Non-Associative
Local
Lie Groups

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Lie’s Theorems

structure constants $C^i_{jk}$

\[ \downarrow \]

Lie algebra $\mathfrak{g}$

\[ \downarrow \]

local Lie group $L$

\[ \downarrow \]

global Lie group $G$

**Theorem.** Every local Lie group $L$ is contained in a global Lie group.

$\Longrightarrow$ The result is only true for sufficiently small local Lie groups!
Some History

Local Lie Groups & Lie Algebras:
  Lie, Killing, Cartan

Smoothness and Analyticity of Group Actions:
  Hilbert’s Fifth Problem

Global Lie Groups:
  Weyl, Cartan, Chevalley

Globalizability of Topological Groups:
  P.A. Smith, Mal’cev
  \( \implies \) associativity

Globalizability of Transformation Groups:
  Mostow, Palais

Hilbert’s Fifth Problem (Global):
  Gleason, Montgomery, Zippin

Hilbert’s Fifth Problem (Local):
  ♠ Jacoby ♠

Hilbert’s Fifth Problem (Semigroups):
  ? Brown, Houston, Hofmann, Weiss ?

Globalizability of Local Groups:
  van Est, Douady, Plaut
  \( \implies \) Isometries & metric convergence
Basic Definitions

Definition. **Global Lie group** \( G \):

(i) group \hspace{1cm} (ii) smooth manifold

Multiplication:

\[
\mu : G \times G \longrightarrow G \quad \mu(g, h) = g \cdot h
\]

Inversion:

\[
\iota : G \longrightarrow G, \quad \iota(g) = g^{-1}
\]

\[\implies \text{ smooth, globally defined.}\]
**Definition.**  *Local Lie group* \( L \):

**Multiplication:**

\[
\mu : \mathcal{U} \longrightarrow L, \quad \mu(x, y) = x \cdot y
\]

\[
\{e\} \times L \cup (L \times \{e\}) \subset \mathcal{U} \subset L \times L
\]

**Inversion:**

\[
\iota : \mathcal{V} \longrightarrow L, \quad \iota(g) = g^{-1}
\]

\[
e \in \mathcal{V} \subset L \quad \mathcal{V} \times \iota(\mathcal{V}), \; \iota(\mathcal{V}) \times \mathcal{V} \subset \mathcal{U}
\]

(i) **Identity:** \( e \cdot x = x = x \in e, \quad x \in L \)

(ii) **Inverse:** \( x^{-1} \cdot x = e = x \cdot x^{-1}, \quad x \in \mathcal{V} \)

(iii) **Associativity:** \( x \cdot (y \cdot z) = (x \cdot y) \cdot z \)

\[
(x, y), \; (y, z), \; (x \cdot y, z), \; (x, y \cdot z) \in \mathcal{U}.
\]
Key Example
of a Local Lie Group

\{e\} \subset N \subset G

\implies \text{Open neighborhood of the identity in a global Lie group.}

Globalizability

**Definition.** A local Lie group \( L \) is called *globalizable* if there exists a local group homeomorphism \( \Phi: L \to N \) mapping \( L \) onto a neighborhood of the identity of a global Lie group \( G \).

\[
\begin{align*}
\Phi(x \cdot y) &= \Phi(x) \cdot \Phi(y) \\
\Phi(x^{-1}) &= \Phi(x)^{-1}
\end{align*}
\]
Infinite Elements

Example. \( L = \mathbb{R} \). Identity: \( e = 0 \)

\[
\begin{align*}
\mathcal{U} &= \{ (x, y) \mid |xy| \neq 1 \} \subset L \times L \\
\mathcal{V} &= \{ x \mid x \neq \frac{1}{2}, x \neq 1 \} \subset L
\end{align*}
\]

\[
\mu(x, y) = \frac{2xy - x - y}{xy - 1} \quad \nu(x) = \frac{x}{2x - 1}
\]

\( \Rightarrow \tilde{L} = \{ |x| < \frac{1}{2} \} \) is globalizable via

\[
\Phi(x) = \frac{x}{x - 1} : \tilde{L} \rightarrow \{ -1 < x < \frac{1}{3} \} \subset \mathbb{R}
\]

\[
\Phi(\mu(x, y)) = \Phi(x) + \Phi(y) \quad \Phi(\nu(x)) = -\Phi(x)
\]
\[
\begin{align*}
\mu(x, y) &= \frac{2xy - x - y}{xy - 1} \\
\iota(x) &= \frac{x}{2x - 1}
\end{align*}
\]

But:

\[\mu(x, 1) = \mu(1, x) = 1 \text{ for all } x \neq 1\]

\[\implies \text{infinite group element}\]

Also: \[\iota(1) = 1, \text{ but } \mu(1, \iota(1)) \text{ not defined.}\]

\[\mu(x, y) = 1 \text{ if and only if } x = 1 \text{ or } y = 1\]

\[\implies \text{inaccessible}\]

Note: \( L \subset \mathbb{RP}^1 \), which is also a local Lie group with an infinite group element, containing a global Lie group as a dense open subset.
Regularity

**Definition.** A local Lie group $L$ is called *regular* if, for each $x \in L$, the left and right multiplication maps

$$\lambda_x(y) = \mu(x, y), \quad \rho_x(y) = \mu(y, x).$$

are diffeomorphisms on their respective domains of definition.
Inversional Local Groups

Given $U \subset L$, let $U^{(n)}$ denote the set of all well-defined $n$-fold products of elements $x_1, \ldots, x_n \in U$.

**Definition.** $U$ generates $L$ if $L = \bigcup_{n=1}^{\infty} U^{(n)}$.

**Definition.** A local Lie group $L$ is called *globally inversional* if the inversion map $\iota$ is defined everywhere, so that $\mathcal{V} = L$.

**Definition.** A local Lie group $L$ is called *inversional* if $\mathcal{V}$ generates $L$, i.e., every $x \in L$ can be written as a product of invertible elements.

**Theorem.** Every inversional local Lie group is regular.
Definition.  $L$ is a \textit{connected local Lie group} if

(i) $L$ is a connected manifold,

(ii) the domains of definition of the multiplication and inversion maps are connected,

(iii) if $U \subset L$ is any neighborhood of the identity, then $U$ generates $L$.

\[ \implies \text{Plaut} \]

Proposition. Any connected local Lie group is inversional, and hence regular.

\[ \implies \text{From now on all local Lie groups are be assumed to be connected.} \]
Higher Associativity

Definition. A local Lie group is

associative to order $n$

if, for every $3 \leq m \leq n$, and every $(x_1, \ldots, x_m) \in L^{\times m}$, all well-defined $m$-fold products are equal.

Example.

\[
x_1 \cdot (x_2 \cdot (x_3 \cdot x_4)) = x_1 \cdot ((x_2 \cdot x_3) \cdot x_4)
\]
\[
= (x_1 \cdot x_2) \cdot (x_3 \cdot x_4) = (x_1 \cdot (x_2 \cdot x_3)) \cdot x_4
\]
\[
= ((x_1 \cdot x_2) \cdot x_3) \cdot x_4
\]

$\implies$ Catalan number $C_n = \frac{1}{n} \binom{2n-2}{n-1}$

A local group is called globally associative if it is associative to every order $n \geq 3$. 
Globalizability

Theorem. A connected local Lie group $L$ is globalizable if and only if it is globally associative.

$\implies$ Mal’cev

★★★★ There exist local Lie groups that are associative to order $n$ but not order $n + 1$!
The Simplest
non-Globalizable Example

\[ G = \mathbb{R}^2 \cong \mathbb{C} \quad M = G \setminus \{-1\} \]

\[ L = \tilde{M} \cong \mathbb{R}^2 \]

— simply connected covering space

\[ \pi : L \rightarrow M \quad \text{— covering map.} \]

\[ \pi(\tilde{z}) = z \quad \tilde{z} = (z, n) \]

\[ L \cong \{ (r, \theta) \mid r > 0 \} \]

\[ z = \pi(r, \theta) = re^{i\theta} - 1 \quad (2n - 1)\pi < \theta \leq (2n + 1)\pi \]
\[ L_0 = \{ (r, \theta) \mid \frac{1}{2} \sec \theta < r < \frac{3}{2} \sec \theta, -\frac{1}{2} \pi < \theta < \frac{1}{2} \pi \} \]

lies above \( M_0 = \{ -\frac{1}{2} < \text{Re} \ z < \frac{1}{2} \} \)

\[ L_1 = \{ (r, \theta) \mid -\frac{1}{2} \pi < \theta < \frac{1}{2} \pi \} \]

lies above \( M_1 = \{ \text{Re} \ z > -1 \} \)

\[ \alpha(z, w) = \arg(w + 1) - \arg(z + 1) \]

\[ -\pi < \alpha(z, w) \leq \pi \]

\[ \implies \text{angle from } z \text{ to } w \text{ wrt } -1 \]

\[ H_z = \{ \hat{w} \in L_0 \mid -\frac{1}{2} \pi < \alpha(z, z + w) < \frac{1}{2} \pi \} \]

Domain of definition of multiplication:

\[ \mathcal{U} = \{ (\hat{z}, \hat{w}) \in L \times L \mid \hat{z} \in H_w \text{ or } \hat{w} \in H_z \} \]

Domain of definition of inversion: \( \mathcal{V} = L_0 \)

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**Theorem.** Under the above constructions, the product \( \mu : \mathcal{U} \to L \) and inversion \( \iota : \mathcal{V} \to L \) endow \( L \) with the structure of a regular, connected, associative, local Lie group which is not globally associative.
General Examples

\[ G \quad \text{— connected, simply connected global Lie group} \]

\[ e \notin S \subset G \quad \text{— closed subset} \]

\[ M = G \setminus S \quad \text{— globalizable local Lie group} \]

\[ L = \widetilde{M} \quad \text{— nontrivial covering group} \]

\[ \implies \text{non-globalizable local Lie group} \]

\[ \implies \text{A (generalized) covering map is a local diffeomorphism} \Phi : L \to \widetilde{M} \]
Frames

\( M \) — smooth \( m \)-dimensional manifold.

**Definition.** A *frame* is an ordered set of vector fields
\[ \{v_1, \ldots, v_m\} \] that form a basis for the tangent space
\( TM|_x \) at each \( x \in M \).

*Structure equations:*

\[
[v_i, v_j] = \sum_{k=1}^{m} C_{ij}^k v_k, \quad i, j = 1, \ldots, m.
\]

The frame has *rank 0* if the structure coefficients \( C_{ij}^k \) are all
constant, and are hence the structure constants of a
Lie algebra \( \mathfrak{g} \).

**Theorem.** If \( L \) is a regular, locally associative, local Lie
group, then it admits a right-invariant frame of
rank 0. Conversely, if \( M \) is a manifold that admits
a rank 0 frame, then \( M \) can be endowed with the
structure of a regular, locally associative local Lie
group having the given frame as right-invariant Lie
algebra elements.
Coframes

Definition. A coframe on $M$ is an ordered set of one-forms $\theta = \{\theta^1, \ldots, \theta^m\}$ which form a basis for the cotangent space $T^*M|_x$ at each $x \in M$:

$$\theta^1 \wedge \theta^2 \wedge \cdots \wedge \theta^m \neq 0$$

Structure equations:

$$d\theta^k = - \sum_{1 \leq i < j \leq m} C^k_{ij} \theta^i \wedge \theta^j,$$

$\Rightarrow$ Maurer–Cartan forms
Main Theorem

Theorem. Let $L$ be a connected local Lie group. Then there exists a local covering group $\bar{L} \to L$ which is also a local covering group $\bar{L} \to M$ of an open subset $e \in M \subset G$ of a global Lie group $G$.

\[ \implies \] The proof is based on the Cartan equivalence method, using the Frobenius Existence Theorem for first order systems of partial differential equations and Cartan’s technique of the graph.
Another Example

\[ L = \{ (r, \varphi) \mid r > 0 \} \]

Frame vector fields:

\[
\mathbf{v}_1 = \cos \varphi \frac{\partial}{\partial r} - \sin \varphi \frac{\partial}{\partial \varphi} \quad \mathbf{v}_2 = \sin \varphi \frac{\partial}{\partial r} + \cos \varphi \frac{\partial}{\partial \varphi}
\]

\[ \implies \text{in rectangular coordinates:} \quad \mathbf{v}_1 \mapsto \frac{\partial}{\partial x}, \quad \mathbf{v}_2 \mapsto \frac{\partial}{\partial y} \]

The vector fields commute: \([\mathbf{v}_1, \mathbf{v}_2] = 0\)

but their flows do not commute!

\[
\exp(\sqrt{2} \mathbf{v}_1) \exp(\sqrt{2} \mathbf{v}_2) (1, \frac{5}{4}\pi) = \exp(\sqrt{2} \mathbf{v}_1) (1, \frac{3}{4}\pi) = (1, \frac{1}{4}\pi)
\]

\[
\exp(\sqrt{2} \mathbf{v}_2) \exp(\sqrt{2} \mathbf{v}_1) (1, \frac{5}{4}\pi) = \exp(\sqrt{2} \mathbf{v}_2) (1, \frac{7}{4}\pi) = (1, \frac{9}{4}\pi)
\]

Indeed,

\[
\exp(s \mathbf{v}_1) \exp(t \mathbf{v}_2)x_0 = \exp(t \mathbf{v}_2) \exp(s \mathbf{v}_1)x_0,
\]

only for \((s, t)\) in the connected component of

\[ V = \{ (s, t) \mid \text{both sides are defined} \} \subset \mathbb{R}^2.\]