Moving Frames

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Moving Frames

Classical contributions:

G. Darboux, É. Cotton, É. Cartan

Modern contributions:

P. Griffiths, M. Green, G. Jensen

"I did not quite understand how he [Cartan] does this in general, though in the examples he gives the procedure is clear."

"Nevertheless, I must admit I found the book, like most of Cartan's papers, hard reading."

— Hermann Weyl

"Cartan on groups and differential geometry"
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Applications of Moving Frames

- Differential geometry
- Equivalence
- Symmetry
- Differential invariants
- Rigidity
- Joint Invariants and Semi-Differential Invariants
- Invariant differential forms and tensors
- Identities and syzygies
- Classical invariant theory
- Computer vision
    - object recognition
    - symmetry detection
- Invariant numerical methods
- Poisson geometry & solitons
- Lie pseudogroups

The Basic Equivalence Problem

M — smooth m-dimensional manifold.

G — transformation group acting on M

- finite-dimensional Lie group
- infinite-dimensional Lie pseudo-group

Equivalence:

Determine when two n-dimensional submanifolds

$$N \quad \text{and} \quad \overline{N} \subset M$$

are *congruent*:

$$\overline{N} = g \cdot N \qquad \text{for} \qquad g \in G$$

Symmetry:

Self-equivalence or *self-congruence*:

$$N = g \cdot N$$

Classical Geometry

Equivalence Problem: Determine whether or not two given submanifolds N and $\overline{N}$ are congruent under a group transformation: $\overline{N} = g \cdot N$.

Symmetry Problem: Given a submanifold N, find all its symmetries (belonging to the group).

- *Euclidean group* — $G = \mathrm{SE}(n)$ or $\mathrm{E}(n)$
  - $\Rightarrow$ isometries of Euclidean space
  - $\Rightarrow$ translations, rotations (& reflections)

$$z \longmapsto R \cdot z + a \qquad \begin{cases} R \in \mathrm{SO}(n) \text{ or } \mathrm{O}(n) \\ a \in \mathbb{R}^n \\ z \in \mathbb{R}^n \end{cases}$$

- *Equi-affine group*: $G = \mathrm{SA}(n)$
  $R \in \mathrm{SL}(n)$ — area-preserving
- *Affine group*: $G = \mathrm{A}(n)$
  $R \in \mathrm{GL}(n)$
- *Projective group*: $G = \mathrm{PSL}(n)$
  acting on $\mathbb{RP}^{n-1}$

$\Longrightarrow$ Applications in computer vision

Classical Invariant Theory

Binary form:

$$Q(x) = \sum_{k=0}^{n} \binom{n}{k} a_k \, x^k$$

Equivalence of polynomials (binary forms):

$$Q(x) = (\gamma x + \delta)^n \, \overline{Q} \left(\frac{\alpha x + \beta}{\gamma x + \delta} \right) \qquad g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GL}(2)$$

$\Rightarrow$ multiplier representation of GL(2)
$\Rightarrow$ modular forms

Transformation group:

$$g \colon (x, u) \longmapsto \left(\frac{\alpha x + \beta}{\gamma x + \delta} \, , \, \frac{u}{(\gamma x + \delta)^n} \right)$$

Equivalence of functions $\Longleftrightarrow$ equivalence of graphs

$$N_Q = \{ \, (x, u) = (x, Q(x)) \, \} \subset \mathbb{C}^2$$

Moving Frames

Definition.

A *moving frame* is a G-equivariant map

$$\rho : M \longrightarrow G$$

Equivariance:

$$\rho(g \cdot z) = \begin{cases} g \cdot \rho(z) & \text{left moving frame} \\ \rho(z) \cdot g^{-1} & \text{right moving frame} \end{cases}$$

$$\rho_{left}(z) = \rho_{right}(z)^{-1}$$

The Main Result

Theorem. A moving frame exists in a neighborhood of a point $z \in M$ if and only if G acts freely and regularly near z.

$G_z = \{ g \mid g \cdot z = z \}$ $\Longrightarrow$ Isotropy subgroup

- free — the only group element $g \in G$ which fixes *one* point $z \in M$ is the identity:
  $\Longrightarrow$ $G_z = \{e\}$ for all $z \in M$.
- locally free — the orbits all have the same dimension as G:
  $\Longrightarrow$ G_z is a discrete subgroup of G.
- regular — all orbits have the same dimension and intersect sufficiently small coordinate charts only once
  $\not\approx$ irrational flow on the torus

Theorem. A moving frame exists in a neighborhood of a point $z \in M$ if and only if G acts freely and regularly near z.

Necessity: Let $z \in M$.

Let $\rho : M \to G$ be a left moving frame.

Freeness: If $g \in G_z$, so $g \cdot z = z$, then by left equivariance:

$$\rho(z) = \rho(g \cdot z) = g \cdot \rho(z).$$

Therefore $g = e$, and hence $G_z = \{e\}$ for all $z \in M$.

Regularity: Suppose

$$z_n = g_n \cdot z \longrightarrow z \qquad \text{as} \qquad n \to \infty$$

By continuity,

$$\rho(z_n) = \rho(g_n \cdot z) = g_n \cdot \rho(z) \longrightarrow \rho(z)$$

Hence $g_n \longrightarrow e$ in G.

Sufficiency: By construction — "normalization".

Q.E.D.

Isotropy

Isotropy subgroup for $z \in M$:

$$G_z = \{ \, g \mid \, g \cdot z = z \, \}$$

- free — the only group element $g \in G$ which fixes *one* point $z \in M$ is the identity:
  $G_z = \{e\}$ for all $z \in M$.

- locally free — the orbits all have the same dimension as G:
  G_z is a discrete subgroup of G.

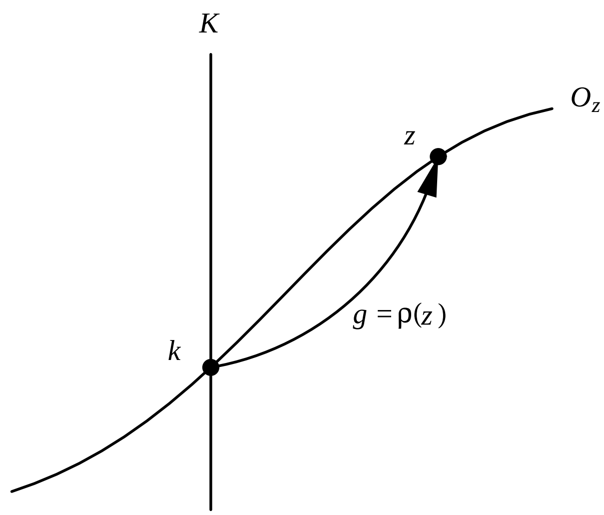
- regular — all orbits have the same dimension and intersect sufficiently small coordinate charts only once
  ($\not\approx$ irrational flow on the torus)

- effective — the only group element $g \in G$ which fixes *every* point $z \in M$ is the identity: $g \cdot z = z$ for all $z \in M$ iff $g = e$:

$$G_M = \bigcap_{z \in M} G_z = \{e\}$$

Geometrical Construction

Normalization = choice of cross-section to the group orbits



K — cross-section to the group orbits

$\mathcal{O}_z$ — orbit through $z \in M$

$k \in K \cap \mathcal{O}_z$ — unique point in the intersection

- k is the *canonical form* of z
- the (nonconstant) coordinates of k are the fundamental invariants

$g \in G$ — *unique* group element mapping k to z

$\Longrightarrow$ freeness

$\rho(z) = g$ left moving frame $\rho(h \cdot z) = h \cdot \rho(z)$

$$k = \rho^{-1}(z) \cdot z = \rho_{right}(z) \cdot z$$

Construction of Moving Frames

$$r = \dim G \;\leq\; m = \dim M$$

Coordinate cross-section

$$K = \{\, z_1 = c_1, \;\ldots\;, z_r = c_r \,\}$$

left	right	
$w(g,z) = g^{-1} \cdot z$	$w(g,z) = g \cdot z$	

Choose $r = \dim G$ components to *normalize*:

$$w_1(g,z) = c_1 \qquad \ldots \qquad w_r(g,z) = c_r$$

The solution

$$g = \rho(z)$$

is a (local) moving frame.

$\Longrightarrow$ Implicit Function Theorem

The Fundamental Invariants

Substituting the moving frame formulae

$$g = \rho(z)$$

into the unnormalized components of $w(g, z)$ produces the fundamental invariants:

$$I_1(z) = w_{r+1}(\rho(z), z) \quad \ldots \quad I_{m-r}(z) = w_m(\rho(z), z)$$

$\Longrightarrow$ These are the coordinates of the canonical form $k \in K$.

Theorem. Every invariant $I(z)$ can be (locally) uniquely written as a function of the fundamental invariants:

$$I(z) = H(I_1(z), \ldots, I_{m-r}(z))$$

Invariantization

Definition. The *invariantization* of a function $F : M \to \mathbb{R}$ with respect to a right moving frame ρ is the the invariant function $I = \iota(F)$ defined by $I(z) = F(\rho(z) \cdot z)$.

$$\iota\,[\,F(z_1, \ldots, z_m)\,] \;=\; F(c_1, , \ldots c_r, I_1(z), \ldots, I_{m-r}(z))$$

Invariantization amounts to restricting F to the cross-section

$$I \mid K = F \mid K$$

and then requiring that $I = \iota(F)$ be constant along the orbits.

In particular, if $I(z)$ is an invariant, then $\iota(I) = I$.

Invariantization defines a canonical projection

$$\iota : \quad \text{functions} \quad \longmapsto \quad \text{invariants}$$

The Rotation Group

$$G = \mathrm{SO}(2) \qquad \text{acting on} \qquad \mathbb{R}^2$$

$$z = (x, u) \longmapsto g \cdot z = (\, x \cos\theta - u \sin\theta \;,\; x \sin\theta + u \cos\theta \,)$$

$$\Longrightarrow \quad \text{Free on } M = \mathbb{R}^2 \setminus \{0\}$$

Left moving frame:

$$w(g, z) = g^{-1} \cdot z = (y, v)$$

$$y = x \cos\theta + u \sin\theta \qquad v = -\, x \sin\theta + u \cos\theta$$

Cross-section

$$K = \{\, u = 0,\; x > 0 \,\}$$

Normalization equation

$$v = -\, x \sin\theta + u \cos\theta = 0$$

Left moving frame:

$$\theta = \tan^{-1} \frac{u}{x} \quad \Longrightarrow \quad \theta = \rho(x, u) \in \mathrm{SO}(2)$$

Fundamental invariant

$$r = \iota(x) = \sqrt{x^2 + u^2}$$

Invariantization

$$\iota[\, F(x, u) \,] = F(r, 0)$$

Prolongation

Most interesting group actions (Euclidean, affine, projective, etc.) are *not* free!

An effective action can usually be made free by:

- Prolonging to derivatives (jet space)

$$G^{(n)} :\ \mathrm{J}^n(M,p) \longrightarrow \mathrm{J}^n(M,p)$$

$\Longrightarrow$ differential invariants

- Prolonging to Cartesian product actions

$$G^{\times n} :\ M \times \cdots \times M \longrightarrow M \times \cdots \times M$$

$\Longrightarrow$ joint invariants

- Prolonging to "multi-space"

$$G^{(n)} :\ M^{(n)} \longrightarrow M^{(n)}$$

$\Longrightarrow$ joint or semi-differential invariants
$\Longrightarrow$ invariant numerical approximations

Jet Space

- Although in use since the time of Lie and Darboux, jet space was first formally defined by Ehresmann in 1950.

- Jet space is the proper setting for the geometry of partial differential equations.

M — smooth m-dimensional manifold

$$1 \leq p \leq m - 1$$

$\mathrm{J}^n = \mathrm{J}^n(M, p)$ — (extended) jet bundle

$\Longrightarrow$ Defined as the space of equivalence classes of p-dimensional submanifolds under the equivalence relation of n^{th} order contact at a single point.

$\Longrightarrow$ Can be identified as the space of n^{th} order Taylor polynomials for submanifolds given as graphs $u = f(x)$

Local Coordinates on Jet Space

$\mathrm{J}^n = \mathrm{J}^n(M, p)$ — n^{th} extended jet bundle for
p-dimensional submanifolds $N \subset M$

Local coordinates:

Assume $N = \{ u = f(x) \}$ is a graph (section).

$x = (x^1, \ldots, x^p)$ — independent variables

$u = (u^1, \ldots, u^q)$ — dependent variables

$$p + q = m = \dim M$$

$$z^{(n)} = (x, u^{(n)}) = (\ \ldots \ x^i \ \ldots \ u^\alpha_J \ \ldots \ )$$

$$u^\alpha_J = \partial_J u^\alpha \qquad 0 \le \# J \le n$$

— induced jet coordinates

- No bundle structure assumed on M.
- Projective completion of $\mathrm{J}^n E$ when $E \to X$ is a bundle.

Prolongation of Group Actions

G — transformation group acting on M

$\Longrightarrow$ G maps submanifolds to submanifolds and preserves the order of contact

$G^{(n)}$ — prolonged action of G on the jet space J^n

The prolonged group formulae

$$w^{(n)} = (y, v^{(n)}) = g^{(n)} \cdot z^{(n)}$$

are obtained by implicit differentiation:

$$dy^i = \sum_{j=1}^{p} P^i_j(g, z^{(1)})\, dx^j$$

$$D_{y^j} = \sum_{i=1}^{p} Q^i_j(g, z^{(1)})\, D_{x^i} \qquad \Longrightarrow \qquad Q = P^{-T}$$

$$v^\alpha_J = D_{y^{j_1}} \cdots D_{y^{j_k}}(v^\alpha)$$

Differential invariant $\quad I : \mathrm{J}^n \to \mathbb{R}$

$$I(g^{(n)} \cdot z^{(n)}) = I(z^{(n)})$$

$\Longrightarrow$ curvatures

Freeness

Theorem. If G acts (locally) effectively on M, then G acts (locally) freely on a dense open subset $\mathcal{V}^n \subset \mathrm{J}^n$ for $n \gg 0$.

Definition. $N \subset M$ is *regular* at order n if $\mathrm{j}_n N \subset \mathcal{V}^n$.

Corollary. Any regular submanifold admits a (local) moving frame.

Theorem. A submanifold is totally singular, $\mathrm{j}_n N \subset \mathrm{J}^n \setminus \mathcal{V}^n$ for all n, if and only if its symmetry group

$$G_N = \{ \, g \mid \, g \cdot N \subset N \, \}$$

does not act freely on N.

Moving Frames on Jet Space

$$w^{(n)} = (y, v^{(n)}) = \begin{cases} g^{(n)} \cdot z^{(n)} & \text{right} \\ (g^{(n)})^{-1} \cdot z^{(n)} & \text{left} \end{cases}$$

Choose $r = \dim G$ jet coordinates

$$z_1, \ldots, z_r \qquad\qquad x^i \text{ or } u^\alpha_J$$

Coordinate cross-section $K \subset \mathrm{J}^n$

$$z_1 = c_1 \quad \ldots \quad z_r = c_r$$

Corresponding lifted differential invariants:

$$w_1, \ldots, w_r \qquad\qquad y^i \text{ or } v^\alpha_J$$

Normalization Equations

$$w_1(g, x, u^{(n)}) = c_1 \quad \ldots \quad w_r(g, x, u^{(n)}) = c_r$$

Solution:

$$g = \rho^{(n)}(z^{(n)}) = \rho^{(n)}(x, u^{(n)}) \qquad \Longrightarrow \quad \text{moving frame}$$

The Fundamental Differential Invariants

$$I^{(n)}(z^{(n)}) = w^{(n)}(\rho^{(n)}(z^{(n)}), z^{(n)})$$

$$\boxed{\begin{aligned} &H^i(x, u^{(n)}) = y^i(\rho^{(n)}(x, u^{(n)}), x, u) \\ &I^\alpha_K(x, u^{(k)}) = v^\alpha_K(\rho^{(n)}(x, u^{(n)}), x, u^{(k)}) \end{aligned}}$$

Phantom differential invariants

$$w_1 = c_1 \;\ldots\; w_r = c_r \qquad \Longrightarrow \quad \text{normalizations}$$

Theorem. Every n^{th} order differential invariant can be locally uniquely written as a function of the non-phantom fundamental differential invariants in $I^{(n)}$.

Invariant Differentiation

Contact-invariant coframe

$$dy^i \quad \longmapsto \quad \omega^i = \sum_{j=1}^{p} P_j^i(\rho^{(n)}(z^{(n)}), z^{(n)}) \; dx^i$$

$\Longrightarrow$ arc length element

Invariant differential operators:

$$D_{y^j} \quad \longmapsto \quad \mathcal{D}_j = \sum_{i=1}^{p} Q_j^i(\rho^{(n)}(z^{(n)}), z^{(n)}) \, D_{x^i}$$

$\Longrightarrow$ arc length derivative

Duality:

$$dF = \sum_{i=1}^{p} \mathcal{D}_i F \cdot \omega^i$$

Theorem. The higher order differential invariants are obtained by invariant differentiation with respect to $\mathcal{D}_1, \ldots, \mathcal{D}_p$.

Euclidean Curves $\qquad G = \mathrm{SE}(2)$

Assume the curve is (locally) a graph:

$$\mathcal{C} = \{u = f(x)\}$$

Prolong to J^3 via implicit differentiation

$$\left.\begin{aligned} y &= \cos\theta\,(x-a) + \sin\theta\,(u-b) \\ v &= -\sin\theta\,(x-a) + \cos\theta\,(u-b) \end{aligned}\right\} \qquad w = R^{-1}(z-b)$$

$$v_y = \frac{-\sin\theta + u_x\cos\theta}{\cos\theta + u_x\sin\theta}$$

$$v_{yy} = \frac{u_{xx}}{(\cos\theta + u_x\sin\theta)^3}$$

$$v_{yyy} = \frac{(\cos\theta + u_x\sin\theta)u_{xxx} - 3u_{xx}^2\sin\theta}{(\cos\theta + u_x\sin\theta)^5}$$

$$\vdots$$

Normalization $\qquad r = \dim G = 3$

$$y = 0, \qquad v = 0, \qquad v_y = 0$$

Left moving frame $\qquad \rho\colon \mathrm{J}^1 \longrightarrow \mathrm{SE}(2)$

$$a = x, \qquad b = u, \qquad \theta = \tan^{-1} u_x$$

Differential invariants

$$\begin{aligned} v_{yy} &\longmapsto \kappa &&= \frac{u_{xx}}{(1+u_x^2)^{3/2}} \\ v_{yyy} &\longmapsto \frac{d\kappa}{ds} &&= \frac{(1+u_x^2)u_{xxx} - 3u_x u_{xx}^2}{(1+u_x^2)^3} \\ v_{yyyy} &\longmapsto \frac{d^2\kappa}{ds^2} - 3\kappa^3 &&= \cdots \end{aligned}$$

Invariant one-form — arc length

$$dy = (\cos\theta + u_x \sin\theta)\, dx \quad \longmapsto \quad ds = \sqrt{1+u_x^2}\ dx$$

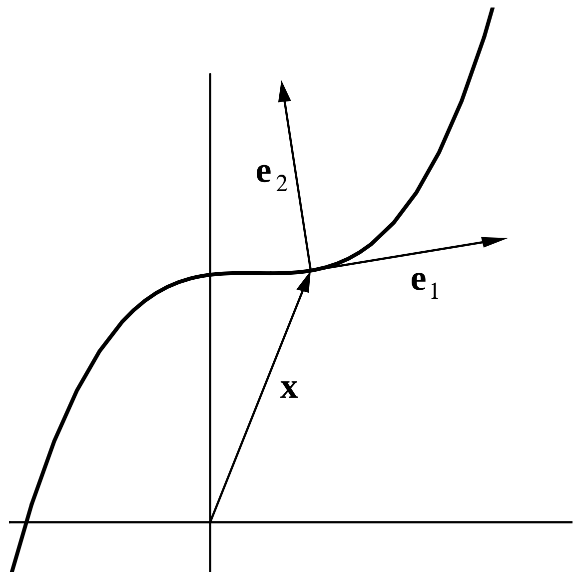
Invariant differential operator

$$\frac{d}{dy} = \frac{1}{\cos\theta + u_x \sin\theta}\ \frac{d}{dx} \quad \longmapsto \quad \frac{d}{ds} = \frac{1}{\sqrt{1+u_x^2}}\ \frac{d}{dx}$$

Theorem. All differential invariants are functions of the derivatives of curvature with respect to arc length:

$$\kappa, \qquad \frac{d\kappa}{ds}, \qquad \frac{d^2\kappa}{ds^2}, \qquad \cdots$$

Euclidean Curves



Moving frame $\qquad \rho : \ (x, u, u_x) \ \longmapsto \ (R, \mathbf{a}) \in \mathrm{SE}(2)$

$$R = \frac{1}{\sqrt{1 + u_x^2}} \begin{pmatrix} 1 & -u_x \\ u_x & 1 \end{pmatrix} = (\mathbf{e}_1, \mathbf{e}_2) \qquad \mathbf{a} = \begin{pmatrix} x \\ u \end{pmatrix}$$

Frenet frame

$$\mathbf{e}_1 = \frac{d\mathbf{x}}{ds} = \begin{pmatrix} x_s \\ y_s \end{pmatrix} \qquad \mathbf{e}_2 = \mathbf{e}_1^{\perp} = \begin{pmatrix} -y_s \\ x_s \end{pmatrix}$$

Frenet equations = Maurer–Cartan equations:

$$\frac{d\mathbf{x}}{ds} = \mathbf{e}_1 \qquad \frac{d\mathbf{e}_1}{ds} = \kappa\, \mathbf{e}_2 \qquad \frac{d\mathbf{e}_2}{ds} = -\,\kappa\, \mathbf{e}_1$$

The Replacement Theorem

Any differential invariant has the form

$$I = F(x, u^{(n)}) = F(y, w^{(n)}) = F(I^{(n)})$$

$\Longrightarrow$ T.Y. Thomas

$$\kappa = \frac{v_{yy}}{(1+v_y^2)^2} = \frac{u_{xx}}{(1+u_x^2)^2}$$

$$\iota(x) = \iota(u) = (u_x) = 0$$

$$\iota(u_{xx}) = \kappa$$

Equi-affine Curves $\qquad G = \mathrm{SA}(2)$

$$z \longmapsto A\,z + b \qquad\qquad A \in \mathrm{SL}(2), \qquad b \in \mathbb{R}^2$$

Prolong to J^3 via implicit differentiation

$$dy = (\delta - u_x \beta)\; dx \qquad\qquad D_y = \frac{1}{\delta - u_x \beta}\, D_x$$

$$\left.\begin{aligned} y &= \delta(x-a) - \beta(u-b) \\ v &= -\,\gamma(x-a) + \alpha(u-b) \end{aligned}\right\} \qquad w = A^{-1}(z-b)$$

$$v_y = -\,\frac{\gamma - \alpha u_x}{\delta - \beta u_x} \qquad\qquad v_{yy} = -\,\frac{u_{xx}}{(\delta - \beta u_x)^3}$$

$$v_{yyy} = -\,\frac{(\delta - \beta u_x) u_{xxx} + 3\beta u_{xx}^2}{(\delta - \beta u_x)^5}$$

$$v_{yyyy} = -\,\frac{u_{xxxx}(\delta - \beta u_x)^2 + 10 u_{xx} u_{xxx} \beta(\delta - \beta u_x) + 15 u_{xx}^3 \beta^2}{(\alpha + \beta u_x)^7}$$

$$\vdots$$

Nondegeneracy $\qquad u_{xx} = 0$

$\Longrightarrow$ Straight lines are totally singular

(three-dimensional equi-affine symmetry group)

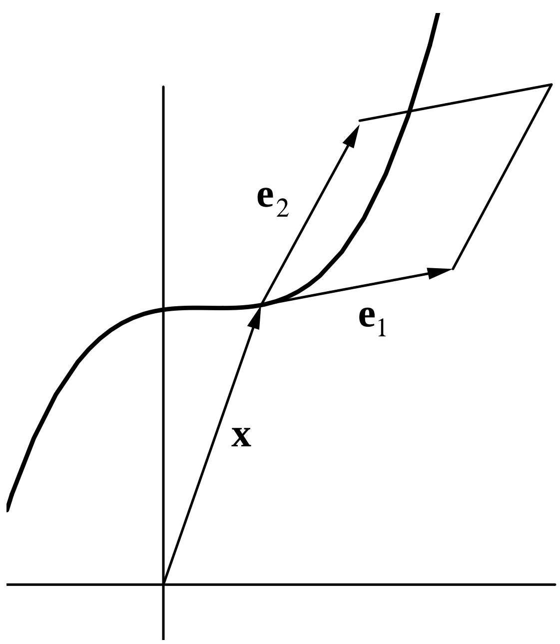
Normalization $\qquad r = \dim G = 5$

$$y = 0, \quad v = 0, \quad v_y = 0, \quad v_{yy} = 1, \quad v_{yyy} = 0.$$

Left Moving frame $\qquad \rho\colon \mathrm{J}^3 \longrightarrow \mathrm{SA}(2)$

$$A = \begin{pmatrix} \sqrt[3]{u_{xx}} & -\frac{1}{3}u_{xx}^{-5/3}u_{xxx} \\ u_x\sqrt[3]{u_{xx}} & u_{xx}^{-1/3} - \frac{1}{3}u_{xx}^{-5/3}u_{xxx} \end{pmatrix} \qquad \mathbf{b} = z = \begin{pmatrix} x \\ u \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{dz}{ds} & \dfrac{d^2z}{ds^2} \end{pmatrix}$$



Frenet frame

$$\mathbf{e}_1 = \frac{dz}{ds} \qquad\qquad \mathbf{e}_2 = \frac{d^2 z}{ds^2}$$

Frenet equations = Maurer–Cartan equations:

$$\frac{dz}{ds} = \mathbf{e}_1 \qquad \frac{d\mathbf{e}_1}{ds} = \mathbf{e}_2 \qquad \frac{d\mathbf{e}_2}{ds} = \kappa\, \mathbf{e}_1$$

Equi-affine arc length

$$dy \quad \longmapsto \quad ds = \sqrt[3]{u_{xx}}\; dx = \sqrt[3]{\dot z \wedge \ddot z}\; dt$$

Invariant differential operator

$$D_y \quad \longmapsto \quad \frac{d}{ds} = \frac{1}{\sqrt[3]{u_{xx}}}\, D_x = \frac{1}{\sqrt[3]{\dot z \wedge \ddot z}}\, D_t$$

Equi-affine curvature

$$v_{4y} \quad \longmapsto \quad \kappa = \frac{5u_{xx}u_{xxxx} - 3u_{xxx}^2}{9u_{xx}^{8/3}} = z_s \wedge z_{ss}$$

$$v_{5y} \quad \longmapsto \quad \frac{d\kappa}{ds} \qquad\qquad v_{6y} \quad \longmapsto \quad \frac{d^2\kappa}{ds^2} - 5\kappa^2$$

Equivalence & Signature

Cartan's main idea: The equivalence and symmetry properties of submanifolds will be found by restricting the differential invariants to the submanifold $J(x) = I(\,\mathrm{j}_n N|_x\,)$.

Equivalent submanifolds should have the same invariants.

However, unless an invariant $J(x)$ is constant, it carries little information by itself, since the equivalence map will typically drastically change the dependence of the invariant on the parameter x.

$\Longrightarrow$ Constant curvature submanifolds

However, a functional dependency or *syzygy* among the invariants *is* intrinsic:

$$J_k(x) = \Phi(J_1(x), \ldots, J_{k-1}(x))$$

The Signature Map

Equivalence and symmetry properties of submanifolds are governed by the functional dependencies — “syzygies” — among the differential invariants.

$$J_k(x) = \Phi(J_1(x), \ldots, J_{k-1}(x))$$

The syzygies are encoded by the *signature map*

$$\Sigma : N \quad \longrightarrow \quad \mathcal{S}$$

of the submanifold N, which is parametrized by the fundamental differential invariants:

$$\Sigma(x) = (J_1(x), \ldots, J_m(x))$$
$$= (\, I_1 \mid N, \; \ldots \; , I_m \mid N \,)$$

The image $\mathcal{S} = \mathrm{Im}\, \Sigma$ is the signature subset (or submanifold) of N.

Geometrically, the signature

$$\mathcal{S} \subset \mathcal{K}$$

is the image of $\mathrm{j}_n N$ in the cross-section $\mathcal{K} \subset \mathrm{J}^n$, where $n \gg 0$ is sufficiently large.

$$\Sigma : N \quad \longrightarrow \quad \mathrm{j}_n N \quad \longrightarrow \quad \mathcal{S} \subset \mathcal{K}$$

Theorem. Two submanifolds are equivalent

$$\overline{N} = g \cdot N$$

if and only if their signatures are identical

$$\mathcal{S} = \overline{\mathcal{S}}$$

Signature Curves

Definition. The *signature curve* $\mathcal{S} \subset \mathbb{R}^2$ of a curve $\mathcal{C} \subset \mathbb{R}^2$ is parametrized by the first two differential invariants κ and κ_s

$$\mathcal{S} = \left\{ \left(\kappa \, , \, \frac{d\kappa}{ds} \right) \right\} \quad \subset \quad \mathbb{R}^2$$

Theorem. Two curves $\mathcal{C}$ and $\overline{\mathcal{C}}$ are equivalent

$$\overline{\mathcal{C}} = g \cdot \mathcal{C}$$

if and only if their signature curves are identical

$$\overline{\mathcal{S}} = \mathcal{S}$$

$\Longrightarrow$ object recognition

Symmetry

Signature map

$$\Sigma \; : \; N \quad \longrightarrow \quad \mathcal{S}$$

Theorem. Let $\mathcal{S}$ denote the signature of the submanifold N. Then the dimension of its symmetry group $G_N = \{ g \,|\, g \cdot N \subset N \}$ equals

$$\dim G_N \;=\; \dim N - \dim \mathcal{S}$$

Corollary. For a regular submanifold $N \subset M$,

$$0 \;\leq\; \dim G_N \;\leq\; \dim N$$

$\Longrightarrow$ Only totally singular submanifolds can have larger symmetry groups!

Maximally Symmetric Submanifolds

Theorem. The following are equivalent:

- The submanifold N has a p-dimensional symmetry group
- The signature $\mathcal{S}$ degenerates to a point

$$\dim \mathcal{S} = 0$$

- The submanifold has all constant differential invariants
- $N = H \cdot \{z_0\}$ is the orbit of a p-dimensional subgroup $H \subset G$

$\Longrightarrow$ In Euclidean geometry, these are the circles, straight lines, spheres & planes.

$\Longrightarrow$ In equi-affine plane geometry, these are the conic sections.

Discrete Symmetries

Definition. The *index* of a submanifold N equals the number of points in N which map to a generic point of its signature $\mathcal{S}$:

$$\iota_N = \min \left\{ \, \# \, \Sigma^{-1}\{w\} \;\middle|\; w \in \mathcal{S} \, \right\}$$

$\Longrightarrow$ Self–intersections

Theorem. The cardinality of the symmetry group of N equals its index ι_N.

$\Longrightarrow$ Approximate symmetries

Transformation Groups and Jets

$(x^1, \ldots, x^p)$ — independent variables

$(u^1, \ldots, u^q)$ — dependent variables

$z^{(n)} = (x, u^{(n)}) \in \mathrm{J}^n$ — n^{th} order jet space

u^α_J — derivative coordinates on J^n

G — transformation group

$G^{(n)}$ — prolonged action on J^n

$\mathbf{v} \in \mathfrak{g}$ — Lie algebra

$\mathbf{v}^{(n)} \in \mathfrak{g}^{(n)}$ — Prolonged inf. gens.

The Prolongation Formula

$$\mathbf{v}^{(n)} = \sum_{i=1}^{p} \xi^i(x,u)\,\frac{\partial}{\partial x^i} + \sum_{\alpha,J}^{n} \varphi^\alpha_J(x,u^{(j)})\,\frac{\partial}{\partial u^\alpha_J}$$

$$\varphi^\alpha_J = D_J Q^\alpha + \sum_{i=1}^{p} \xi^i\, u^\alpha_{J,i}$$

Characteristic

$$Q^\alpha(x,u^{(1)}) = \varphi^\alpha - \sum_{i=1}^{p} \xi^i\,\frac{\partial u^\alpha}{\partial x^i}$$

Rotation group — SO(2)

$$(x,u) \longmapsto (x\cos\theta - u\sin\theta,\ x\sin\theta + u\cos\theta)$$

Transformed function $\quad v = \bar f(y)$:

$$y = x\cos\theta - f(x)\sin\theta,$$

$$v = x\sin\theta + f(x)\cos\theta,$$

Second prolongation

$$(x,u,u_x,u_{xx}) \longmapsto \left(x\cos\theta - u\sin\theta, x\sin\theta + u\cos\theta,\right.$$
$$\left.\frac{\sin\theta + u_x\cos\theta}{\cos\theta - u_x\sin\theta},\ \frac{u_{xx}}{(\cos\theta - u_x\sin\theta)^3}\right)$$

Infinitesimal generator

$$\mathbf{v} = -u\,\frac{\partial}{\partial x} + x\,\frac{\partial}{\partial u}$$

Second prolongation

$$\mathbf{v}^{(2)} = -u\,\frac{\partial}{\partial x} + x\,\frac{\partial}{\partial u} + (1+u_x^2)\,\frac{\partial}{\partial u_x} + 3u_xu_{xx}\,\frac{\partial}{\partial u_{xx}}$$

$$Q = x + uu_x$$

$$\varphi^x = D_xQ + \xi u_{xx} = D_x(x+uu_x) - uu_{xx} = 1 + u_x^2$$

$$\varphi^{xx} = D_x^2Q + \xi u_{xxx} = D_x^2(x+uu_x) - uu_{xxx} = 3u_xu_{xx}$$

Differential invariant:

$$I(g^{(n)} \cdot (x, u^{(n)})) = I(x, u^{(n)})$$

Infinitesimal criterion:

$$\mathbf{v}^{(n)}(I) = 0 \qquad \text{for all} \qquad \mathbf{v}^{(n)} \in \mathfrak{g}^{(n)}$$

$\Longrightarrow$ Solve the first order linear partial differential equation by the method of characteristics.

$\Longrightarrow$ Moving frames avoids integration!

Note: If $I_1, \ldots, I_k$ are differential invariants, so is $\Phi(I_1, \ldots, I_k)$.

$\Longrightarrow$ Classify differential invariants up to functional independence.

$\Longrightarrow$ Local results on open subsets of jet space.

Theorem. Any transformation group admits a finite system of fundamental differential invariants

$$J_1, \ldots, J_\ell$$

and p invariant differential operators

$$\mathcal{D}_1, \ldots, \mathcal{D}_p$$

such that every differential invariant is a function of the differentiated invariants:

$$I = \Phi(\ \ldots \ \mathcal{D}_K J_\nu \ \ldots \ )$$

Classification Problem.

How many fundamental differential invariants $J_1, \ldots, J_\ell$ are required?

$\Longrightarrow$ For curves ($p = 1$), we have $\ell = q$.

Syzygy Problem.

Determine the algebraic relations

$$\Phi(\ \ldots \ \mathcal{D}_K J_\nu \ \ldots \ ) = 0$$

among the differentiated invariants.

Commutation Formulae.

The order of invariant differentiation matters

$$[\, \mathcal{D}_i \, , \mathcal{D}_j \,] \ = \ ???$$

$\Longrightarrow$ Only an issue when $p > 1$.

The Fundamental Differential Invariants

$$I^{(n)}(z^{(n)}) = \rho^{(n)}(z^{(n)})^{-1} \cdot z^{(n)}$$

$$\boxed{\begin{aligned} H^i(x,u^{(n)}) &= y^i(\rho^{(n)}(x,u^{(n)}),x,u) \\ I^\alpha_K(x,u^{(k)}) &= v^\alpha_K(\rho^{(n)}(x,u^{(n)}),x,u^{(k)}) \end{aligned}}$$

Recurrence Formulae:

$$\boxed{\begin{aligned} \mathcal{D}_j H^i &= \delta^i_j + M^i_j \\ \mathcal{D}_j I^\alpha_K &= I^\alpha_{K,j} + M^\alpha_{K,j} \end{aligned}}$$

$$M^i_j, M^\alpha_{K,j} \qquad \text{—} \qquad \text{correction terms}$$

Commutation Formulae:

$$\boxed{[\mathcal{D}_i, \mathcal{D}_j] = \sum_{i=1}^{p} A^k_{ij}\, \mathcal{D}_k}$$

- The correction terms can be computed directly from the infinitesimal generators!

Generating Invariants

Theorem. A generating system of differential invariants consists of

- all non-phantom differential invariants H^i and I^α coming from the un-normalized zero$^{\text{th}}$ order lifted invariants y^i, v^α, and
- all non-phantom differential invariants of the form $I^\alpha_{J,i}$ where I^α_J is a phantom differential invariant.

$$\text{order} \;\leq\; \text{order}\,\rho + 1$$

In other words, every other differential invariant can, locally, be written as a function of the generating invariants and their invariant derivatives, $\mathcal{D}_K H^i$, $\mathcal{D}_K I^\alpha_{J,i}$.

$\Longrightarrow$ Not necessarily a minimal set!

Syzygies

A syzygy is a functional relation among differentiated invariants:

$$H(\ \ldots\ \mathcal{D}_J I_\nu\ \ldots\ )\ \equiv\ 0$$

Derivatives of syzygies are syzygies
$\Longrightarrow$ find a minimal basis

Remark: There are no syzygies among the normalized differential invariants $I^{(n)}$ except for the "phantom syzygies"

$$I_\nu = c_\nu$$

corresponding to the normalizations.

Classification of Syzygies

Theorem. All syzygies among the differentiated invariants are differential consequences of the following three fundamental types:

$$\boxed{\mathcal{D}_j H^i = \delta^i_j + M^i_j}$$

— H^i non-phantom

$$\boxed{\mathcal{D}_J I^\alpha_K = c_\nu + M^\alpha_{K,J}}$$

— I^α_K generating

— $I^\alpha_{J,K} = w_\nu = c_\nu$ phantom

$$\boxed{\mathcal{D}_J I^\alpha_{LK} - \mathcal{D}_K I^\alpha_{LJ} = M^\alpha_{LK,J} - M^\alpha_{LJ,K}}$$

— I^α_{LK}, I^α_{LJ} generating, $K \cap J = \varnothing$

$\Longrightarrow$ Not necessarily a minimal system!

Invariant Variational Problems

$$\mathcal{I}[u] = \int L(x, u^{(n)})\, d\mathbf{x} = \int P(\ \ldots \ \mathcal{D}_K I^\alpha \ \ \ldots \ ) \ \boldsymbol{\omega}$$

$I_1, \ldots, I_\ell$ — fundamental differential invariants

$\mathcal{D}_K I^\alpha$ — differentiated invariants

$\boldsymbol{\omega} = \omega^1 \wedge \cdots \wedge \omega^p$ — contact-invariant volume form

Invariant Euler-Lagrange equations

$$\mathbf{E}(L) = F(\ \ldots \ \mathcal{D}_K I^\alpha \ \ \ldots \ ) = 0$$

Problem.

Construct F directly from P.

$\Longrightarrow$ P. Griffiths, I. Anderson

Example. Planar Euclidean group $G = \mathrm{SE}(2)$

Invariant variational problem

$$\int P(\kappa, \kappa_s, \kappa_{ss}, \ \ldots\ )\, ds$$

Euler-Lagrange equations

$$\mathbf{E}(L) = F(\kappa, \kappa_s, \kappa_{ss}, \ \ldots\ ) = 0$$

The Elastica (Euler):

$$\mathcal{I}[u] = \int \tfrac{1}{2}\,\kappa^2\, ds = \int \frac{u_{xx}^2\, dx}{(1 + u_x^2)^{5/2}}$$

Euler-Lagrange equation

$$\mathbf{E}(L) = \kappa_{ss} + \tfrac{1}{2}\,\kappa^3 = 0$$

$\Longrightarrow$ elliptic functions

$$\int P(\kappa, \kappa_s, \kappa_{ss}, \ \ldots\ ) \, ds$$

Invariantized Euler operator

$$\mathcal{E} = \sum_{n=0}^{\infty} (-\mathcal{D})^n \frac{\partial}{\partial \kappa_n} \qquad \mathcal{D} = \frac{d}{ds}$$

Invariantized Hamiltonian operator

$$\mathcal{H}(P) = \sum_{i>j} \kappa_{i-j} \, (-\mathcal{D})^j \frac{\partial P}{\partial \kappa_i} - P$$

Invariant Euler-Lagrange formula

$$\mathbf{E}(L) = (\mathcal{D}^2 + \kappa^2) \, \mathcal{E}(P) + \kappa \, \mathcal{H}(P).$$

Elastica

$$P = \tfrac{1}{2}\kappa^2 \qquad \mathcal{E}(P) = \kappa \qquad \mathcal{H}(P) = -P = -\tfrac{1}{2}\kappa^2$$

$$\mathbf{E}(L) = \kappa_{ss} + \tfrac{1}{2}\kappa^3 = 0$$

Euler-Lagrange Equations

Integration by Parts:

$$\pi : \Omega^{p,1} \longrightarrow \mathcal{F}^1 = \Omega^{p,1} \,/\, d_H\, \Omega^{p-1,1}$$

$$\Longrightarrow \quad \text{Source forms}$$

Variational derivative or Euler operator:

$$\delta = \pi \circ d_V : \Omega^{p,0} \quad \longrightarrow \quad \mathcal{F}^1$$

Variational Problems $\quad \longrightarrow \quad$ Source Forms

$$\delta : \lambda = L\, d\mathbf{x} \quad \longrightarrow \quad \sum_{\alpha=1}^{q} \mathbf{E}_\alpha(L)\ \theta^\alpha \wedge\ d\mathbf{x}$$

Hamiltonian

$$\mathbf{H}(L) = \sum_{\alpha=1}^{m} \sum_{i>j\geq 0} u^\alpha_{i-j} (-D_x)^j \frac{\partial L}{\partial u^\alpha_i} - L$$

The Simplest Example. $M = \mathbb{R}^2 \quad x, u \in \mathbb{R}$

Lagrangian form

$$\lambda = L(x, u^{(n)})\, dx$$

Vertical derivative

$$\begin{aligned} d\lambda &= d_V\, \lambda \\ &= \left(\frac{\partial L}{\partial u}\, \theta + \frac{\partial L}{\partial u_x}\, \theta_x + \frac{\partial L}{\partial u_{xx}}\, \theta_{xx} + \cdots \right) \wedge\, dx \in \Omega^{1,1} \end{aligned}$$

Integration by parts

$$\begin{aligned} d_H\,(A\,\theta) &= (D_x A)\, dx \wedge \theta - A\,\theta_x \wedge dx \\ &= -[\,(D_x A)\,\theta + A\,\theta_x\,] \wedge dx \end{aligned}$$

Variational derivative

$$\begin{aligned} \delta\lambda &= \left(\frac{\partial L}{\partial u} - D_x\, \frac{\partial L}{\partial u_x} + D_x^2\, \frac{\partial L}{\partial u_{xx}} - \cdots \right) \theta \wedge\, dx \\ &= \mathbf{E}(L)\,\theta \wedge\, dx \in \mathcal{F}^1 \end{aligned}$$

Plane Curves

Invariant Lagrangian

$$\int P(\kappa, \kappa_s, \ldots)\, \varpi$$

κ — fundamental differential invariant (curvature)

$\varpi = \omega + \eta$ — fully invariant horizontal form

$\omega = ds$ — contact-invariant arc length

Invariant integration by parts

$$d_{\mathcal{V}}\,(P\,\varpi) = \mathcal{E}(P)\ d_{\mathcal{V}}\,\kappa \wedge \varpi - \mathcal{H}(P)\ d_{\mathcal{V}}\,\varpi$$

Vertical differentiation formulae

$d_{\mathcal{V}}\,\kappa = \mathcal{A}(\vartheta)$ $\qquad$ $\mathcal{A}$ — Eulerian operator

$d_{\mathcal{V}}\,\varpi = \mathcal{B}(\vartheta) \wedge \varpi$ $\qquad$ $\mathcal{B}$ — Hamiltonian operator

$\Longrightarrow$ The explicit formulae follow from our fundamental recurrence formula, based on the infinitesimal generators of the action.

Invariant Euler-Lagrange equation

$$\boxed{\mathcal{A}^*\mathcal{E}(P) - \mathcal{B}^*\mathcal{H}(P) = 0}$$

General Framework

Fundamental differential invariants

$$I^1, \ldots, I^\ell$$

Invariant horizontal coframe

$$\varpi^1, \ldots, \varpi^p$$

Dual invariant differential operators

$$\mathcal{D}_1, \ldots, \mathcal{D}_p$$

Invariant volume form

$$\boldsymbol{\varpi} = \varpi^1 \wedge \cdots \wedge \varpi^p$$

Differentiated invariants

$$I^\alpha_{,K} = \mathcal{D}^K J^\alpha = \mathcal{D}_{k_1} \cdots \mathcal{D}_{k_n} J^\alpha$$

$\Longrightarrow$ order is important!

Eulerian operator

$$d_{\mathcal{V}}\, I^\alpha = \sum_{\beta=1}^{q} \mathcal{A}^\alpha_\beta(\vartheta^\beta) \qquad \mathcal{A} = (\, \mathcal{A}^\alpha_\beta \,)$$

$\Longrightarrow$ $m \times q$ matrix of invariant differential operators

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Hamiltonian operator complex

$$d_{\mathcal{V}}\, \varpi^j = \sum_{\beta=1}^{q} \mathcal{B}^j_{i,\beta}(\vartheta^\beta) \wedge \varpi^i \qquad \mathcal{B}^j_i = (\, \mathcal{B}^j_{i,\beta} \,)$$

$\Longrightarrow$ p^2 row vectors of invariant differential operators

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$$\boldsymbol{\varpi}_{(i)} = (-1)^{i-1}\, \varpi^1 \wedge \cdots \wedge \varpi^{i-1} \wedge \varpi^{i+1} \wedge \cdots \wedge \varpi^p$$

Twist invariants

$$d_{\mathcal{H}}\, \boldsymbol{\varpi}_{(i)} = Z_i\, \boldsymbol{\varpi}$$

Twisted adjoint

$$\mathcal{D}_i^\dagger = -(\mathcal{D}_i + Z_i)$$

Invariant variational problem

$$\int P(I^{(n)})\,\boldsymbol{\varpi}$$

Invariant Eulerian

$$\mathcal{E}_\alpha(P) = \sum_K \mathcal{D}_K^\dagger \frac{\partial P}{\partial I^\alpha_{,K}}$$

Invariant Hamiltonian tensor

$$\mathcal{H}^i_j(P) = -\,P\,\delta^i_j + \sum_{\alpha=1}^{q} \sum_{J,K} I^\alpha_{,J,j}\,\mathcal{D}_K^\dagger \frac{\partial P}{\partial I^\alpha_{,J,i,K}}\,,$$

Invariant Euler-Lagrange equations

$$\mathcal{A}^\dagger\,\mathcal{E}(P) - \sum_{i,j=1}^{p} (\mathcal{B}^j_i)^\dagger\,\mathcal{H}^i_j(P) = 0.$$

Euclidean Surfaces

$S \subset M = \mathbb{R}^3$ coordinates $z = (x, y, u)$

Group: $G = \mathrm{E}(3)$

$$z \longmapsto R\,z + a, \qquad R \in \mathrm{O}(3)$$

Normalization — coordinate cross-section

$$x = y = u = u_x = u_y = u_{xy} = 0.$$

Left moving frame

$$a = z \qquad R = (\ \mathbf{t}_1\ \mathbf{t}_2\ \mathbf{n}\ )$$

- $\mathbf{t}_1, \mathbf{t}_2 \in TS$ — Frenet frame
- $\mathbf{n}$ — unit normal

Fundamental differential invariants

$$\kappa^1 = \iota(u_{xx}) \qquad \kappa^2 = \iota(u_{yy})$$

$\Longrightarrow$ principal curvatures

Frenet coframe

$$\varpi^1 = \iota(dx^1) = \omega^1 + \eta^1 \qquad \varpi^2 = \iota(dx^2) = \omega^2 + \eta^2$$

Invariant differential operators

$$\mathcal{D}_1 \qquad \mathcal{D}_2$$

$\Longrightarrow$ Frenet differentiation

Fundamental Syzygy:

Use the recurrence formula to compare

$$\iota(u_{xxyy}) \qquad \text{with} \qquad \begin{aligned} \kappa^1_{,22} &= \mathcal{D}_2^2 \iota(u_{xx}) \\ \kappa^2_{,11} &= \mathcal{D}_1^2 \iota(u_{yy}) \end{aligned}$$

$$\kappa^1_{,22} - \kappa^2_{,11} + \frac{\kappa^1_{,1}\kappa^2_{,1} + \kappa^1_{,2}\kappa^2_{,2} - 2(\kappa^2_{,1})^2 - 2(\kappa^1_{,2})^2}{\kappa^1 - \kappa^2} - \kappa^1\kappa^2(\kappa^1 - \kappa^2) = 0$$

$\Longrightarrow$ Codazzi equations

Twisted adjoints

$$\mathcal{D}_1^\dagger = -\,(\mathcal{D}_1 + Z_1) \qquad Z_1 = \frac{\kappa^2_{,1}}{\kappa^1 - \kappa^2}$$

$$\mathcal{D}_2^\dagger = -(\mathcal{D}_2 + Z_2) \qquad Z_2 = \frac{\kappa^1_{,2}}{\kappa^2 - \kappa^1}$$

Gauss curvature — Codazzi equations:

$$K = \kappa^1\kappa^2 = \mathcal{D}_1^\dagger(Z_1) + \mathcal{D}_2^\dagger(Z_2)$$
$$= -\,(\mathcal{D}_1 + Z_1)Z_1 - (\mathcal{D}_2 + Z_2)Z_2$$

K is an invariant divergence

$\Longrightarrow$ Gauss–Bonnet Theorem!

Invariant contact form

$$\vartheta = \iota(\theta) = \iota(du - u_x\,dx - u_y\,dy)$$

Invariant vertical derivatives

$$d_{\mathcal{V}}\,\kappa^1 = \iota(\theta_{xx}) = (\,\mathcal{D}_1^2 + Z_2\,\mathcal{D}_2 + (\kappa^1)^2\,)\,\vartheta$$

$$d_{\mathcal{V}}\,\kappa^2 = \iota(\theta_{yy}) = (\,\mathcal{D}_2^2 + Z_1\,\mathcal{D}_1 + (\kappa^2)^2\,)\,\vartheta$$

Eulerian operator

$$\mathcal{A} = \begin{pmatrix} \mathcal{D}_1^2 + Z_2\,\mathcal{D}_2 + (\kappa^1)^2 \\ \mathcal{D}_2^2 + Z_1\,\mathcal{D}_1 + (\kappa^2)^2 \end{pmatrix}$$

$$d_{\mathcal{V}}\,\varpi^1 = -\,\kappa^1\,\vartheta\wedge\varpi^1 + \frac{1}{\kappa^1-\kappa^2}(\,\mathcal{D}_1\mathcal{D}_2 - Z_2\mathcal{D}_1\,)\vartheta\wedge\varpi^2,$$

$$d_{\mathcal{V}}\,\varpi^2 = \frac{1}{\kappa^2-\kappa^1}(\,\mathcal{D}_2\mathcal{D}_1 - Z_1\mathcal{D}_2\,)\vartheta\wedge\varpi^1 - \kappa^2\,\vartheta\wedge\varpi^2,$$

Hamiltonian operator complex

$$\begin{aligned}\mathcal{B}_1^1 &= -\kappa^1, \\ \mathcal{B}_2^2 &= -\kappa^2,\end{aligned} \qquad \mathcal{B}_2^1 = \frac{1}{\kappa^1-\kappa^2}(\,\mathcal{D}_1\mathcal{D}_2 - Z_2\mathcal{D}_1\,) = -\,\mathcal{B}_1^2$$

Euclidean-invariant variational problem

$$\int P(\kappa^{(n)})\,\omega^1 \wedge \omega^2 = \int P(\kappa^{(n)})\; dA$$

Euler-Lagrange equations

$$\mathbf{E}(L) = \mathcal{A}^\dagger\,\mathcal{E}(P) - \mathcal{B}^\dagger\,\mathcal{H}(P) = 0,$$

Special case: $P(\kappa^1, \kappa^2)$

$$\mathbf{E}(L) = [\,(\mathcal{D}_1^\dagger)^2 + \mathcal{D}_2^\dagger \cdot Z_2 + (\kappa^1)^2\,]\,\frac{\partial \widetilde{L}}{\partial \kappa^1} +$$

$$+ [\,(\mathcal{D}_2^\dagger)^2 + \mathcal{D}_1^\dagger \cdot Z_1 + (\kappa^2)^2\,]\,\frac{\partial \widetilde{L}}{\partial \kappa^2} - (\kappa^1 + \kappa^2)\widetilde{L}.$$

Minimal surfaces: $P = 1$

$$-(\kappa^1 + \kappa^2) = -2H = 0$$

Minimizing mean curvature: $P = H = \frac{1}{2}(\kappa^1 + \kappa^2)$

$$\tfrac{1}{2}\left[\,(\kappa^1)^2 + (\kappa^2)^2 - (\kappa^1 + \kappa^2)^2\,\right] = -\,\kappa^1\,\kappa^2 = -K = 0.$$

Willmore surfaces: $P = \frac{1}{2}(\kappa^1)^2 + \frac{1}{2}(\kappa^2)^2$

$$\Delta(\kappa^1 + \kappa^2) + \tfrac{1}{2}(\kappa^1 + \kappa^2)(\kappa^1 - \kappa^2)^2 = 2\,\Delta H + 4(H^2 - K)H = 0$$

Laplace–Beltrami operator

$$\Delta = (\mathcal{D}_1 + Z_1)\mathcal{D}_1 + (\mathcal{D}_2 + Z_2)\mathcal{D}_2 = -\,\mathcal{D}_1^\dagger \cdot \mathcal{D}_1 - \mathcal{D}_2^\dagger \cdot \mathcal{D}_2$$