Invariant Histograms and Signatures for Object Recognition and Symmetry Detection

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North Carolina, October, 2011

References

Boutin, M., & Kemper, G., On reconstructing n-point configurations from the distribution of distances or areas, *Adv. Appl. Math.* **32** (2004) 709–735

Brinkman, D., & Olver, P.J., Invariant histograms, *Amer. Math. Monthly* **118** (2011) 2–24.

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The Distance Histogram

Definition. The distance histogram of a finite set of points $P = \{z_1, \ldots, z_n\} \subset V$ is the function

$$\eta_P(r) = \#\left\{ (i,j) \;\middle|\; 1 \le i < j \le n,\; d(z_i, z_j) = r \right\}.$$

The Distance Set

The support of the histogram function,

$$\operatorname{supp} \eta_P = \Delta_P \subset \mathbb{R}^+$$

is the distance set of P.

Erdös' distinct distances conjecture (1946):

$$\text{If } P \subset \mathbb{R}^m, \text{ then } \#\,\Delta_P \geq c_{m,\varepsilon}\,(\#\,P)^{2/m-\varepsilon}$$

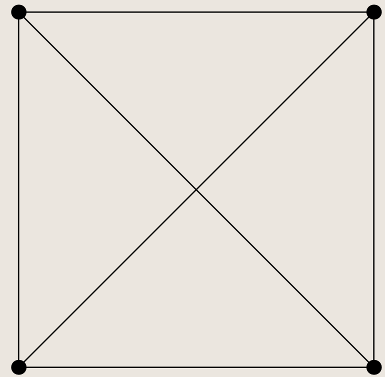
Characterization of Point Sets

Note: If $\tilde{P} = g \cdot P$ is obtained from $P \subset \mathbb{R}^m$ by a rigid motion $g \in \mathrm{E}(n)$, then they have the same distance histogram: $\eta_P = \eta_{\tilde{P}}$.

Question: Can one uniquely characterize, up to rigid motion, a set of points $P\{z_1, \ldots, z_n\} \subset \mathbb{R}^m$ by its distance histogram?

$\Longrightarrow$ Tinkertoy problem.

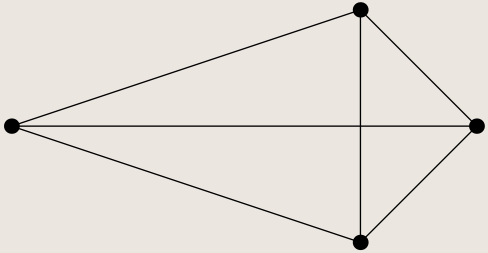
Yes:



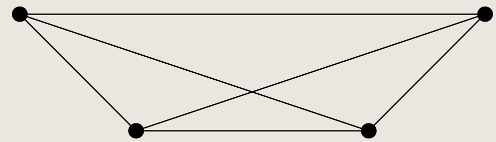
$$\eta = 1,\ 1,\ 1,\ 1,\ \sqrt{2},\ \sqrt{2}.$$

No:

Kite



Trapezoid



$$\eta = \sqrt{2},\ \sqrt{2},\ 2,\ \sqrt{10},\ \sqrt{10},\ 4.$$

No:

$$\left.\begin{array}{l} P = \{0, 1, 4, 10, 12, 17\} \\ Q = \{0, 1, 8, 11, 13, 17\} \end{array}\right. \subset \mathbb{R}$$

$$\eta = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 16, 17$$

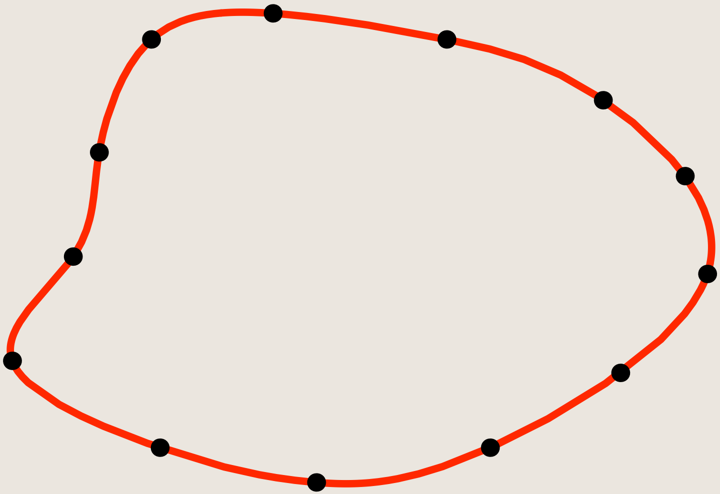
$\Longrightarrow$ G. Bloom, *J. Comb. Theory, Ser. A* **22** (1977) 378–379

Theorem. (*Boutin–Kemper*) Suppose $n \leq 3$ or $n \geq m + 2$. Then there is a Zariski dense open subset in the space of n point configurations in $\mathbb{R}^m$ that are uniquely characterized, up to rigid motion, by their distance histograms.

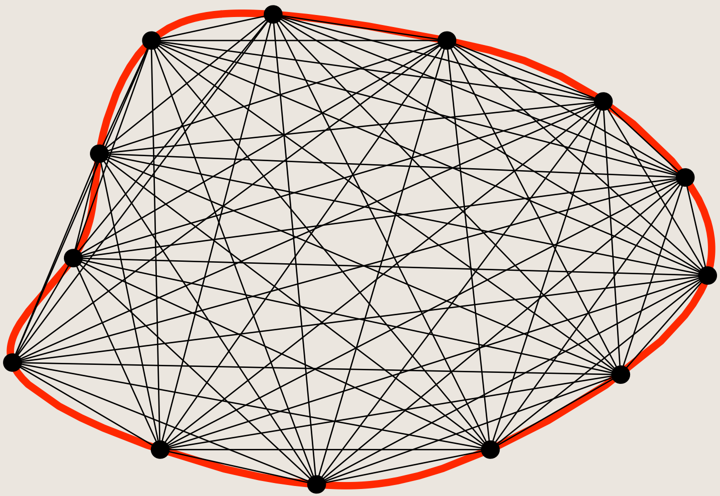
$\Longrightarrow$ M. Boutin, G. Kemper, *Adv. Appl. Math.* **32** (2004) 709–735

Limiting Curve Histogram

Limiting Curve Histogram



Limiting Curve Histogram



Sample Point Histograms

Cumulative distance histogram: $\quad n = \#P$:

$$\Lambda_P(r) = \frac{1}{n} + \frac{2}{n^2} \sum_{s \le r} \eta_P(s) = \frac{1}{n^2} \# \left\{ (i,j) \;\middle|\; d(z_i, z_j) \le r \right\},$$

Note

$$\eta(r) = \tfrac{1}{2} n^2 [\, \Lambda_P(r) - \Lambda_P(r - \delta) \,] \qquad \delta \ll 1.$$

Local distance histogram:

$$\lambda_P(r, z) = \frac{1}{n} \# \left\{ j \;\middle|\; d(z, z_j) \le r \right\} = \frac{1}{n} \#(P \cap B_r(z))$$

Ball of radius r centered at z:

$$B_r(z) = \{ v \in V \mid d(v, z) \le r \}$$

Note:

$$\Lambda_P(r) = \frac{1}{n} \sum_{z \in P} \lambda_P(r, z) = \frac{1}{n^2} \sum_{z \in P} \#(P \cap B_r(z)).$$

Limiting Curve Histogram Functions

Length of a curve

$$l(C) = \int_C \, ds < \infty$$

Local curve distance histogram function $\quad z \in V$

$$h_C(r, z) = \frac{l(C \cap B_r(z))}{l(C)}$$

$\Longrightarrow$ The fraction of the curve contained in the ball of radius r centered at z.

Global curve distance histogram function:

$$H_C(r) = \frac{1}{l(C)} \int_C h_C(r, z(s)) \, ds.$$

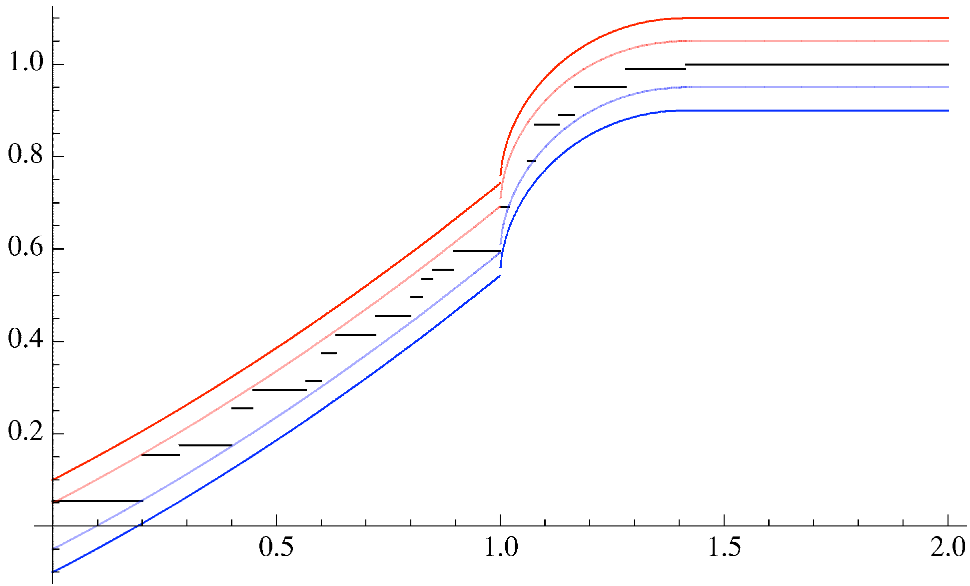
Convergence

Theorem. Let C be a regular plane curve. Then, for both uniformly spaced and randomly chosen sample points $P \subset C$, the cumulative local and global histograms converge to their continuous counterparts:

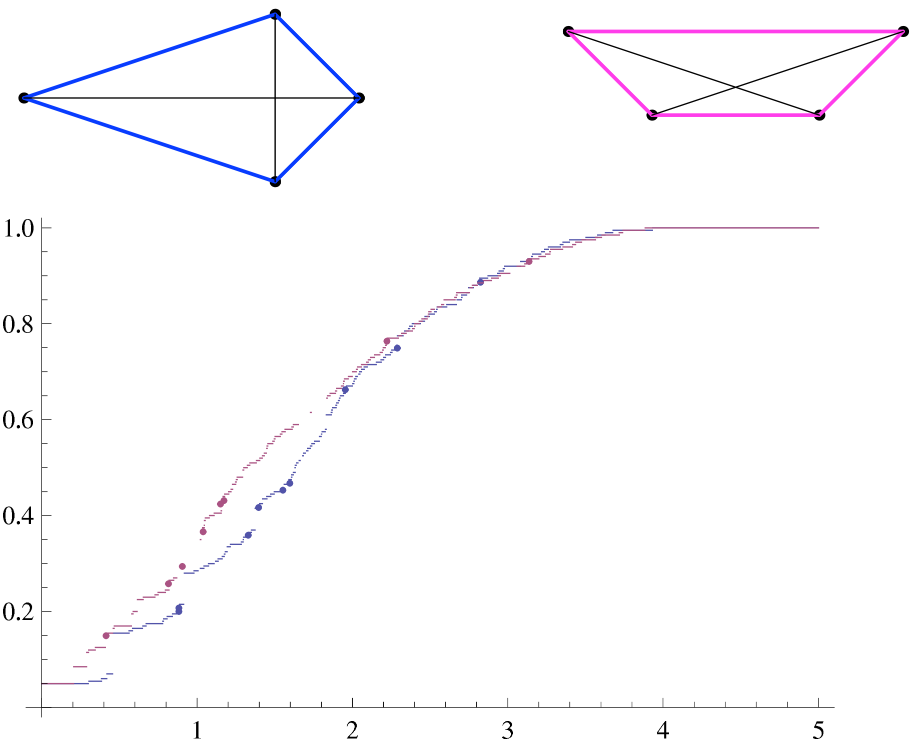
$$\lambda_P(r, z) \longrightarrow h_C(r, z), \qquad \Lambda_P(r) \longrightarrow H_C(r),$$

as the number of sample points goes to infinity.

Square Curve Histogram with Bounds



Kite and Trapezoid Curve Histograms



Histogram–Based Shape Recognition

500 sample points

Shape	(a)	(b)	(c)	(d)	(e)	(f)
(a) triangle	2.3	20.4	66.9	81.0	28.5	76.8
(b) square	28.2	.5	81.2	73.6	34.8	72.1
(c) circle	66.9	79.6	.5	137.0	89.2	138.0
(d) 2×3 rectangle	85.8	75.9	141.0	2.2	53.4	9.9
(e) 1×3 rectangle	31.8	36.7	83.7	55.7	4.0	46.5
(f) star	81.0	74.3	139.0	9.3	60.5	.9

Curve Histogram Conjecture

Two sufficiently regular plane curves C and $\widetilde{C}$ have identical global distance histogram functions, so $H_C(r) = H_{\widetilde{C}}(r)$ for all $r \geq 0$, if and only if they are rigidly equivalent: $C \simeq \widetilde{C}$.

"Proof Strategies"

- Show that any polygon obtained from (densely) discretizing a curve does not lie in the Boutin–Kemper exceptional set.
- Polygons with obtuse angles: taking r small, one can recover (i) the set of angles and (ii) the shortest side length from $H_C(r)$. Further increasing r leads to further geometric information about the polygon ...
- Expand $H_C(r)$ in a Taylor series at $r = 0$ and show that the corresponding integral invariants characterize the curve.

Taylor Expansions

Local distance histogram function:

$$L\,h_C(r,z) = 2\,r + \tfrac{1}{12}\,\kappa^2\,r^3 + \left(\tfrac{1}{40}\,\kappa\,\kappa_{ss} + \tfrac{1}{45}\,\kappa_s^2 + \tfrac{3}{320}\,\kappa^4\right) r^5 + \;\cdots\;.$$

Global distance histogram function:

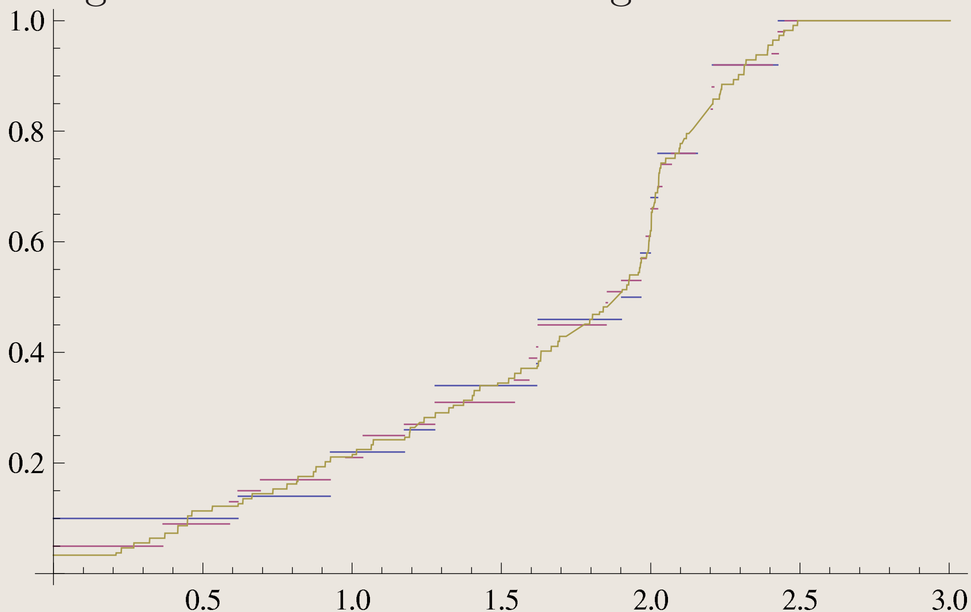
$$H_C(r) = \frac{2\,r}{L} + \frac{r^3}{12\,L^2}\oint_C \kappa^2\,ds + \frac{r^5}{40\,L^2}\oint_C \left(\tfrac{3}{8}\,\kappa^4 - \tfrac{1}{9}\,\kappa_s^2\right) ds + \;\cdots\;.$$

Space Curves

Saddle curve:

$$z(t) = (\cos t, \sin t, \cos 2t), \qquad 0 \le t \le 2\pi.$$

Convergence of global curve distance histogram function:

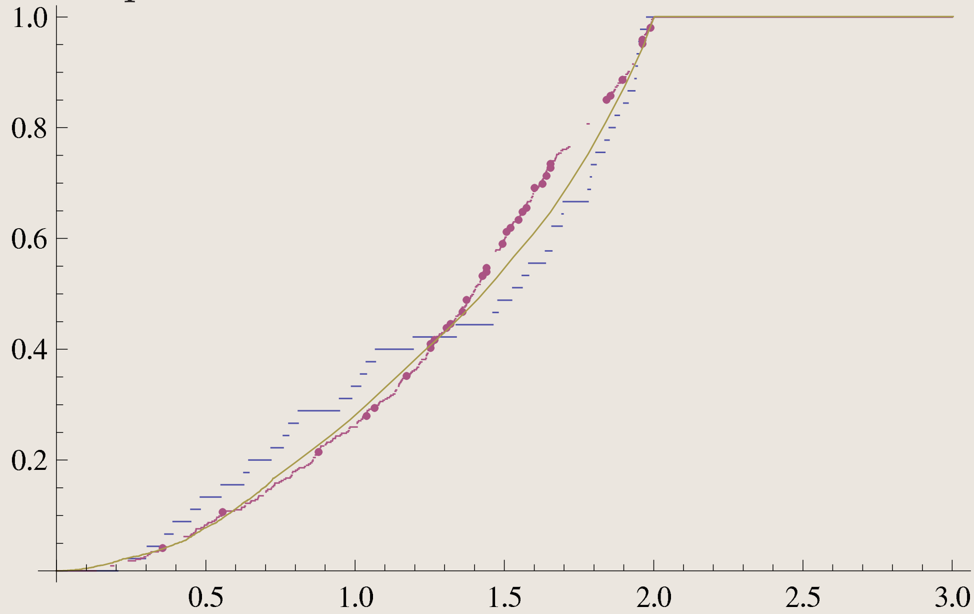


Surfaces

Local and global surface distance histogram functions:

$$h_S(r, z) = \frac{\text{area}\,(S \cap B_r(z))}{\text{area}\,(S)}, \qquad H_S(r) = \frac{1}{\text{area}\,(S)} \iint_S h_S(r, z)\, dS.$$

Convergence for sphere:



Area Histograms

Rewrite global curve distance histogram function:

$$H_C(r) = \frac{1}{L}\oint_C h_C(r, z(s))\, ds = \frac{1}{L^2}\oint_C \oint_C \chi_r(d(z(s), z(s')))\, ds\, ds'$$

$$\text{where} \qquad \chi_r(t) = \begin{cases} 1, & t \le r, \\ 0, & t > r, \end{cases}$$

Global curve area histogram function

$$A_C(r) = \frac{1}{L^3}\oint_C \oint_C \oint_C \chi_r(\text{area}\,(z(\widehat{s}), z(\widehat{s}'), z(\widehat{s}'')))\, d\widehat{s}\, d\widehat{s}'\, d\widehat{s}'',$$

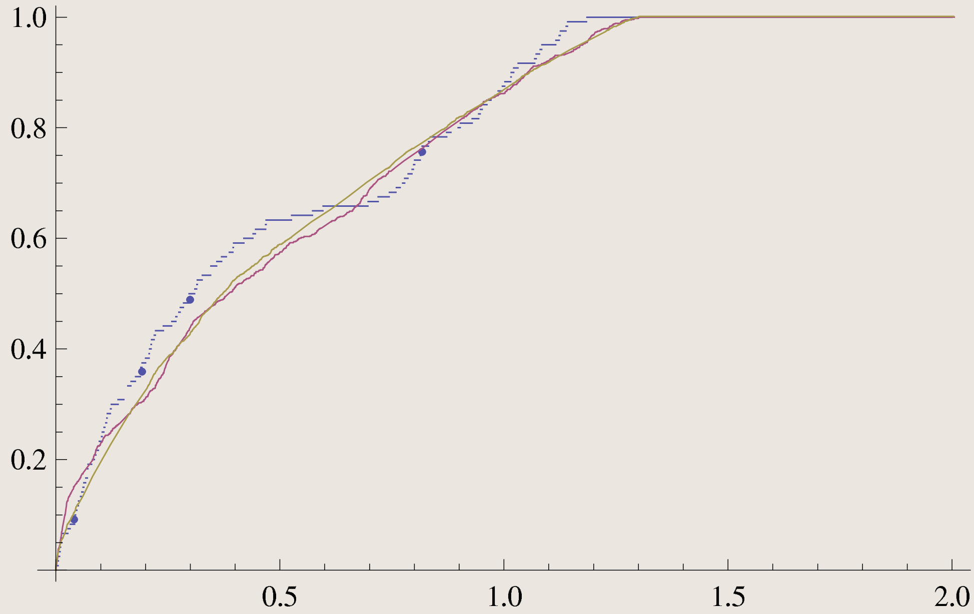
$d\widehat{s}$ — equi-affine arc length element $\qquad L = \int_C d\widehat{s}$

Discrete cumulative area histogram

$$A_P(r) = \frac{1}{n(n-1)(n-2)} \sum_{z \ne z' \ne z'' \in P} \chi_r(\text{area}\,(z, z', z'')),$$

Boutin & Kemper: the area histogram uniquely determines generic point sets $P \subset \mathbb{R}^2$ up to equi-affine motion

Area Histogram for Circle



★ ★ Joint invariant histograms — convergence???

Triangle Distance Histograms

$Z = (\ldots z_i \ldots) \subset M$ — sample points on a subset $M \subset \mathbb{R}^n$ (curve, surface, etc.)

$T_{i,j,k}$ — triangle with vertices z_i, z_j, z_k.

Side lengths:

$$\sigma(T_{i,j,k}) = (\, d(z_i, z_j), d(z_i, z_k), d(z_j, z_k)\,)$$

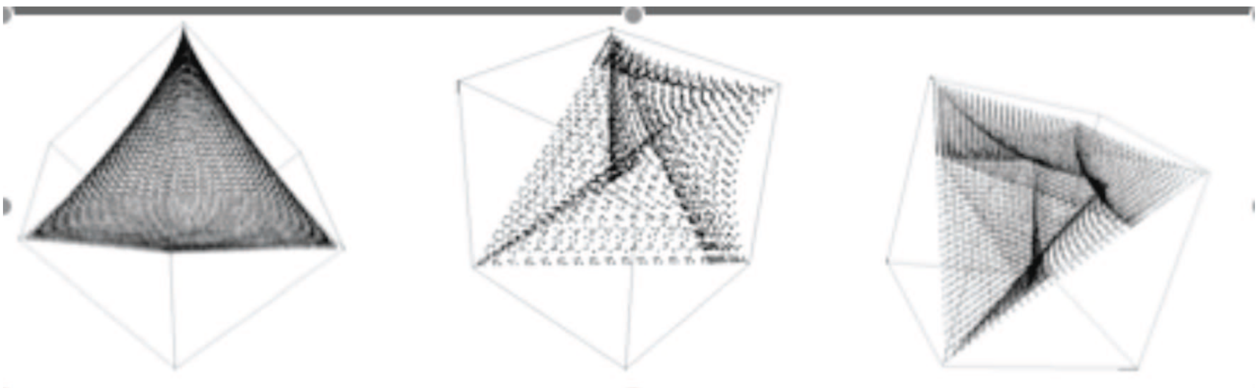
Discrete triangle histogram:

$$\mathcal{S} = \sigma(\mathcal{T}) \subset K$$

Triangle inequality cone

$K = \{\,(x, y, z) \mid\ x, y, z \geq 0,\ x + y \geq z,\ x + z \geq y,\ y + z \geq x\,\} \subset \mathbb{R}^3.$

Triangle Histogram Distributions



Circle Triangle Square

$\Longrightarrow$ Madeleine Kotzagiannidis

Practical Object Recognition

- Scale-invariant feature transform (SIFT) (Lowe)
- Shape contexts (Belongie–Malik–Puzicha)
- Integral invariants (Krim, Kogan, Yezzi, Pottman, . . .)
- Shape distributions (Osada–Funkhouser–Chazelle–Dobkin)
  Surfaces: distances, angles, areas, volumes, etc.
- Gromov–Hausdorff and Gromov-Wasserstein distances (Mémoli)
  $\Longrightarrow$ lower bounds

Signature Curves

Definition. The *signature curve* $\mathcal{S} \subset \mathbb{R}^2$ of a curve $\mathcal{C} \subset \mathbb{R}^2$ is parametrized by the two lowest order differential invariants

$$\mathcal{S} = \left\{ \left(\kappa \, , \, \frac{d\kappa}{ds} \right) \right\} \quad \subset \quad \mathbb{R}^2$$

$\Longrightarrow$ One can recover the signature curve from the Taylor expansion of the local distance histogram function.

Other Signatures

Euclidean space curves: $\mathcal{C} \subset \mathbb{R}^3$

$$\mathcal{S} = \{\, (\,\kappa\,,\,\kappa_s\,,\,\tau\,)\,\} \quad \subset \quad \mathbb{R}^3$$

- κ — curvature, τ — torsion

Euclidean surfaces: $\mathcal{S} \subset \mathbb{R}^3$ (generic)

$$\mathcal{S} = \left\{\, \Big(\, H\,,\, K\,,\, H_{,1}\,,\, H_{,2}\,,\, K_{,1}\,,\, K_{,2} \Big)\,\right\} \quad \subset \quad \mathbb{R}^3$$

- H — mean curvature, K — Gauss curvature

Equi–affine surfaces: $\mathcal{S} \subset \mathbb{R}^3$ (generic)

$$\mathcal{S} = \left\{\, \Big(\, P\,,\, P_{,1}\,,\, P_{,2},\, P_{,11} \,\Big)\,\right\} \quad \subset \quad \mathbb{R}^3$$

- P — Pick invariant

Equivalence and Signature Curves

Theorem. Two regular curves $\mathcal{C}$ and $\overline{\mathcal{C}}$ are equivalent:

$$\overline{\mathcal{C}} = g \cdot \mathcal{C}$$

if and only if their signature curves are identical:

$$\overline{\mathcal{S}} = \mathcal{S}$$

$\Longrightarrow$ object recognition

Symmetry and Signature

Theorem. The dimension of the symmetry group

$$G_N = \{ \, g \mid g \cdot N \subset N \, \}$$

of a nonsingular submanifold $N \subset M$ equals the codimension of its signature:

$$\dim G_N = \dim N - \dim \mathcal{S}$$

Discrete Symmetries

Definition. The index of a submanifold N equals the number of points in N which map to a generic point of its signature:

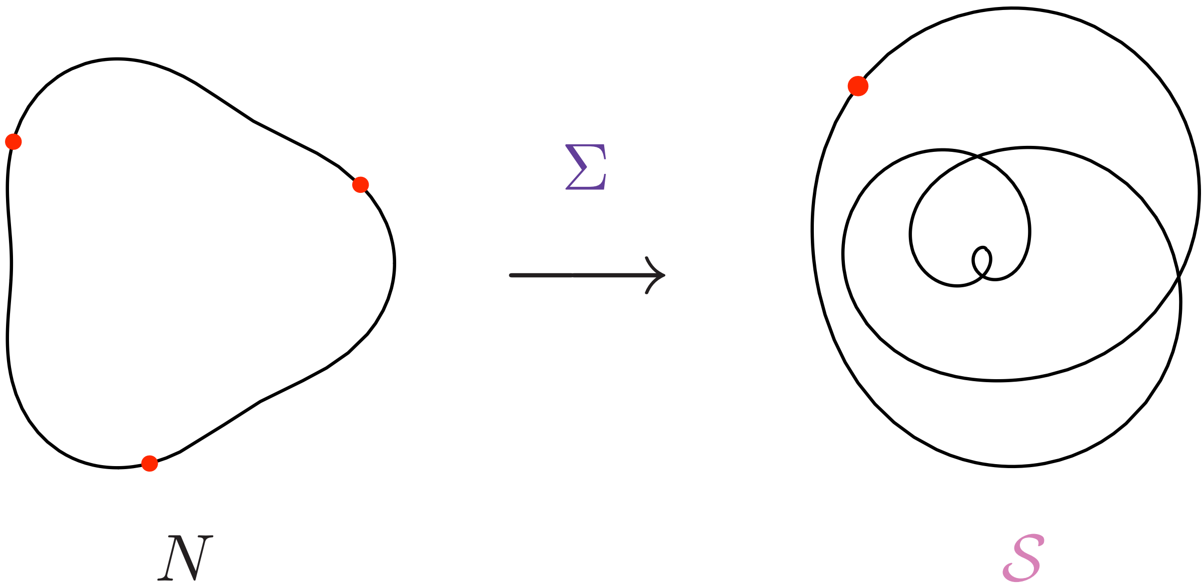
$$\iota_N = \min \left\{ \# \, \Sigma^{-1}\{w\} \;\middle|\; w \in \mathcal{S} \right\}$$

$\Longrightarrow$ Self–intersections

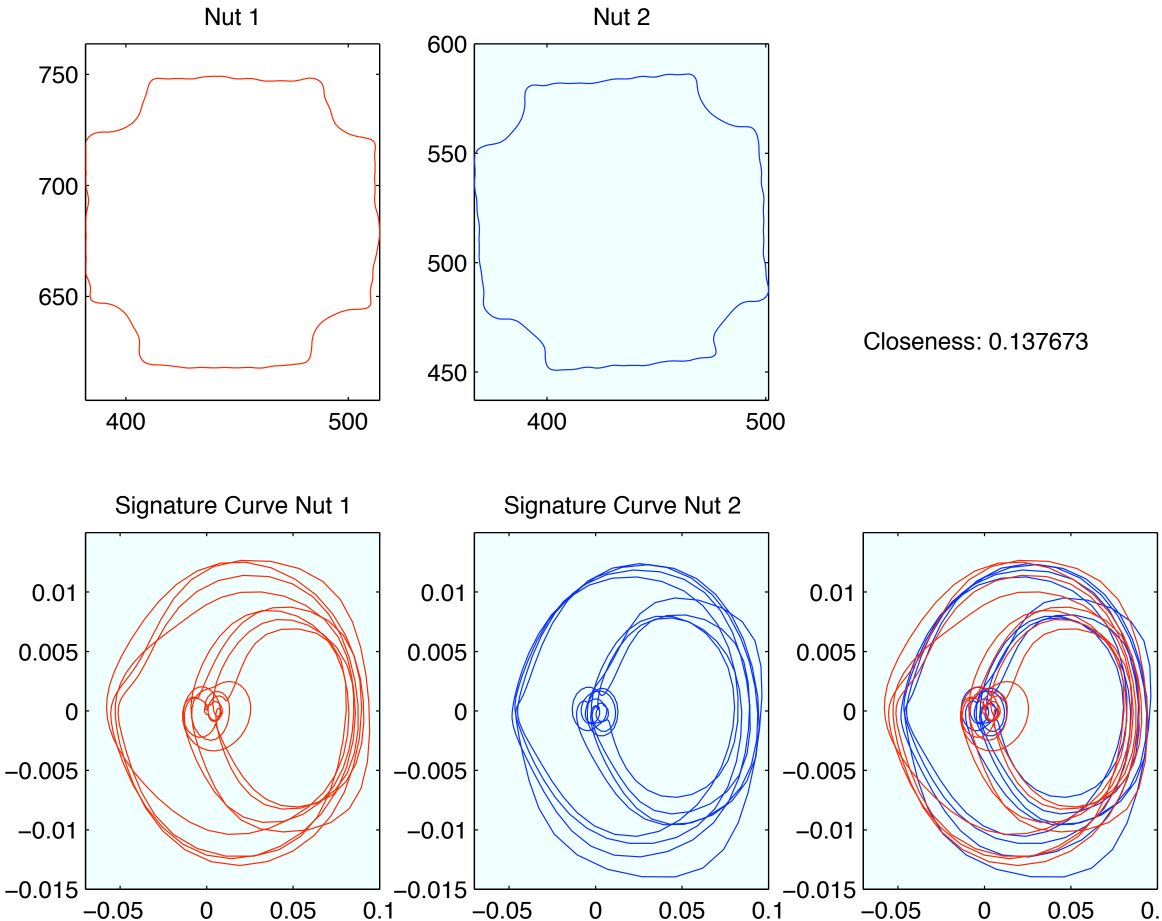
Theorem. The cardinality of the symmetry group of a submanifold N equals its index ι_N.

$\Longrightarrow$ Approximate symmetries

The Index



$\Longrightarrow$ Steve Haker


Nut 1
Nut 2
Closeness: 0.137673
Signature Curve Nut 1
Signature Curve Nut 2
750
700
650
400
500
600
550
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0.05
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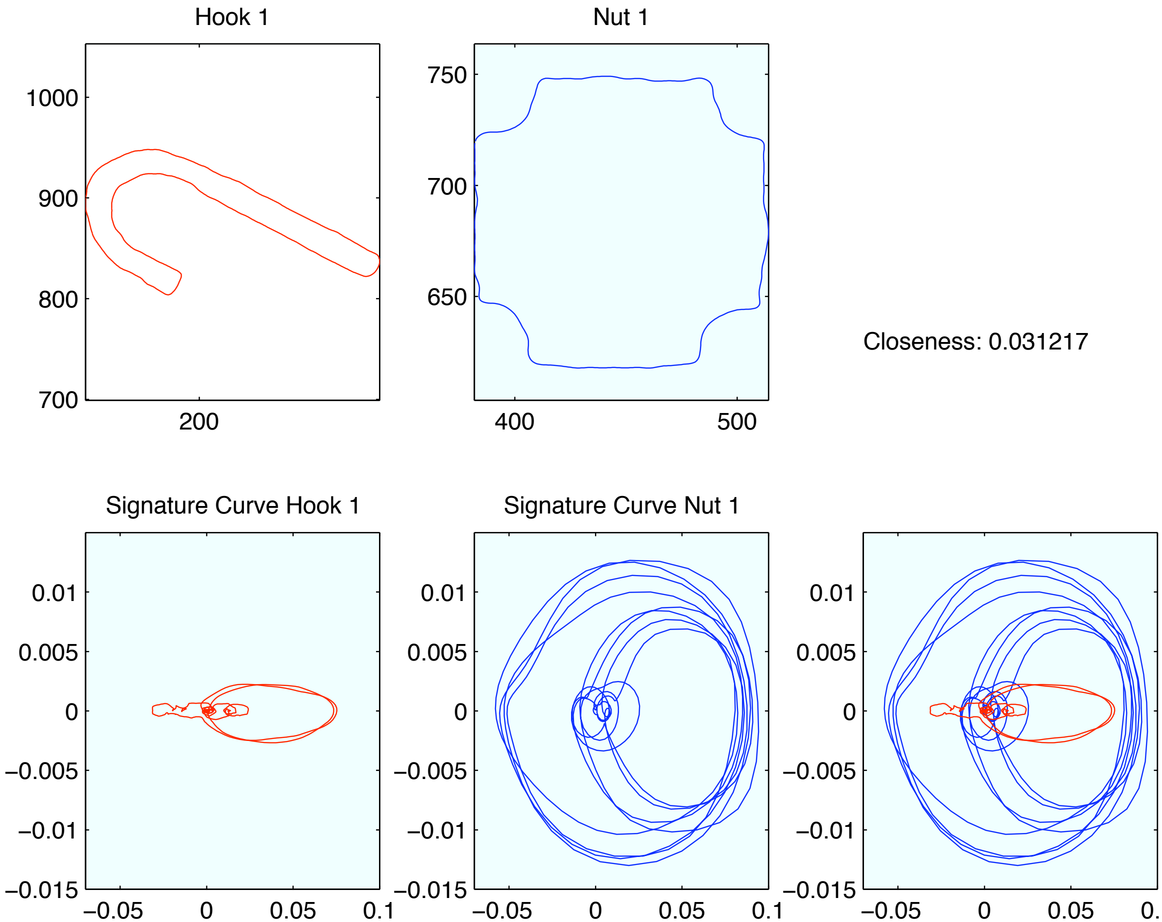
Hook 1
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Nut 1
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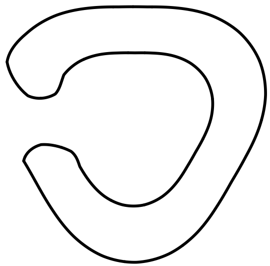
Signature Curve Hook 1
Signature Curve Nut 1
0.01
0.005
0
−0.005
−0.01
−0.015
−0.05
0
0.05
0.1

Signature Metrics

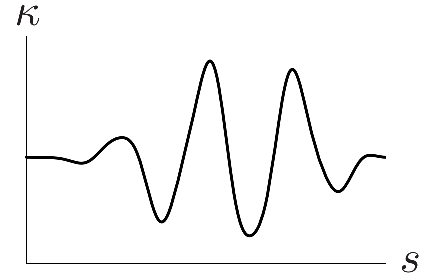
- Hausdorff
- Monge–Kantorovich transport
- Electrostatic repulsion
- Latent semantic analysis (Shakiban)
- Histograms (Kemper–Boutin)
- Diffusion metric
- Gromov–Hausdorff

Signatures

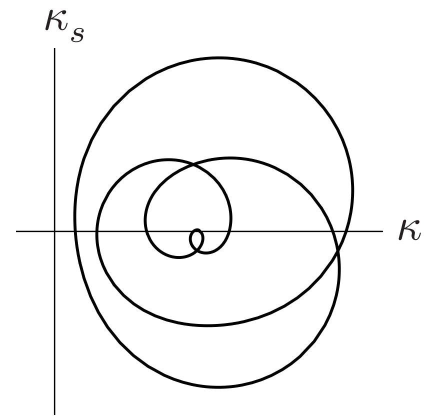


Original curve

$\longrightarrow$

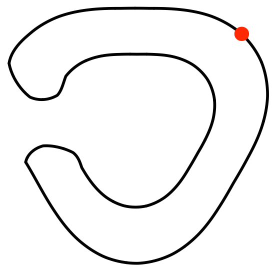


Classical Signature

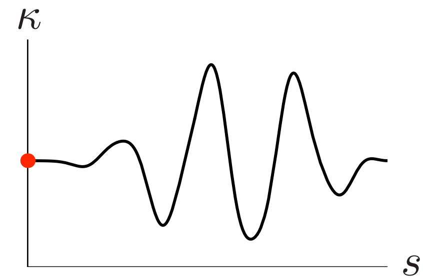


Differential invariant signature

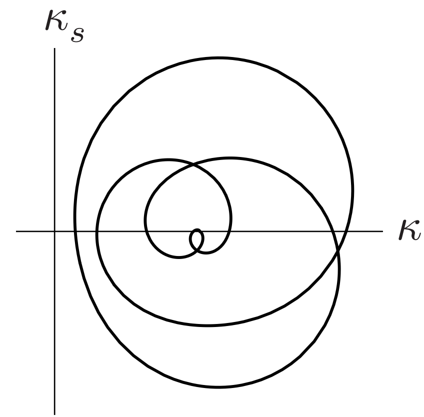
Signatures



Original curve

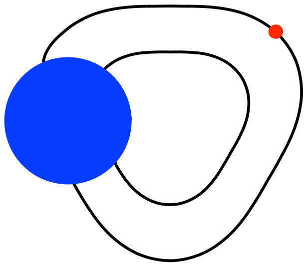


Classical Signature

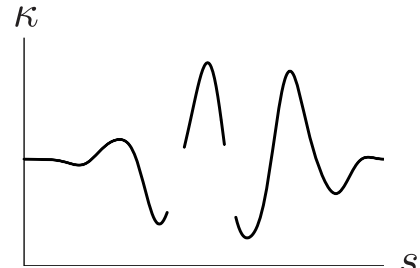


Differential invariant signature

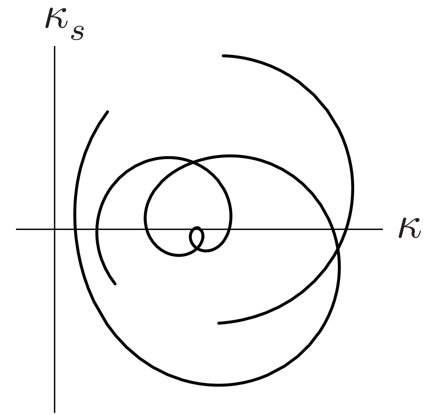
Occlusions



Original curve

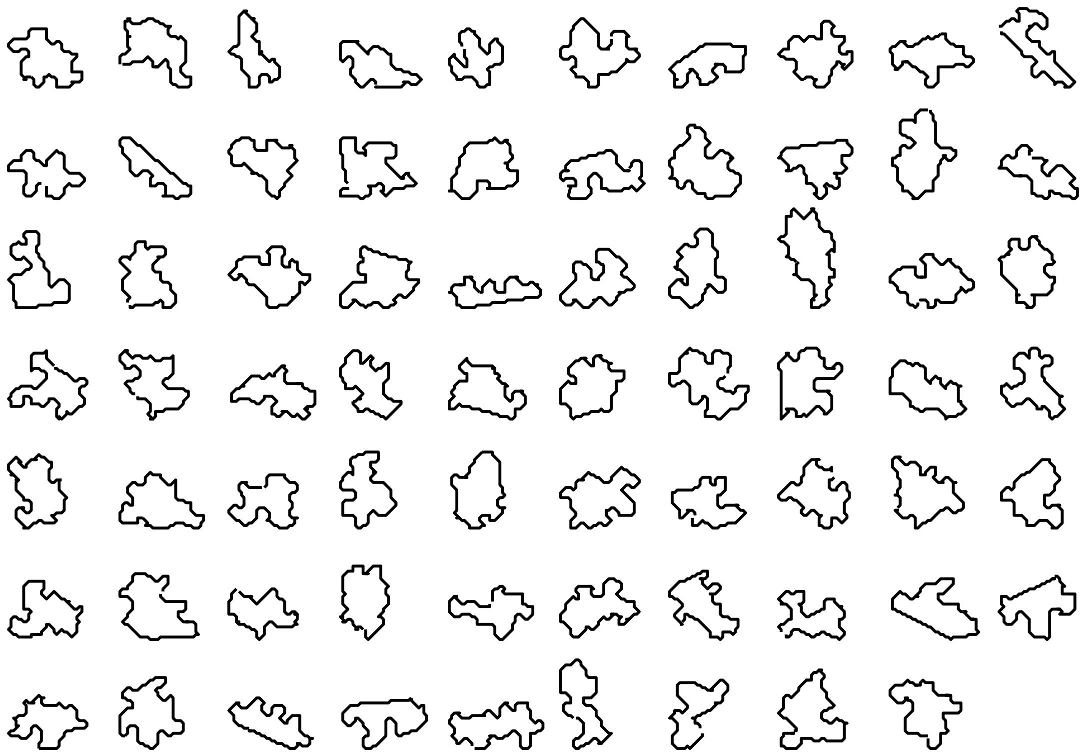


Classical Signature

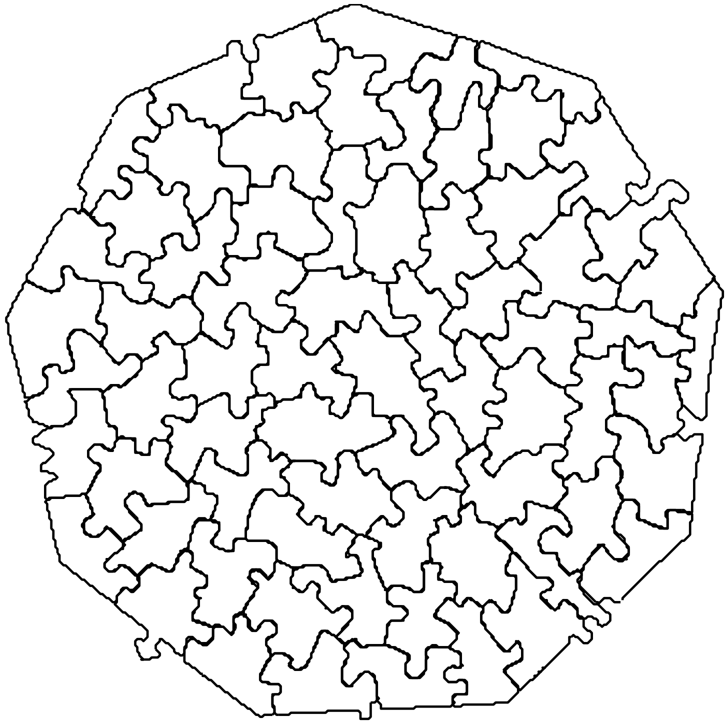


Differential invariant signature

The Baffler Jigsaw Puzzle



The Baffler Solved



⟹ Dan Hoff

Advantages of the Signature Curve

- Purely local — no ambiguities
- Symmetries and approximate symmetries
- Extends to surfaces and higher dimensional submanifolds
- Occlusions and reconstruction

Main disadvantage: Noise sensitivity due to dependence on high order derivatives.

Noise Reduction

⋆ Use lower order invariants to construct a signature:

- joint invariants
- joint differential invariants
- integral invariants
- topological invariants
- . . .

Joint Invariants

A joint invariant is an invariant of the k-fold Cartesian product action of G on $M \times \cdots \times M$:

$$\boxed{I(g \cdot z_1, \ldots, g \cdot z_k) \;=\; I(z_1, \ldots, z_k)}$$

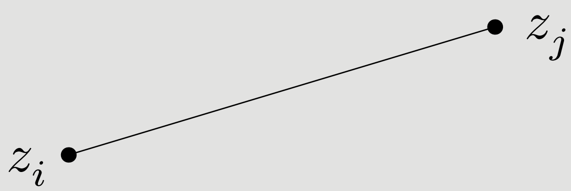
A joint differential invariant or semi-differential invariant is an invariant depending on the derivatives at several points $z_1, \ldots, z_k \in N$ on the submanifold:

$$\boxed{I(g \cdot z_1^{(n)}, \ldots, g \cdot z_k^{(n)}) \;=\; I(z_1^{(n)}, \ldots, z_k^{(n)})}$$

Joint Euclidean Invariants

Theorem. Every joint Euclidean invariant is a function of the interpoint distances

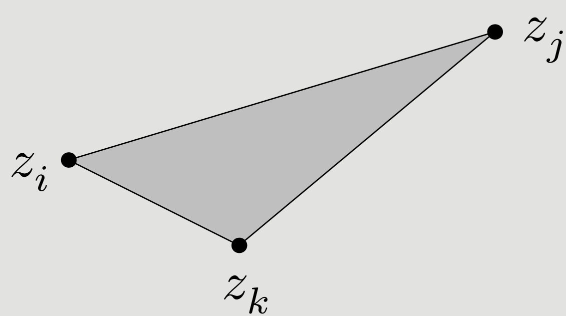
$$d(z_i, z_j) = \| z_i - z_j \|$$



Joint Equi–Affine Invariants

Theorem. Every planar joint equi–affine invariant is a function of the triangular areas

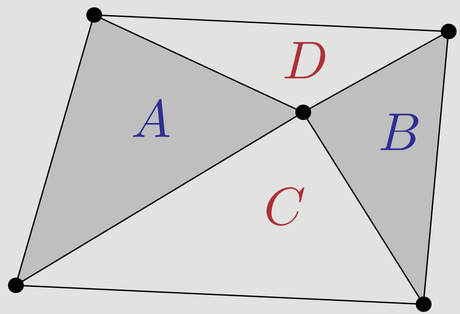
$$[\, i \; j \; k \,] = \tfrac{1}{2}\,(z_i - z_j) \wedge (z_i - z_k)$$



Joint Projective Invariants

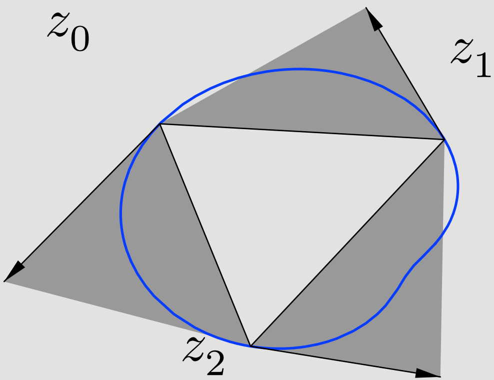
Theorem. Every joint projective invariant is a function of the planar cross-ratios

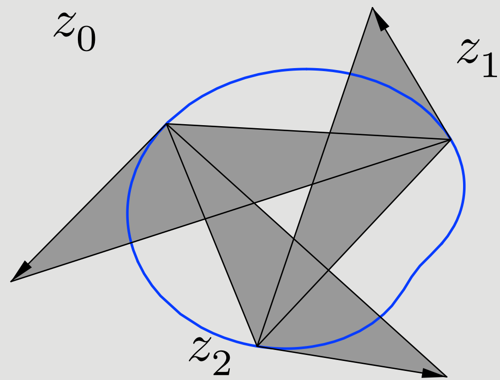
$$[\, z_i, z_j, z_k, z_l, z_m \,] \;=\; \frac{A\,B}{C\,D}$$



- Three–point projective joint differential invariant
  — tangent triangle ratio:

$$\frac{[\,0\ 2\ \dot{0}\,]\,[\,0\ 1\ \dot{1}\,]\,[\,1\ 2\ \dot{2}\,]}{[\,0\ 1\ \dot{0}\,]\,[\,1\ 2\ \dot{1}\,]\,[\,0\ 2\ \dot{2}\,]}$$





Joint Invariant Signatures

If the invariants depend on k points on a p-dimensional submanifold, then you need at least

$$\ell > k\,p$$

distinct invariants $I_1, \ldots, I_\ell$ in order to construct a syzygy. Typically, the number of joint invariants is

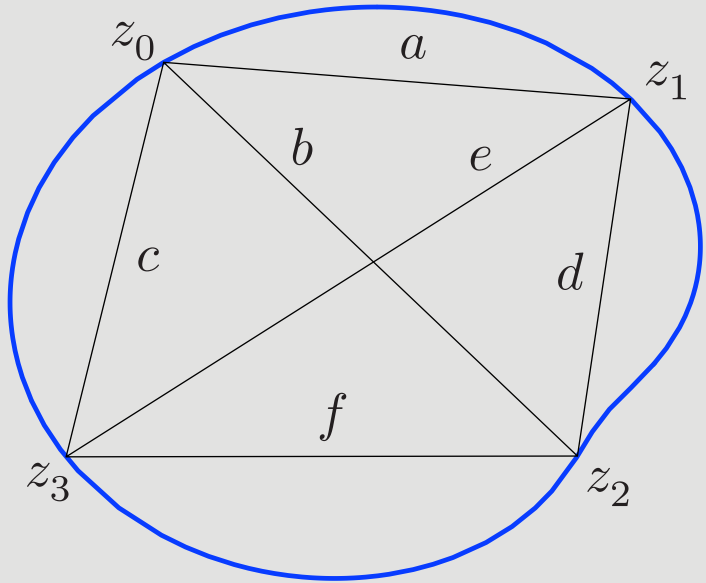
$$\ell = k\,m - r = (\#\text{points})\,(\dim M) - \dim G$$

Therefore, a purely joint invariant signature requires at least

$$k \;\geq\; \frac{r}{m-p} + 1$$

points on our p-dimensional submanifold $N \subset M$.

Joint Euclidean Signature



Joint signature map:

$$\Sigma : \mathcal{C}^{\times 4} \longrightarrow \mathcal{S} \subset \mathbb{R}^6$$

$$a = \| z_0 - z_1 \| \qquad b = \| z_0 - z_2 \| \qquad c = \| z_0 - z_3 \|$$

$$d = \| z_1 - z_2 \| \qquad e = \| z_1 - z_3 \| \qquad f = \| z_2 - z_3 \|$$

$\Longrightarrow$ six functions of four variables

Syzygies: $\Phi_1(a,b,c,d,e,f) = 0 \quad \Phi_2(a,b,c,d,e,f) = 0$

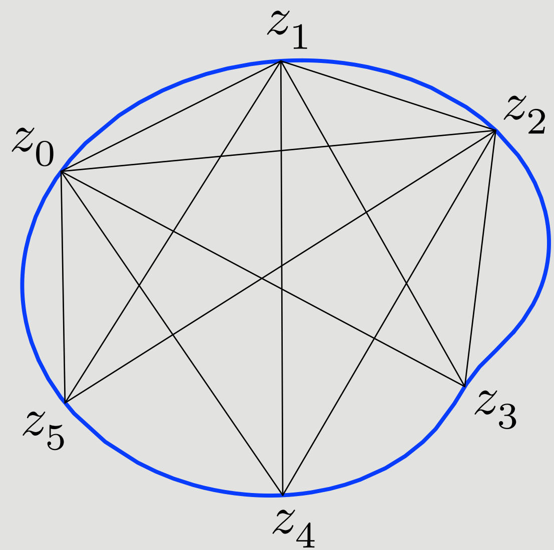
Universal Cayley–Menger syzygy $\iff$ $\mathcal{C} \subset \mathbb{R}^2$

$$\det \begin{vmatrix} 2\,a^2 & a^2+b^2-d^2 & a^2+c^2-e^2 \\ a^2+b^2-d^2 & 2\,b^2 & b^2+c^2-f^2 \\ a^2+c^2-e^2 & b^2+c^2-f^2 & 2\,c^2 \end{vmatrix} = 0$$

Joint Equi–Affine Signature

Requires 7 triangular areas:

$[\,0\;1\;2\,],\; [\,0\;1\;3\,],\; [\,0\;1\;4\,],\; [\,0\;1\;5\,],\; [\,0\;2\;3\,],\; [\,0\;2\;4\,],\; [\,0\;2\;5\,]$



Joint Invariant Signatures

- The joint invariant signature subsumes other signatures, but resides in a higher dimensional space and contains a lot of redundant information.
- Identification of landmarks can significantly reduce the redundancies (Boutin)
- It includes the differential invariant signature and semi-differential invariant signatures as its “coalescent boundaries”.
- Invariant numerical approximations to differential invariants and semi-differential invariants are constructed (using moving frames) near these coalescent boundaries.

Statistical Sampling

Idea: Replace high dimensional joint invariant signatures by increasingly dense point clouds obtained by multiply sampling the original submanifold.

- The equivalence problem requires direct comparison of signature point clouds.
- Continuous symmetry detection relies on determining the underlying dimension of the signature point clouds.
- Discrete symmetry detection relies on determining densities of the signature point clouds.