ is a 1-ghost:

$$v_1(u_x) = D_x(1) = 0$$

$$v_1(u_n) = D^n_x(1) = 0 \quad n \geq 1$$

$\implies$ positive (local) ghost
Example. A non-local ghost:

\[
\gamma(u_k) = \chi_{k+1}(x) = D_x^{k+1}(1) = \begin{cases} 
\frac{x^{-k-1}}{(-k-1)!} & k \leq -1 \\
0 & \text{otherwise}
\end{cases}
\]

Characteristic:

\[
\begin{align*}
\gamma(u) &= Q = 0 \\
\gamma(u_n) &= D_x^n(Q) = 0 & n \geq 0 \\
\gamma(u_{-1}) &= 1 \\
\gamma(u_{-n}) &= D_x^{-n+1}(1)
\end{align*}
\]

\[\implies \quad \text{Only polynomial } u\text{-independent vector fields can be ghosts!}
\]

\[
v(u_\alpha^\alpha) = q_\alpha^\alpha(x) \in A
\]
Jacobi Revisited

\[ [1, [u_x, D_x^{-1}u]] + [u_x, [D_x^{-1}u, 1]] + [D_x^{-1}u, [1, u_x]] \]

Generalized vector fields \( \sim \) characteristics

\[ v \sim 1, \quad w \sim u_x, \quad z \sim D_x^{-1}u. \]

 Surprise: The problem is with the local commutator

\[ [\partial_u, \partial_x] \sim [1, u_x] \sim [v, w] = \gamma \neq 0 \]

On the nonlocal differential algebra, \( \gamma \) is a ghost vector field!

\[ v(u_k) = D_x^k(1) = \chi_k(x) = \begin{cases} \frac{x^{-k}}{(-k)!}, & k \leq 0, \\ 0 & \text{otherwise}, \end{cases} \]

\[ w(u_k) = D_x(u_k) = u_{k+1} \]

\[ w(v(u_k)) = 0, \]

\[ v(w(u_k)) = v(u_{k+1}) = \chi_{k+1}(x) \]

\[ \gamma(u_k) = v(w(u_k)) - w(v(u_k)) = \chi_{k+1}(x) \]
\[ [1, [u_x, D_x^{-1}u]] + [u_x, [D_x^{-1}u, 1]] + [D_x^{-1}u, [1, u_x]] \]
\[ \mathbf{v} \sim 1, \quad \mathbf{w} \sim u_x, \quad \mathbf{z} \sim D_x^{-1}u. \]
\[ [1, u_x] \sim [\mathbf{v}, \mathbf{w}] = \gamma \]

This ghost provides the missing terms in the Jacobi identity:
\[ [\mathbf{z}, \gamma] = -\mathbf{v} \]
because
\[ [\mathbf{z}, \gamma](u) = -\gamma(\mathbf{z}(u)) = -\gamma(D_x^{-1}u) = -1. \]

\[ [\mathbf{z}, [\mathbf{v}, \mathbf{w}]] = [\mathbf{z}, \gamma] = -\mathbf{v} \]
\[ [\mathbf{v}, [\mathbf{w}, \mathbf{z}]] = [\mathbf{v}, \mathbf{v}_c] = 0 \]
\[ [\mathbf{w}, [\mathbf{z}, \mathbf{v}]] = [\mathbf{w}, \mathbf{v}_{-x+d}] = \mathbf{v} \]
The Ghost Calculus

Assume $q = 1$ — one dependent variable $u$
Restrict to polynomial vector fields.

*Only vector fields that do not depend on $u_K$ can be ghosts!*

Define

$$\chi_K = D^K(1) = \begin{cases} 
\frac{x^{-K}}{(-K)!}, & -K \geq 0, \\
0 & \text{otherwise},
\end{cases}$$

Basis ghost vector field:

$$\gamma_J(u_K) = \chi_{J+K},$$

---

**Theorem.** Every ghost vector field is a constant coefficient linear combination of the basis ghosts

$$\gamma = \sum_J c_J \gamma_J.$$

$\implies$ The summation can be infinite, but with certain restrictions.
Theorem. Any evolutionary vector field on a polynomial differential algebra can be written a linear combination of basis ghosts and a polynomial $u$-dependent vector field with characteristic $Q$

$$v = v_Q + \sum J c_J \gamma_J$$

$$v(u) = Q + \sum J c_J \chi_J$$

$$= Q + \sum_{-J \geq 0} c_J \frac{x^{-J}}{(-J)!}$$

$$v(u_K) = D^K Q + \sum J c_J \chi_{K+J}$$

$$= D^K Q + \sum_{-J-K \geq 0} c_J \frac{x^{-J-K}}{(-J-K)!}$$

$\Rightarrow$ sums must be finite
Ghost calculus rules

Uniform notation: Replace

\[ x^J \implies J! \chi_{-J} \quad J \geq 0 \]
\[ \gamma_K \implies \chi_K \]

Every differential polynomial is a sum of monomials

\[ \chi_L u_{K_1} \cdots u_{K_m} \sim \frac{x^{-L}}{(-L)!} u_{K_1} \cdots u_{K_m} \quad -L \geq 0 \]
\[ \chi_L \chi_K = \left( \begin{array}{c} -K - L \\ -L \end{array} \right) \chi_{K+L} \]

\[
\begin{align*}
\{ \chi_J, \chi_K \} &= 0 \\
\{ \chi_J, u_{K} \} &= \chi_{J+K} \\
\{ \chi_J, \chi_L u_{K_1} \cdots u_{K_m} \} &= \sum \left( \begin{array}{c} -K_\nu - L \\ -L \end{array} \right) \chi_{J+K_\nu + L} u_{K_1} \cdots \widehat{u_{K_\nu}} \cdots u_{K_m} \\
\{ Q, R \} &= \nu_Q(R) - \nu_R(Q)
\end{align*}
\]

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Jacobi paradox revisited

\[
[1, [u_x, D_x^{-1}u]] + [u_x, [D_x^{-1}u, 1]] \\
+ [D_x^{-1}u, [1, u_x]]
\]

The three ghost characteristics are

\[
1 \implies \chi_0 \quad u_x \implies u_1 \quad D_x^{-1}u \implies u_{-1}
\]

\[
[\chi_0, [u_1, u_{-1}]] = 0
\]

\[
[u_1, [u_{-1}, \chi_0]] = -[u_1, \chi_{-1}] = \chi_0
\]

\[
[u_{-1}, [\chi_0, u_1]] = [u_{-1}, \chi_1] = -\chi_0
\]

The sum of these three terms is 0.
The Original Jacobi Paradox

Independent variables     —  \( x, y \)
Dependent variable       —  \( u \)

Characteristics:  \( y, yu_x, u_x D_x^{-1} u_y \).

Without ghost terms:

\[
[ y, [ u_x D_x^{-1} u_y, yu_x ] ] + [ yu_x, [ y, u_x D_x^{-1} u_y ] ] \\
+ [ u_x D_x^{-1} u_y, [ yu_x, y ] ] = -2yu_x \neq 0
\]

Ghost characteristics

\( y \mapsto \chi_{0,-1} \quad yu_x \mapsto \chi_{0,-1} u_{1,0} \quad u_x D_x^{-1} u_y \mapsto u_{1,0} u_{-1,1} \)

The three terms are

\[
[ \chi_{0,-1}, \chi_{0,-1} u_{1,0} ] = \chi_{0,-1} \chi_{1,-1} = 2 \chi_{1,-2} \\
[ u_{1,0} u_{-1,1}, 2 \chi_{1,-2} ] = -2 \chi_{0,-1} u_{1,0}, \\
[ \chi_{0,-1} u_{1,0}, u_{1,0} u_{-1,1} ] = u_{0,0} u_{1,0} \\
[ \chi_{0,-1}, u_{0,0} u_{1,0} ] = \chi_{0,-1} u_{1,0} \\
[ u_{1,0} u_{-1,1}, \chi_{0,-1} ] = -\chi_{1,-1} u_{-1,1} - \chi_{-1,0} u_{1,0} = -\chi_{-1,0} u_{1,0} \\
[ \chi_{0,-1} u_{1,0}, -\chi_{-1,0} u_{1,0} ] = \chi_{0,-1} u_{1,0}
\]

The sum of these three terms is 0, and so the Jacobi identity is valid in the ghost framework.

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The Kadomtsev–Petviashvili equation

\[ u_t = D_x^{-1} u_{yy} - 6 u u_x - u_{xxx} \]

The following known KP symmetries

\[ M = \frac{1}{12} D_x^{-2} u_{yyy} - \frac{1}{4} u_{xxy} - uu_y - \frac{1}{2} u_x D_x^{-1} u_y \]

\[ N = y \]

\[ H = yu_y + \frac{1}{2} xu_x + u, \]

span an \( \mathfrak{sl}(2, \mathbb{R}) \) Lie algebra:

\[ [ M, N ] = H, \quad [ H, M ] = 2M, \quad [ H, N ] = -2N. \]

Paradox:

\[ [ M, u_x ] = [ N, u_x ] = 0, \quad \text{but} \quad [ H, u_x ] = \frac{1}{2} u_x \]

\[ \implies \text{The vector space spanned by } u_x \text{ is a one-dimensional representation of } \mathfrak{sl}(2, \mathbb{R}), \text{ but representation theory requires that } \]

\[ [ H, u_x ] = 0. \]
Ghosts to the Rescue!

The symmetries do not, in fact, form a Lie algebra, but must be modified by appending a ghost to

\[
[ M, N ] = yu_y + \frac{1}{2} x u_x + u + \frac{1}{4} \gamma_{2,0} = H + \frac{1}{4} \gamma_{2,0} = \hat{H}
\]

Now, their Lie brackets are correct:

\[
[ M, N ] = \hat{H}, \ [ \hat{H}, M ] = 2M, \ [ \hat{H}, N ] = -2N.
\]

We have

\[
[ \hat{H}, u_x ] = \frac{1}{2} u_x + \frac{1}{4} \gamma_{3,0}
\]

Thus \( u_x \) is not an eigenvector for \( H \), but actually belongs to a two-dimensional \( \mathfrak{sl}(2, \mathbb{R}) \) representation space spanned by \( u_x \) and \( \gamma_{3,0} \).