Applications of Moving Frames

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Differential Invariants
Variational Problems
Geometric Flows

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Basic Notation

\[ x = (x^1, \ldots, x^p) \text{ — independent variables} \]

\[ u = (u^1, \ldots, u^q) \text{ — dependent variables} \]

\[ u^\alpha_j = \partial_j u^\alpha \text{ — partial derivatives} \]

\[ F(x, u^{(n)}) = F(\ldots x^k \ldots u^\alpha_j \ldots) \text{ — differential function} \]

\[ G \text{ — transformation group acting on the space of independent and dependent variables} \]
Variational Problems

\[ \mathcal{I}[u] = \int L(x, u^{(n)}) \, dx \quad \text{— variational problem} \]

\[ L(x, u^{(n)}) \quad \text{— Lagrangian} \]

Variational derivative — Euler-Lagrange equations: \[ \mathbf{E}(L) = 0 \]

components: \[ \mathbf{E}_\alpha(L) = \sum_J (-D)^J \frac{\partial L}{\partial u^\alpha_J} \]

\[ D_kF = \frac{\partial F}{\partial x^k} + \sum_{\alpha, J} u^\alpha_{J,k} \frac{\partial F}{\partial u^\alpha_J} \quad \text{— total derivative of } F \text{ with respect to } x^k \]
Invariant Variational Problems

According to Lie, any $G$–invariant variational problem can be written in terms of the differential invariants:

\[
\mathcal{I}[u] = \int L(x, u^{(n)}) \, dx = \int P(\ldots \mathcal{D}_K I^\alpha \ldots) \, \omega
\]

$I^1, \ldots, I^\ell$ — fundamental differential invariants

$\mathcal{D}_1, \ldots, \mathcal{D}_p$ — invariant differential operators

$\mathcal{D}_K I^\alpha$ — differentiated invariants

$\omega = \omega^1 \wedge \cdots \wedge \omega^p$ — invariant volume form
If the variational problem is $G$-invariant, so
\[ \mathcal{I}[u] = \int L(x, u^{(n)}) \, dx = \int P(\ldots D_K I^\alpha \ldots) \, \omega \]
then its Euler–Lagrange equations admit $G$ as a symmetry group, and hence can also be expressed in terms of the differential invariants:
\[ \mathbf{E}(L) \simeq F(\ldots D_K I^\alpha \ldots) = 0 \]

Main Problem:

Construct $F$ directly from $P$.

(P. Griffiths, I. Anderson)
Planar Euclidean group \[ G = \text{SE}(2) \]

\[ \kappa = \frac{u_{xx}}{(1 + u_x^2)^{3/2}} \quad \text{— curvature (differential invariant)} \]

\[ ds = \sqrt{1 + u_x^2} \, dx \quad \text{— arc length} \]

\[ D = \frac{d}{ds} = \frac{1}{\sqrt{1 + u_x^2}} \frac{d}{dx} \quad \text{— arc length derivative} \]

Euclidean–invariant variational problem

\[ \mathcal{I}[u] = \int L(x, u^{(n)}) \, dx = \int P(\kappa, \kappa_s, \kappa_{ss}, \ldots) \, ds \]

Euler-Lagrange equations

\[ \mathbf{E}(L) \approx F(\kappa, \kappa_s, \kappa_{ss}, \ldots) = 0 \]
Euclidean Curve Examples

Minimal curves (geodesics):

\[ I[u] = \int ds = \int \sqrt{1 + u_x^2} \, dx \]

\[ E(L) = -\kappa = 0 \quad \Rightarrow \text{straight lines} \]

The Elastica (Euler):

\[ I[u] = \int \frac{1}{2} \kappa^2 \, ds = \int \frac{u_{xx}^2 \, dx}{(1 + u_x^2)^{5/2}} \]

\[ E(L) = \kappa_{ss} + \frac{1}{2} \kappa^3 = 0 \quad \Rightarrow \text{elliptic functions} \]
General Euclidean–invariant variational problem

\[ \mathcal{I}[u] = \int L(x, u^{(n)}) \, dx = \int P(\kappa, \kappa_s, \kappa_{ss}, \ldots) \, ds \]
General Euclidean–invariant variational problem

\[ \mathcal{I}[u] = \int L(x, u^{(n)}) \, dx = \int P(\kappa, \kappa_s, \kappa_{ss}, \ldots) \, ds \]

Invariantized Euler–Lagrange expression

\[ \mathcal{E}(P) = \sum_{n=0}^{\infty} (-D)^n \frac{\partial P}{\partial \kappa_n} \quad \text{with} \quad D = \frac{d}{ds} \]
General Euclidean–invariant variational problem

\[ \mathcal{I}[u] = \int L(x, u^{(n)}) \, dx = \int P(\kappa, \kappa_s, \kappa_{ss}, \ldots) \, ds \]

Invariantized Euler–Lagrange expression

\[ \mathcal{E}(P) = \sum_{n=0}^{\infty} (-D)^n \frac{\partial P}{\partial \kappa_n} \quad \mathcal{D} = \frac{d}{ds} \]

Invariantized Hamiltonian

\[ \mathcal{H}(P) = \sum_{i>j} \kappa_{i-j} (-D)^j \frac{\partial P}{\partial \kappa_i} - P \]
\[ \mathcal{I}[u] = \int L(x, u^{(n)}) \, dx = \int P(\kappa, \kappa_s, \kappa_{ss}, \ldots) \, ds \]

Euclidean–invariant Euler-Lagrange formula

\[ \mathbf{E}(L) = (\mathcal{D}^2 + \kappa^2) \mathcal{E}(P) + \kappa \mathcal{H}(P) = 0 \]
\[ \mathcal{I}[u] = \int L(x, u^{(n)}) \, dx = \int P(\kappa, \kappa_s, \kappa_{ss}, \ldots) \, ds \]

Euclidean–invariant Euler-Lagrange formula

\[ \mathbf{E}(L) = (\mathcal{D}^2 + \kappa^2) \mathcal{E}(P) + \kappa \mathcal{H}(P) = 0 \]

The Elastica: \[ \mathcal{I}[u] = \int \frac{1}{2} \kappa^2 \, ds \quad P = \frac{1}{2} \kappa^2 \]

\[ \mathcal{E}(P) = \kappa \quad \mathcal{H}(P) = -P = -\frac{1}{2} \kappa^2 \]

\[ \mathbf{E}(L) = (\mathcal{D}^2 + \kappa^2) \kappa + \kappa \left( -\frac{1}{2} \kappa^2 \right) \]

\[ = \kappa_{ss} + \frac{1}{2} \kappa^3 = 0 \]
The shape of a Möbius strip

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The Möbius strip, obtained by taking a rectangular strip of plastic or paper, twisting one end through 180°, and then joining the ends, is the canonical example of a one-sided surface. Finding its characteristic developable shape has been an open problem ever since its first formulation in refs 1, 2. Here we use the invariant variational bicomplex formalism to derive the first equilibrium equations for a wide developable strip undergoing large deformations, thereby giving the first nontrivial demonstration of the potential of this approach. We then formulate the boundary-value problem for the Möbius strip and solve it numerically. Solutions for increasing width show the formation of creases bounding nearly flat triangular regions, a feature also familiar from fabric draping and paper crumpling. This could give new insight into energy localization phenomena in unstretchable sheets, which might help to predict points of onset of tearing. It could also aid our understanding of the relationship between geometry and physical properties of nanometre- and microscale Möbius strip structures.

It is fair to say that the Möbius strip is one of the few icons of mathematics that have been absorbed into wider culture. It has mathematical beauty and inspired artists such as Escher. In engineering, pulley belts are often used in the form of Möbius strips to wear both sides equally. At a much smaller scale, Möbius strips have nearly been formed in ribbon-shaped NbN, crystals under certain growth conditions involve a low-temperature transition. 

Figure 1 Photo of a paper Möbius strip of aspect ratio 2 to. The strip adopts a characteristic shape. Irregularity in the material causes the surface to be developable. Its straight projections are sheared, and the colouring varies according to the bending energy density.
Figure 2: Computed Moiré strips. The left panel shows their three-dimensional shapes for \( \alpha = 0.1 \), \( 0.2 \), \( 0.5 \), \( 0.8 \), \( 1.0 \) \& \( 1.3 \), \( f \), and the right panel the corresponding developments on the plane. The colouring changes according to the local bending energy density, from white for regions of low bending to red for regions of high bending (loci are individually adjusted). Solutions may be compared with the paper model in Fig. 1 on which the generator field and density coloring have been printed.
Applications of Moving Frames

- Differential geometry
- Equivalence
- Symmetry
- Differential invariants
- Rigidity
- Joint invariants and semi-differential invariants
- Integral invariants
- Symmetries of differential equations
- Factorization of differential operators
- Invariant differential forms and tensors
• Identities and syzygies
• Classical invariant theory
• Computer vision
  ○ object recognition
  ○ symmetry detection
  ○ structure from motion
• Invariant variational problems
• Invariant numerical methods
• Poisson geometry & solitons
• Killing tensors in relativity
• Invariants of Lie algebras in quantum mechanics
• Lie pseudo-groups
Moving Frames

$G$ — $r$-dimensional Lie group acting on $M$

$J^n = J^n(M, p)$ — $n^{\text{th}}$ order jet bundle for $p$-dimensional submanifolds $N = \{ u = f(x) \} \subset M$

$z^{(n)} = (x, u^{(n)}) = (\ldots x^i \ldots u^\alpha_j \ldots)$ — coordinates on $J^n$

$G$ acts on $J^n$ by prolongation (chain rule)
Definition.
An $n^{\text{th}}$ order moving frame is a $G$-equivariant map

$$\rho = \rho^{(n)} : V \subset J^n \rightarrow G$$

Equivariance:

$$\rho(g^{(n)} \cdot z^{(n)}) = \begin{cases} 
g \cdot \rho(z^{(n)}) & \text{left moving frame} \\
\rho(z^{(n)}) \cdot g^{-1} & \text{right moving frame} \end{cases}$$

Note: $\rho_{\text{left}}(z^{(n)}) = \rho_{\text{right}}(z^{(n)})^{-1}$
**Theorem.** A moving frame exists in a neighborhood of a point \( z^{(n)} \in J^n \) if and only if \( G \) acts *freely* and *regularly* near \( z^{(n)} \).

- **free** — the only group element \( g \in G \) which fixes *one* point \( z \in M \) is the identity: \( g \cdot z = z \) if and only if \( g = e \).

- **locally free** — the orbits have the same dimension as \( G \).

- **regular** — all orbits have the same dimension and intersect sufficiently small coordinate charts only once
  \( \nRightarrow \) irrational flow on the torus)
Geometric Construction

\[ \mathcal{O}_{\dot{z}(n)} \]

Normalization = choice of cross-section to the group orbits
Geometric Construction

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The Normalization Construction

1. Write out the explicit formulas for the prolonged group action:

\[ w^{(n)}(g, z^{(n)}) = g^{(n)} \cdot z^{(n)} \]

\[ \implies \text{Implicit differentiation} \]

2. From the components of \( w^{(n)} \), choose \( r = \dim G \) normalization equations:

\[ w_1(g, z^{(n)}) = c_1 \quad \ldots \quad w_r(g, z^{(n)}) = c_r \]
3. Solve the normalization equations for the group parameters $g = (g_1, \ldots, g_r)$:

$$g = \rho(z^{(n)}) = \rho(x, u^{(n)})$$

The solution is the right moving frame.

4. Invariantization: Substitute the moving frame formulas

$$g = \rho(z^{(n)}) = \rho(x, u^{(n)})$$

for the group parameters into the un-normalized components of $w^{(n)}$ to produce a complete system of functionally independent differential invariants:

$$I^{(n)}(x, u^{(n)}) = \iota(z^{(n)}) = w^{(n)}(\rho(z^{(n)}), z^{(n)})$$
Euclidean plane curves \( G = \text{SE}(2) \)

Assume the curve is (locally) a graph:
\[
\mathcal{C} = \{ u = f(x) \}
\]

Write out the group transformations
\[
\begin{align*}
y &= x \cos \phi - u \sin \phi + a \\
v &= x \cos \phi + u \sin \phi + b \\
w &= R z + c
\end{align*}
\]
Prolong to $J^n$ via implicit differentiation

\[ y = x \cos \phi - u \sin \phi + a \]
\[ v = x \cos \phi + u \sin \phi + b \]
\[ v_y = \frac{\sin \phi + u_x \cos \phi}{\cos \phi - u_x \sin \phi} \]
\[ v_{yy} = \frac{u_{xx}}{(\cos \phi - u_x \sin \phi)^3} \]
\[ v_{yyy} = \frac{(\cos \phi - u_x \sin \phi) u_{xxx} - 3 u_{xx}^2 \sin \phi}{(\cos \phi - u_x \sin \phi)^5} \]

Choose a cross-section, or, equivalently a set of $r = \dim G = 3$ normalization equations:
\[ y = 0 \quad v = 0 \quad v_y = 0 \]

Solve the normalization equations for the group parameters:
\[ \phi = - \tan^{-1} u_x \quad a = - \frac{x + uu_x}{\sqrt{1 + u_x^2}} \quad b = \frac{xu_x - u}{\sqrt{1 + u_x^2}} \]

The result is the right moving frame $\rho : J^1 \rightarrow \mathbb{SE}(2)$
Substitute into the moving frame formulas for the group parameters into the remaining prolonged transformation formulae to produce the basic differential invariants:

\[
\begin{align*}
v_{yy} \mapsto & \quad \kappa \quad \Rightarrow \quad \frac{u_{xx}}{(1 + u_x^2)^{3/2}} \\
v_{yyy} \mapsto & \quad \frac{d\kappa}{ds} \quad \Rightarrow \quad \frac{(1 + u_x^2)u_{xxx} - 3u_x u_{xx}^2}{(1 + u_x^2)^3} \\
v_{yyyy} \mapsto & \quad \frac{d^2\kappa}{ds^2} + 3\kappa^3 = \cdots
\end{align*}
\]

**Theorem.** All differential invariants are functions of the derivatives of curvature with respect to arc length:

\[
\begin{align*}
\kappa \quad \frac{d\kappa}{ds} \quad \frac{d^2\kappa}{ds^2} \quad \cdots
\end{align*}
\]
The invariant differential operators and invariant differential forms are also substituting the moving frame formulas for the group parameters:

**Invariant one-form — arc length**

\[ dy = (\cos \phi - u_x \sin \phi) \, dx \quad \mapsto \quad ds = \sqrt{1 + u_x^2} \, dx \]

**Invariant differential operator — arc length derivative**

\[ \frac{d}{dy} = \frac{1}{\cos \phi - u_x \sin \phi} \frac{d}{dx} \quad \mapsto \quad \frac{d}{ds} = \frac{1}{\sqrt{1 + u_x^2}} \frac{d}{dx} \]
Euclidean Curves

Left moving frame \( \tilde{\rho}(x, u^{(1)}) = \rho(x, u^{(1)})^{-1} \)

\[
\tilde{a} = x \quad \tilde{b} = u \quad \tilde{\phi} = \tan^{-1} u_x
\]

\[
R = \frac{1}{\sqrt{1 + u_x^2}} \begin{pmatrix} 1 & -u_x \\ u_x & 1 \end{pmatrix} = \begin{pmatrix} t \\ n \end{pmatrix} \quad \tilde{a} = \begin{pmatrix} x \\ u \end{pmatrix}
\]
Invariantization

The process of replacing group parameters in transformation rules by their moving frame formulae is known as invariantization:

\[ \iota : \begin{cases} 
\text{Functions} & \rightarrow & \text{Invariants} \\
\text{Forms} & \rightarrow & \text{Invariant Forms} \\
\text{Differential Operators} & \rightarrow & \text{Invariant Differential Operators} \\
\vdots & & \vdots 
\end{cases} \]

- The invariantization \( I = \iota(F) \) is the unique invariant function that agrees with \( F \) on the cross-section: \( I | K = F | K \).
- Invariantization defines an (exterior) algebra morphism.
- Invariantization does not affect invariants: \( \iota(I) = I \)
The Fundamental Differential Invariants

Invariantized jet coordinate functions:

\[
\begin{align*}
H^i(x, u^{(n)}) &= \iota(x^i) \\
I^K_{\alpha}(x, u^{(l)}) &= \iota(u^K_{\alpha})
\end{align*}
\]

- The constant differential invariants, as dictated by the moving frame normalizations, are known as the **phantom invariants**.
- The remaining non-constant differential invariants are the **basic invariants** and form a complete system of functionally independent differential invariants for the prolonged group action.
Invariantization of general differential functions:

\[ \iota \left[ F(\ldots x^i \ldots u_j^\alpha \ldots) \right] = F(\ldots H^i \ldots I_j^\alpha \ldots) \]
Invariantization of general differential functions:

\[
\imath \left[ F( \ldots x^i \ldots u^\alpha_\beta \ldots ) \right] = F( \ldots H^i \ldots I^\alpha_\beta \ldots )
\]

The Replacement Theorem:

If \( J \) is a differential invariant, then \( \imath (J) = J \).

\[
J( \ldots x^i \ldots u^\alpha_\beta \ldots ) = J( \ldots H^i \ldots I^\alpha_\beta \ldots )
\]
Invariantization of general differential functions:

\[
\iota \left[ F(\ldots x^i \ldots u^\alpha_j \ldots) \right] = F(\ldots H^i \ldots I^\alpha_j \ldots)
\]

The Replacement Theorem:

If \( J \) is a differential invariant, then \( \iota(J) = J \).

\[
J(\ldots x^i \ldots u^\alpha_j \ldots) = J(\ldots H^i \ldots I^\alpha_j \ldots)
\]

Key fact: Invariantization and differentiation do not commute:

\[
\iota(D_i F) \neq D_i \iota(F)
\]

★★ Recurrence Formulae ★★
Infinitesimal Generators

Infinitesimal generators of action of $G$ on $M$:

$$v_\kappa = \sum_{i=1}^{p} \xi_i^\kappa(x, u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \varphi^\alpha_\kappa(x, u) \frac{\partial}{\partial u^\alpha} \quad \kappa = 1, \ldots, r$$

Prolonged infinitesimal generators on $J^n$:

$$v^{(n)}_\kappa = v_\kappa + \sum_{\alpha=1}^{q} \sum_{j=\#J=1}^{n} \varphi^\alpha_{\kappa,j}(x, u^{(j)}) \frac{\partial}{\partial u^\alpha_j}$$

Prolongation formula:

$$\varphi^\alpha_{\kappa,j} = D_K\left( \varphi^\alpha_\kappa - \sum_{i=1}^{p} u^\alpha_i \xi_i^\kappa \right) + \sum_{i=1}^{p} u^\alpha_{\kappa,i} \xi_i^\kappa$$

$$D_1, \ldots, D_p \quad \text{— total derivatives}$$
Recurrence Formulae

\[ \mathcal{D}_j \iota(F) = \iota(D_j F) + \sum_{\kappa=1}^{r} R^\kappa_j \iota(v^{(n)}_\kappa(F)) \]

\[ \omega^i = \iota(dx^i) \quad \text{— invariant coframe} \]

\[ \mathcal{D}_i = \iota(D_{x^i}) \quad \text{— dual invariant differential operators} \]

\[ R^\kappa_j \quad \text{— Maurer–Cartan invariants} \]

\[ v_1, \ldots \ v_r \in \mathfrak{g} \quad \text{— infinitesimal generators} \]

\[ \mu^1, \ldots \ \mu^r \in \mathfrak{g}^* \quad \text{— dual Maurer–Cartan forms} \]
The Maurer–Cartan Invariants

Invariantized Maurer–Cartan forms:

\[ \gamma^\kappa = \rho^*(\mu^\kappa) \equiv \sum_{j=1}^{p} R_j^\kappa \omega^j \]

Remark: When \( G \subset GL(N) \), the Maurer–Cartan invariants \( R_j^\kappa \) are the entries of the Frenet matrices.

Theorem. (E. Hubert) The Maurer–Cartan invariants and, in the intransitive case, the order zero invariants generate the differential invariant algebra \( I(G) \).
The Maurer–Cartan Invariants

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Remark: When \( G \subset \text{GL}(N) \), the Maurer–Cartan invariants \( R_j^\kappa \) are the entries of the Frenet matrices

\[ D_i \rho(x, u^{(n)}) \cdot \rho(x, u^{(n)})^{-1} \]
The Maurer–Cartan Invariants

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\[ D_i \rho(x, u^{(n)}) \cdot \rho(x, u^{(n)})^{-1} \]

Theorem. (E. Hubert) The Maurer–Cartan invariants and, in the intransitive case, the order zero invariants serve to generate the differential invariant algebra \( \mathcal{I}(G) \).
Recurrence Formulae

\[ D_j \iota(F) = \iota(D_j F) + \sum_{\kappa=1}^{r} R_j^\kappa \iota(v^{(n)}_\kappa(F)) \]
Recurrence Formulae

\[ D_j \iota(F) = \iota(D_j F) + \sum_{\kappa=1}^{r} R_{j}^{\kappa} \iota(v_{\kappa}^{(n)}(F)) \]

If \( \iota(F) = c \) is a phantom differential invariant, then the left hand side of the recurrence formula is zero. The collection of all such phantom recurrence formulae form a linear algebraic system of equations that can be uniquely solved for the Maurer–Cartan invariants \( R_{j}^{\kappa} \)!
Recurrence Formulae

\[ D_j \iota(F') = \iota(D_jF') + \sum_{\kappa=1}^{r} R^\kappa_j \iota(\nu^{(n)}_{\kappa}(F')) \]

♠ If \( \iota(F') = c \) is a phantom differential invariant, then the left hand side of the recurrence formula is zero. The collection of all such phantom recurrence formulae form a linear algebraic system of equations that can be uniquely solved for the Maurer–Cartan invariants \( R^\kappa_j \)!

♥ Once the Maurer–Cartan invariants are replaced by their explicit formulae, the induced recurrence relations completely determine the structure of the differential invariant algebra \( \mathcal{I}(G) \)!
The Universal Recurrence Formula

Let $\Omega$ be any differential form on $J^n$.

\[ d \iota(\Omega) = \iota(d\Omega) + \sum_{\kappa=1}^{r} \gamma^{\kappa} \wedge \iota[v_{\kappa}(\Omega)] \]

\[ \implies \text{The invariant variational bicomplex} \]
The Universal Recurrence Formula

Let $\Omega$ be any differential form on $J^n$.

\[
d \iota(\Omega) = \iota(d\Omega) + \sum_{\kappa=1}^{r} \gamma^\kappa \wedge \iota[v_\kappa(\Omega)]
\]

\[\implies \text{The invariant variational bicomplex}\]

Commutator invariants:

\[
d \omega^i = d[\iota(dx^i)] = \iota(d^2 x^i) + \sum_{\kappa=1}^{r} \gamma^\kappa \wedge \iota[v_\kappa(dx^i)]
\]

\[= - \sum_{j<k} Y^i_{jk} \omega^j \wedge \omega^k + \cdots\]

\[[D_j, D_k] = \sum_{i=1}^{p} Y^i_{jk} D_i\]
The Differential Invariant Algebra

Thus, remarkably, the structure of $\mathcal{I}(G)$ can be determined without knowing the explicit formulae for either the moving frame, or the differential invariants, or the invariant differential operators!

The only required ingredients are the specification of the cross-section, and the standard formulae for the prolonged infinitesimal generators.

**Theorem.** If $G$ acts transitively on $M$, or if the infinitesimal generator coefficients depend rationally in the coordinates, then all recurrence formulae are rational in the basic differential invariants and so $\mathcal{I}(G)$ is a rational, non-commutative differential algebra.
Euclidean Surfaces

\[ M = \mathbb{R}^3 \quad G = \text{SE}(3) = \text{SO}(3) \ltimes \mathbb{R}^3 \quad \dim G = 6. \]

\[ g \cdot z = Rz + b, \quad R^T R = I, \quad z = \begin{pmatrix} x \\ y \\ u \end{pmatrix} \in \mathbb{R}^3. \]

Assume (for simplicity) that \( S \subset \mathbb{R}^3 \) is the graph of a function:

\[ u = f(x, y) \]

Cross-section to prolonged action on \( J^2 \):

\[ x = y = u = u_x = u_y = u_{xy} = 0, \quad u_{xx} \neq u_{yy}. \]
Invariantization — differential invariants: \[ I_{jk} = \iota(u_{jk}) \]

Phantom differential invariants:
\[
\iota(x) = \iota(y) = \iota(u) = \iota(u_x) = \iota(u_y) = \iota(u_{xy}) = 0.
\]

Principal curvatures:
\[
\kappa_1 = I_{20} = \iota(u_{xx}), \quad \kappa_2 = I_{02} = \iota(u_{yy}),
\]

★★ non-umbilic point: \( \kappa_1 \neq \kappa_2 \) ★★

Mean and Gauss curvatures:
\[
H = \frac{1}{2}(\kappa_1 + \kappa_2), \quad K = \kappa_1 \kappa_2.
\]

Invariant differential operators:
\[
\mathcal{D}_1 = \iota(D_x), \quad \mathcal{D}_2 = \iota(D_y).
\]

\[ \implies \text{diagonalizing Frenet frame} \]
To obtain the recurrence formulae for the higher order differential invariants, we need the infinitesimal generators of $\mathfrak{g} = \mathfrak{se}(3)$:

\[
\begin{align*}
\mathbf{v}_1 &= -y \partial_x + x \partial_y \\
\mathbf{v}_2 &= -u \partial_x + x \partial_u \\
\mathbf{v}_3 &= -u \partial_y + y \partial_u
\end{align*}
\]

\[
\begin{align*}
\mathbf{w}_1 &= \partial_x \\
\mathbf{w}_2 &= \partial_y \\
\mathbf{w}_3 &= \partial_u
\end{align*}
\]

- The translations will be ignored, as they play no role in the higher order recurrence formulae.
Recurrence formulae

\[ \mathcal{D}_i \iota(u_{jk}) = \iota(D_i u_{jk}) + \sum_{\nu=1}^{3} \iota[\varphi_{\nu}^{jk}(x, y, u^{(j+k)})] R_{i}^{\nu}, \quad j + k \geq 1 \]

\[ \mathcal{D}_1 I_{jk} = I_{j+1,k} + \sum_{\nu=1}^{3} \varphi_{\nu}^{jk}(0, 0, I^{(j+k)}) R_{1}^{\nu} \]

\[ \mathcal{D}_2 I_{jk} = I_{j,k+1} + \sum_{\nu=1}^{3} \varphi_{\nu}^{jk}(0, 0, I^{(j+k)}) R_{2}^{\nu} \]

\[ \varphi_{\nu}^{jk}(0, 0, I^{(j+k)}) = \iota[\varphi_{\nu}^{jk}(x, y, u^{(j+k)})] \quad \text{— invariantized prolonged infinitesimal generator coefficients} \]

\[ R_{i}^{\nu} \quad \text{— Maurer–Cartan invariants} \]
Phantom recurrence formulae:

\[
0 = \mathcal{D}_1 I_{10} = I_{20} + R_1^2 \quad 0 = \mathcal{D}_2 I_{10} = R_2^2 \\
0 = \mathcal{D}_1 I_{01} = R_1^3 \quad 0 = \mathcal{D}_2 I_{01} = I_{02} + R_2^3 \\
0 = \mathcal{D}_1 I_{11} = I_{21} + (I_{20} - I_{02}) R_1^l \quad 0 = \mathcal{D}_2 I_{11} = I_{12} + (I_{20} - I_{02}) R_2^l
\]

Maurer–Cartan invariants:

\[
R_1 = (Y_2, -\kappa_1, 0) \quad R_2 = (-Y_1, 0, -\kappa_2)
\]

where

\[
Y_1 = \frac{I_{12}}{I_{20} - I_{02}} = \frac{\mathcal{D}_1 \kappa_2}{\kappa_1 - \kappa_2} \quad Y_2 = \frac{I_{21}}{I_{02} - I_{20}} = \frac{\mathcal{D}_2 \kappa_1}{\kappa_2 - \kappa_1}
\]

are also the commutator invariants:

\[
[\mathcal{D}_1, \mathcal{D}_2] = \mathcal{D}_1 \mathcal{D}_2 - \mathcal{D}_2 \mathcal{D}_1 = Y_2 \mathcal{D}_1 - Y_1 \mathcal{D}_2.
\]
Second order recurrence formulae:

\[ I_{30} = D_1 I_{20} = \kappa_{1,1} \quad I_{21} = D_2 I_{20} = \kappa_{1,2} \]

\[ I_{12} = D_1 I_{02} = \kappa_{2,1} \quad I_{03} = D_2 I_{02} = \kappa_{2,2} \]

The fourth order recurrence formulae

\[ D_2 I_{21} + \frac{I_{30} I_{12} - 2 I_{12}^2}{\kappa_1 - \kappa_2} + \kappa_1 \kappa_2 = I_{22} = D_1 I_{12} - \frac{I_{21} I_{03} - 2 I_{21}^2}{\kappa_1 - \kappa_2} + \kappa_1 \kappa_2 \]

lead to the Codazzi syzygy

\[ \kappa_{1,22} - \kappa_{2,11} + \frac{\kappa_{1,1} \kappa_{2,1} + \kappa_{1,2} \kappa_{2,2} - 2 \kappa_{2,1}^2 - 2 \kappa_{1,2}^2}{\kappa_1 - \kappa_2} - \kappa_1 \kappa_2 (\kappa_1 - \kappa_2) = 0 \]

- The principal curvatures \( \kappa_1, \kappa_2 \), or, equivalently, the Gauss and mean curvatures \( H, K \), form a generating system for the differential invariant algebra.
Second order recurrence formulae:
\[ I_{30} = \mathcal{D}_1 I_{20} = \kappa_{1,1} \quad I_{21} = \mathcal{D}_2 I_{20} = \kappa_{1,2} \]
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The fourth order recurrence formulae
\[ \mathcal{D}_2 I_{21} + \frac{I_{30} I_{12} - 2 I_{12}^2}{\kappa_1 - \kappa_2} + \kappa_1 \kappa_2^2 = I_{22} = \mathcal{D}_1 I_{12} - \frac{I_{21} I_{03} - 2 I_{21}^2}{\kappa_1 - \kappa_2} + \kappa_2^2 \kappa_2 \]
lead to the Codazzi syzygy
\[ \kappa_{1,22} - \kappa_{2,11} + \frac{\kappa_{1,1} \kappa_{2,1} + \kappa_{1,2} \kappa_{2,2} - 2 \kappa_{2,1}^2 - 2 \kappa_{1,2}^2}{\kappa_1 - \kappa_2} - \kappa_1 \kappa_2 (\kappa_1 - \kappa_2) = 0 \]

- The principal curvatures \( \kappa_1, \kappa_2 \), or, equivalently, the Gauss and mean curvatures \( H, K \), form a generating system for the differential invariant algebra.
  
  \[ \star \star \quad \text{Neither is a minimal generating set!} \quad \star \star \]
Codazzi syzygy:

\[ K = \kappa_1 \kappa_2 = -(D_1 + Y_1)Y_1 - (D_2 + Y_2)Y_2 \]
Codazzi syzygy:

\[ K = \kappa_1 \kappa_2 = - (\mathcal{D}_1 + Y_1)Y_1 - (\mathcal{D}_2 + Y_2)Y_2 \]

Gauss’ Theorema Egregium

The Gauss curvature is intrinsic.
Codazzi syzygy:

\[ K = \kappa_1 \kappa_2 = - (\mathcal{D}_1 + Y_1)Y_1 - (\mathcal{D}_2 + Y_2)Y_2 \]

---

Gauss’ Theorema Egregium

The Gauss curvature is intrinsic.

**Proof:** The Frenet frome is intrinsic, hence so are the invariant differentiations and also commutator invariants. \( Q.E.D. \)
Codazzi syzygy:

\[ K = \kappa_1 \kappa_2 = - (D_1 + Y_1)Y_1 - (D_2 + Y_2)Y_2 \]

---

**Gauss’ Theorema Egregium**

The Gauss curvature is intrinsic.

*Proof*: The Frenet frame is intrinsic, hence so are the invariant differentiations and also commutator invariants. \hspace{1cm} \textit{Q.E.D.}

---

**Theorem.** For suitably nondegenerate surfaces, the mean curvature \( H \) is a generating differential invariant, i.e., all other Euclidean surface differential invariants can be expressed as functions of \( H \) and its invariant derivatives.
Proof: Since $H, K$ generate the differential invariant algebra, it suffices to express the Gauss curvature $K$ as a function of $H$ and its derivatives. For this, the Codazzi syzygy implies that we need only express the commutator invariants in terms of $H$.

The commutator identity can be applied to any differential invariant. In particular,

$$
\mathcal{D}_1 \mathcal{D}_2 H - \mathcal{D}_2 \mathcal{D}_1 H = Y_2 \mathcal{D}_1 H - Y_1 \mathcal{D}_2 H \\
\mathcal{D}_1 \mathcal{D}_2 \mathcal{D}_j H - \mathcal{D}_2 \mathcal{D}_1 \mathcal{D}_j H = Y_2 \mathcal{D}_1 \mathcal{D}_j H - Y_1 \mathcal{D}_2 \mathcal{D}_j H
$$

Provided the nondegeneracy condition

$$(\mathcal{D}_1 H)(\mathcal{D}_2 \mathcal{D}_j H) \neq (\mathcal{D}_2 H)(\mathcal{D}_1 \mathcal{D}_j H), \quad \text{for } j = 1 \text{ or } 2$$

holds, we can solve (*) for the commutator invariants as rational functions of invariant derivatives of $H$. Q.E.D.

Note: Constant Mean Curvature surfaces are degenerate. Are there others?
Theorem. \( G = SA(3) = SL(3) \ltimes \mathbb{R}^3 \) acts on \( S \subset M = \mathbb{R}^3 \):

The algebra of differential invariants of generic equiaffine surfaces is generated by a single third order invariant, the Pick invariant.

Theorem. \( G = SO(4,1) \) acts on \( S \subset M = \mathbb{R}^3 \):

The algebra of differential invariants of generic conformal surfaces is generated by a single third order invariant.

Theorem. \( G = PSL(4) \) acts on \( S \subset M = \mathbb{R}^3 \):

The algebra of differential invariants of generic projective surfaces is generated by a single fourth order invariant.
The Infinite Jet Bundle

Jet bundles

\[ M = J^0 \leftarrow J^1 \leftarrow J^2 \leftarrow \cdots \]

Inverse limit

\[ J^\infty = \lim_{n \to \infty} J^n \]

Local coordinates

\[ z^{(\infty)} = (x, u^{(\infty)}) = (\ldots x^i \ldots u_j^\alpha \ldots) \]

\[ \implies \text{Taylor series} \]
Differential Forms

Coframe — basis for the cotangent space $T^\ast J^\infty$:

- Horizontal one-forms
  
  $dx^1, \ldots, dx^p$

- Contact (vertical) one-forms

  \[
  \theta^\alpha_j = du^\alpha_j - \sum_{i=1}^{p} u^\alpha_{j,i} \, dx^i
  \]

Intrinsic definition of contact form

\[
\theta \mid j_{\infty}N = 0 \iff \theta = \sum A^\alpha_j \theta^\alpha_j
\]
The Variational Bicomplex

⇒ Dedecker, Vinogradov, Tsujishita, I. Anderson, …

Bigrading of the differential forms on $J^\infty$:

$$\Omega^\ast = \bigoplus_{r,s} \Omega^{r,s}$$

$r = \# \text{ horizontal forms}$

$s = \# \text{ contact forms}$


d = d_H + d_V

$d_H : \Omega^{r,s} \longrightarrow \Omega^{r+1,s}$

$d_V : \Omega^{r,s} \longrightarrow \Omega^{r,s+1}$
Vertical and Horizontal Differentials

\[ F(x, u^{(n)}) \quad \text{— differential function} \]

\[ d_H F = \sum_{i=1}^{p} (D_i F) \; dx^i \quad \text{— total differential} \]

\[ d_V F = \sum_{\alpha, J} \frac{\partial F}{\partial u^\alpha_J} \; \theta^\alpha_J \quad \text{— first variation} \]

\[ d_H (dx^i) = d_V (dx^i) = 0, \]

\[ d_H (\theta^\alpha_J) = \sum_{i=1}^{p} dx^i \wedge \theta^\alpha_{J,i} \] \quad \[ d_V (\theta^\alpha_J) = 0 \]
The Simplest Example

\[(x, u) \in M = \mathbb{R}^2\]

\[x \quad \text{— independent variable}\]
\[u \quad \text{— dependent variable}\]

**Horizontal form**

\[dx\]

**Contact (vertical) forms**

\[\theta = du - u_x \, dx\]
\[\theta_x = du_x - u_{xx} \, dx\]
\[\theta_{xx} = du_{xx} - u_{xxx} \, dx\]
\[\vdots\]
\[ \theta = du - u_x \, dx, \quad \theta_x = du_x - u_{xx} \, dx, \quad \theta_{xx} = du_{xx} - u_{xxx} \, dx \]

Differential:
\[
dF = \frac{\partial F}{\partial x} \, dx + \frac{\partial F}{\partial u} \, du + \frac{\partial F}{\partial u_x} \, du_x + \frac{\partial F}{\partial u_{xx}} \, du_{xx} + \cdots
\]
\[= (D_x F) \, dx + \frac{\partial F}{\partial u} \, \theta + \frac{\partial F}{\partial u_x} \, \theta_x + \frac{\partial F}{\partial u_{xx}} \, \theta_{xx} + \cdots
\]
\[= d_H F + d_V F
\]

Total derivative:
\[
D_x F = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial u} \, u_x + \frac{\partial F}{\partial u_x} \, u_{xx} + \frac{\partial F}{\partial u_{xx}} \, u_{xxx} + \cdots
\]
The Variational Bicomplex

\[
\begin{array}{ccccccccc}
\Omega^0,3 & \Omega^1,3 & \Omega^p-1,3 & \Omega^p,3 \\
\downarrow d_V & \downarrow d_V & \downarrow d_V & \downarrow d_V & \downarrow d_V & \downarrow \delta \\
\Omega^0,2 & \Omega^1,2 & \Omega^p-1,2 & \Omega^p,2 \\
\downarrow d_V & \downarrow d_V & \downarrow d_V & \downarrow d_V & \downarrow \delta \\
\Omega^0,1 & \Omega^1,1 & \Omega^p-1,1 & \Omega^p,1 \\
\downarrow d_V & \downarrow d_V & \downarrow d_V & \downarrow d_V & \downarrow \delta \\
\mathbb{R} & \Omega^0,0 & \Omega^1,0 & \Omega^p-1,0 & \Omega^p,0 \\
\end{array}
\]
The Variational Bicomplex

\[ \begin{align*}
\Omega^0,0 & \xrightarrow{d_V} \Omega^1,0 & \xrightarrow{d_H} \Omega^2,0 & \xrightarrow{d_H} \cdots & \xrightarrow{d_H} \Omega^{p-1,0} & \xrightarrow{d_H} \Omega^{p,0} \\
\Omega^0,1 & \xrightarrow{d_V} \Omega^1,1 & \xrightarrow{d_H} \Omega^2,1 & \xrightarrow{d_H} \cdots & \xrightarrow{d_H} \Omega^{p-1,1} & \xrightarrow{d_H} \Omega^{p,1} \\
\Omega^0,2 & \xrightarrow{d_V} \Omega^1,2 & \xrightarrow{d_H} \Omega^2,2 & \xrightarrow{d_H} \cdots & \xrightarrow{d_H} \Omega^{p-1,2} & \xrightarrow{d_H} \Omega^{p,2} \\
\Omega^0,3 & \xrightarrow{d_V} \Omega^1,3 & \xrightarrow{d_H} \Omega^2,3 & \xrightarrow{d_H} \cdots & \xrightarrow{d_H} \Omega^{p-1,3} & \xrightarrow{d_H} \Omega^{p,3} \\
\end{align*} \]

\[ \begin{align*}
\delta & \quad \pi & \quad F^3 \\
\delta & \quad \pi & \quad F^2 \\
\delta & \quad \pi & \quad F^1 \\
E & & \\
\end{align*} \]

Lagrangians
The Variational Bicomplex

\[ \begin{array}{cccccccccc}
\Omega^0,3 & \xrightarrow{d_H} & \Omega^1,3 & \xrightarrow{d_H} & \cdots & \xrightarrow{d_H} & \Omega^{p-1,3} & \xrightarrow{d_H} & \Omega^{p,3} & \xrightarrow{\pi} \mathcal{F}^3 \\
\Omega^0,2 & \xrightarrow{d_H} & \Omega^1,2 & \xrightarrow{d_H} & \cdots & \xrightarrow{d_H} & \Omega^{p-1,2} & \xrightarrow{d_H} & \Omega^{p,2} & \xrightarrow{\pi} \mathcal{F}^2 \\
\Omega^0,1 & \xrightarrow{d_H} & \Omega^1,1 & \xrightarrow{d_H} & \cdots & \xrightarrow{d_H} & \Omega^{p-1,1} & \xrightarrow{d_H} & \Omega^{p,1} & \xrightarrow{\pi} \mathcal{F}^1 \\
\mathbb{R} & \rightarrow & \Omega^0,0 & \xrightarrow{d_H} & \Omega^1,0 & \xrightarrow{d_H} & \cdots & \xrightarrow{d_H} & \Omega^{p-1,0} & \xrightarrow{d_H} & \Omega^{p,0} \\
\end{array} \]

Lagrangians \quad PDEs (Euler–Lagrange)
The Variational Bicomplex

The diagram illustrates the Variational Bicomplex, which is a mathematical structure that arises in the study of Lagrangians, PDEs (Euler–Lagrange), and Helmholtz conditions. The bicomplex is given by a sequence of differential forms and differential operators, leading to the formulation of conservation laws in physics.
The Variational Bicomplex

\[ \Omega^0,3 \xrightarrow{d_H} \Omega^1,3 \xrightarrow{d_H} \cdots \xrightarrow{d_H} \Omega^{p-1,3} \xrightarrow{d_H} \Omega^{p,3} \xrightarrow{\delta} \mathcal{F}^3 \]

\[ \Omega^0,2 \xrightarrow{d_H} \Omega^1,2 \xrightarrow{d_H} \cdots \xrightarrow{d_H} \Omega^{p-1,2} \xrightarrow{d_H} \Omega^{p,2} \xrightarrow{\pi} \mathcal{F}^2 \]

\[ \Omega^0,1 \xrightarrow{d_H} \Omega^1,1 \xrightarrow{d_H} \cdots \xrightarrow{d_H} \Omega^{p-1,1} \xrightarrow{d_H} \Omega^{p,1} \xrightarrow{\pi} \mathcal{F}^1 \]

\[ \mathbb{R} \rightarrow \Omega^0,0 \xrightarrow{d_H} \Omega^1,0 \xrightarrow{d_H} \cdots \xrightarrow{d_H} \Omega^{p-1,0} \xrightarrow{d_H} \Omega^{p,0} \]

conservation laws  Lagrangians  PDEs (Euler–Lagrange)  Helmholtz conditions
The Variational Derivative

\[ E = \pi \circ d_V \]

- \( d_V \) — first variation
- \( \pi \) — integration by parts = mod out by image of \( d_H \)

\[
\begin{align*}
\Omega^{p,0} & \xrightarrow{d_V} \Omega^{p,1} \quad \pi \quad \mathcal{F}^1 = \Omega^{p,1} / d_H \Omega^{p-1,1} \\
\lambda = L \, dx & \quad \sum_{\alpha,J} \frac{\partial L}{\partial u^\alpha_J} \theta^\alpha_J \wedge dx \quad \sum_{\alpha=1}^q E_\alpha(L) \theta^\alpha \wedge dx \\
\text{Variational} & \quad \text{First} \quad \text{Euler–Lagrange} \\
\text{problem} & \quad \text{variation} \quad \text{source form}
\end{align*}
\]
The Simplest Example: \((x, u) \in M = \mathbb{R}^2\)

Lagrangian form: \(\lambda = L(x, u^{(n)}) \, dx \in \Omega^{1,0}\)
The Simplest Example: \((x, u) \in M = \mathbb{R}^2\)

Lagrangian form: \(\lambda = L(x, u^{(n)}) \, dx \in \Omega^{1,0}\)

First variation — vertical derivative:
\[
d\lambda = d_V \lambda = d_V L \wedge dx
\]
\[
= \left( \frac{\partial L}{\partial u} \theta + \frac{\partial L}{\partial u_x} \theta_x + \frac{\partial L}{\partial u_{xx}} \theta_{xx} + \cdots \right) \wedge dx \in \Omega^{1,1}
\]
**The Simplest Example:**  \((x, u) \in M = \mathbb{R}^2\)

Lagrangian form:  \(\lambda = L(x, u^{(n)}) dx \in \Omega^{1,0}\)

First variation — vertical derivative:

\[
d\lambda = d_V \lambda = d_V L \wedge dx = \left( \frac{\partial L}{\partial u} \theta + \frac{\partial L}{\partial u_x} \theta_x + \frac{\partial L}{\partial u_{xx}} \theta_{xx} + \cdots \right) \wedge dx \in \Omega^{1,1}
\]

Integration by parts — compute modulo im \(d_H\):

\[
d\lambda \sim \delta \lambda = \left( \frac{\partial L}{\partial u} \right. - D_x \frac{\partial L}{\partial u_x} + D^2_x \frac{\partial L}{\partial u_{xx}} - \cdots ) \theta \wedge dx \in \mathcal{F}^1
\]

\[
= E(L) \theta \wedge dx \implies \text{Euler-Lagrange source form.}
\]
To analyze invariant variational problems, invariant conservation laws, invariant flows, etc., we apply the moving frame invariantization process to the variational bicomplex:
Differential Invariants and Invariant Differential Forms

\( \iota \) — invariantization associated with moving frame \( \rho \).

- Fundamental differential invariants
  \[ H^i(x, u^{(n)}) = \iota(x^i) \quad I^K(x, u^{(n)}) = \iota(u^K) \]

- Invariant horizontal forms
  \[ \varpi^i = \iota(dx^i) \]

- Invariant contact forms
  \[ \varphi^\alpha_j = \iota(\theta^\alpha_j) \]
The Invariant “Quasi–Tricomplex”

Differential forms

\[ \Omega^* = \bigoplus_{r,s} \hat{\Omega}^{r,s} \]

Differential

\[ d = d_{\mathcal{H}} + d_{\mathcal{V}} + d_{\mathcal{W}} \]

\[ d_{\mathcal{H}} : \quad \hat{\Omega}^{r,s} \rightarrow \hat{\Omega}^{r+1,s} \]

\[ d_{\mathcal{V}} : \quad \hat{\Omega}^{r,s} \rightarrow \hat{\Omega}^{r,s+1} \]

\[ d_{\mathcal{W}} : \quad \hat{\Omega}^{r,s} \rightarrow \hat{\Omega}^{r-1,s+2} \]

Key fact: invariantization and differentiation do not commute:

\[ d \iota(\Omega) \neq \iota(d\Omega) \]
The Universal Recurrence Formula

\[ d \iota(\Omega) = \iota(d\Omega) + \sum_{\kappa=1}^{r} \nu^\kappa \wedge \iota[\nu^\kappa(\Omega)] \]

\[ \nu^1, \ldots, \nu^r \] — invariantized dual Maurer–Cartan forms

\[ \nu^1, \ldots, \nu^r \] — invariantized dual Maurer–Cartan forms

\[ \nu^1, \ldots, \nu^r \] — invariantized dual Maurer–Cartan forms

⇒ uniquely determined by the recurrence formulae for the phantom differential invariants
\[ d\nu(\Omega) = \nu(d\Omega) + \sum_{\kappa=1}^{r} \nu^\kappa \wedge \nu[v_\kappa(\Omega)] \]

All identities, commutation formulae, syzygies, etc., among differential invariants and, more generally, the invariant variational bicomplex follow from this universal formula by letting \( \Omega \) range over the basic functions and differential forms!

Moreover, determining the structure of the differential invariant algebra and invariant variational bicomplex requires only linear differential algebra, and not any explicit formulas for the moving frame, the differential invariants, the invariant differential forms, or the group transformations!
Euclidean plane curves

Fundamental normalized differential invariants

\[
\begin{align*}
\iota(x) &= H = 0 \\
\iota(u) &= I_0 = 0 \\
\iota(u_x) &= I_1 = 0 \\
\phantom{=} &\quad \text{phantom diff. invs.} \\
\iota(u_{xx}) &= I_2 = \kappa \\
\iota(u_{xxx}) &= I_3 = \kappa_s \\
\iota(u_{xxxx}) &= I_4 = \kappa_{ss} + 3\kappa^3
\end{align*}
\]

In general:

\[
\iota(F(x, u, u_x, u_{xx}, u_{xxx}, u_{xxxx}, \ldots)) = F(0, 0, 0, \kappa, \kappa_s, \kappa_{ss} + 3\kappa^3, \ldots)
\]
Invariant arc length form

\[ dy = (\cos \phi - u_x \sin \phi) \, dx - (\sin \phi) \, \theta \]

\[ \varpi = \iota(dx) = \omega + \eta \]

\[ = \sqrt{1 + u_x^2} \, dx + \frac{u_x}{\sqrt{1 + u_x^2}} \, \theta \]

\[ \implies \theta = du - u_x \, dx \]

Invariant contact forms

\[ \vartheta = \iota(\theta) = \frac{\theta}{\sqrt{1 + u_x^2}} \]

\[ \vartheta_1 = \iota(\theta_x) = \frac{(1 + u_x^2) \theta_x - u_x u_{xx} \theta}{(1 + u_x^2)^2} \]
Prolonged infinitesimal generators

\[ \mathbf{v}_1 = \partial_x, \quad \mathbf{v}_2 = \partial_u, \quad \mathbf{v}_3 = -u \partial_x + x \partial_u + (1 + u^2_x) \partial_{ux} + 3u_x u_{xx} \partial_{uxx} + \cdots \]

Basic recurrence formula

\[ dt(F) = \iota(dF) + \iota(\mathbf{v}_1(F)) \nu^1 + \iota(\mathbf{v}_2(F)) \nu^2 + \iota(\mathbf{v}_3(F)) \nu^3 \]

Use phantom invariants

\[ 0 = dH = \iota(dx) + \iota(\mathbf{v}_1(x)) \nu^1 + \iota(\mathbf{v}_2(x)) \nu^2 + \iota(\mathbf{v}_3(x)) \nu^3 = \varpi + \nu^1, \]
\[ 0 = dI_0 = \iota(du) + \iota(\mathbf{v}_1(u)) \nu^1 + \iota(\mathbf{v}_2(u)) \nu^2 + \iota(\mathbf{v}_3(u)) \nu^3 = \vartheta + \nu^2, \]
\[ 0 = dI_1 = \iota(du_x) + \iota(\mathbf{v}_1(u_x)) \nu^1 + \iota(\mathbf{v}_2(u_x)) \nu^2 + \iota(\mathbf{v}_3(u_x)) \nu^3 = \kappa \varpi + \vartheta_1 + \nu^3, \]

to solve for the Maurer–Cartan forms:

\[ \nu^1 = -\varpi, \quad \nu^2 = -\vartheta, \quad \nu^3 = -\kappa \varpi - \vartheta_1. \]
Recurrence formulae:

\[ d\kappa = dt(u_{xx}) = t(du_{xx}) + t(v_1(u_{xx}))\nu^1 + t(v_2(u_{xx}))\nu^2 + t(v_3(u_{xx}))\nu^3 = t(u_{xxx}dx + \theta_{xx}) - t(3u_xu_{xx})(\kappa w + \vartheta_1) = I_3w + \vartheta_2. \]

Therefore,

\[ D\kappa = \kappa_s = I_3, \quad d\vartheta = I_3, \quad d\vartheta_2 = (D^2 + \kappa^2)\vartheta. \]

where the final formula follows from the contact form recurrence formulae

\[ d\vartheta = dt(\theta_x) = w \wedge \vartheta_1, \quad d\vartheta_1 = dt(\theta) = w \wedge (\vartheta_2 - \kappa^2 \vartheta) - \kappa \vartheta_1 \wedge \vartheta \]

which imply

\[ \vartheta_1 = D\vartheta, \quad \vartheta_2 = D\vartheta_1 + \kappa^2 \vartheta = (D^2 + \kappa^2)\vartheta. \]
Similarly,
\[d\varpi = \iota(d^2x) + \nu^1 \wedge \iota(v_1(dx)) + \nu^2 \wedge \iota(v_2(dx)) + \nu^3 \wedge \iota(v_3(dx))\]
\[= (\kappa \varpi + \vartheta_1) \wedge \iota(u_x dx + \theta) = \kappa \varpi \wedge \vartheta + \vartheta_1 \wedge \vartheta.\]

In particular,
\[d_Y \varpi = -\kappa \vartheta \wedge \varpi\]

---

**Key recurrence formulae:**

\[d_Y \kappa = (D^2 + \kappa^2) \vartheta\]
\[d_Y \varpi = -\kappa \vartheta \wedge \varpi\]
Plane Curves

Invariant Lagrangian:
\[ \tilde{\lambda} = L(x, u^{(n)}) dx = P(\kappa, \kappa_s, \ldots) \, \varpi \]

Euler–Lagrange form:
\[ d_V \tilde{\lambda} \sim E(L) \vartheta \wedge \varpi \]

Invariant Integration by Parts Formula
\[ F d_V (\mathcal{D}H) \wedge \varpi \sim - (DF) d_V H \wedge \varpi - (F \cdot \mathcal{D}H) d_V \varpi \]

\[
\begin{align*}
d_V \tilde{\lambda} &= d_V P \wedge \varpi + P d_V \varpi \\
&= \sum_n \frac{\partial P}{\partial \kappa_n} d_V \kappa_n \wedge \varpi + P d_V \varpi \\
&\sim \mathcal{E}(P) \, d_V \kappa \wedge \varpi + \mathcal{H}(P) \, d_V \varpi
\end{align*}
\]
Vertical differentiation formulae

\[ d_V \kappa = A(\vartheta) \] \hspace{1cm} \( A \) — “Eulerian operator”

\[ d_V \varpi = B(\vartheta) \wedge \varpi \] \hspace{1cm} \( B \) — “Hamiltonian operator”

\[ d_V \tilde{\lambda} \sim \mathcal{E}(P) A(\vartheta) \wedge \varpi + \mathcal{H}(P) B(\vartheta) \wedge \varpi \]
\[ \sim \left[ A^* \mathcal{E}(P) - B^* \mathcal{H}(P) \right] \vartheta \wedge \varpi \]

Invariant Euler-Lagrange equation

\[ A^* \mathcal{E}(P) - B^* \mathcal{H}(P) = 0 \]
Euclidean Plane Curves

\[ d_V \kappa = (D^2 + \kappa^2) \vartheta \]

Eulerian operator

\[ A = D^2 + \kappa^2 \quad \quad A^* = D^2 + \kappa^2 \]

\[ d_V \varpi = -\kappa \vartheta \wedge \varpi \]

Hamiltonian operator

\[ B = -\kappa \quad \quad B^* = -\kappa \]

Euclidean–invariant Euler-Lagrange formula

\[ E(L) = A^* \mathcal{E}(P) - B^* \mathcal{H}(P) = (D^2 + \kappa^2) \mathcal{E}(P) + \kappa \mathcal{H}(P). \]
Invariant Plane Curve Flows

$G$ — Lie group acting on $\mathbb{R}^2$

$C(t)$ — parametrized family of plane curves

$G$-invariant curve flow:

$$\frac{dC}{dt} = V = I \mathbf{t} + J \mathbf{n}$$

- $I, J$ — differential invariants
- $\mathbf{t}$ — “unit tangent”
- $\mathbf{n}$ — “unit normal”
t, n — basis of the invariant vector fields dual to the invariant one-forms:

\[ \langle t; \varpi \rangle = 1, \quad \langle n; \varpi \rangle = 0, \]
\[ \langle t; \vartheta \rangle = 0, \quad \langle n; \vartheta \rangle = 1. \]

\[ C_t = V = I t + J n \]

- The tangential component \( I t \) only affects the underlying parametrization of the curve. Thus, we can set \( I \) to be anything we like without affecting the curve evolution.
- There are two principal choices of tangential component:
Normal Curve Flows

\[ C_t = J n \]

Examples — Euclidean–invariant curve flows

- \( C_t = n \) — geometric optics or grassfire flow;
- \( C_t = \kappa n \) — curve shortening flow;
- \( C_t = \kappa^{1/3} n \) — equi-affine invariant curve shortening flow:
  \[ C_t = n_{\text{equi–affine}} \]
- \( C_t = \kappa_s n \) — modified Korteweg–deVries flow;
- \( C_t = \kappa_{ss} n \) — thermal grooving of metals.
Intrinsic Curve Flows

**Theorem.** The curve flow generated by

\[ \mathbf{v} = I \mathbf{t} + J \mathbf{n} \]

preserves arc length if and only if

\[ B(J) + D I = 0. \]

\( D \) — invariant arc length derivative

\[ d_\nu \overline{\omega} = B(\vartheta) \wedge \overline{\omega} \]

\( B \) — invariant Hamiltonian operator
Normal Evolution of Differential Invariants

**Theorem.** Under a normal flow $C_t = J n$,

\[
\frac{\partial \kappa}{\partial t} = A_\kappa(J), \quad \frac{\partial \kappa_s}{\partial t} = A_{\kappa_s}(J).
\]

Invariant variations:

\[
d_{\mathcal{V}} \kappa = A_\kappa(\mathcal{V}), \quad d_{\mathcal{V}} \kappa_s = A_{\kappa_s}(\mathcal{V}).
\]

$A_\kappa = A$ — invariant linearization operator of curvature;

$A_{\kappa_s} = \mathcal{D} A_\kappa + \kappa \kappa_s$ — invariant linearization operator of $\kappa_s$. 
Euclidean–invariant Curve Evolution

Normal flow: \( C_t = J \, n \)

\[
\frac{\partial \kappa}{\partial t} = A_\kappa (J) = (D^2 + \kappa^2) \, J,
\]

\[
\frac{\partial \kappa_s}{\partial t} = A_{\kappa_s} (J) = (D^3 + \kappa^2 D + 3 \kappa \kappa_s) \, J.
\]

Warning: For non-intrinsic flows, \( \partial_t \) and \( \partial_s \) do not commute!

Grassfire flow: \( J = 1 \)

\[
\frac{\partial \kappa}{\partial t} = \kappa^2, \quad \frac{\partial \kappa_s}{\partial t} = 3 \kappa \kappa_s, \quad \ldots
\]

\( \implies \) caustics
Signature Curves

Definition. The signature curve $S \subset \mathbb{R}^2$ of a curve $C \subset \mathbb{R}^2$ is parametrized by the two lowest order differential invariants

$$S = \left\{ \left( \kappa, \frac{d\kappa}{ds} \right) \right\} \subset \mathbb{R}^2$$
Equivalence and Signature Curves

**Theorem.** Two curves $C$ and $\overline{C}$ are equivalent:

$$\overline{C} = g \cdot C$$

if and only if their signature curves are identical:

$$\overline{S} = S$$

$$\implies$$ object recognition
Euclidean Signature Evolution

Evolution of the Euclidean signature curve

\[ \kappa_s = \Phi(t, \kappa). \]

Grassfire flow:

\[ \frac{\partial \Phi}{\partial t} = 3 \kappa \Phi - \kappa^2 \frac{\partial \Phi}{\partial \kappa}. \]

Curve shortening flow:

\[ \frac{\partial \Phi}{\partial t} = \Phi^2 \Phi_{\kappa \kappa} - \kappa^3 \Phi_{\kappa} + 4 \kappa^2 \Phi. \]

Modified Korteweg-deVries flow:

\[ \frac{\partial \Phi}{\partial t} = \Phi^3 \Phi_{\kappa \kappa \kappa} + 3 \Phi^2 \Phi_{\kappa} \Phi_{\kappa \kappa} + 3 \kappa \Phi^2. \]
Canine Left Ventricle Signature

Original Canine Heart MRI Image

Boundary of Left Ventricle
Smoothed Ventricle Signature

[Diagrams of smoothed ventricle signatures]

- [First diagram showing a complex ventricle shape]
- [Second diagram showing a simpler ventricle shape]
- [Third diagram showing a basic ventricle shape]

- [Fourth diagram showing a complex ventricle signature]
- [Fifth diagram showing a simpler ventricle signature]
- [Sixth diagram showing a basic ventricle signature]
Intrinsic Evolution of Differential Invariants

Theorem.

Under an arc-length preserving flow,

$$\kappa_t = \mathcal{R}(J) \quad \text{where} \quad \mathcal{R} = \mathcal{A} - \kappa_s D^{-1} \mathcal{B} \quad (\ast)$$

In surprisingly many situations, (\ast) is a well-known integrable evolution equation, and \(\mathcal{R}\) is its recursion operator!

\[\implies\text{Hasimoto}\]

\[\implies\text{Langer, Singer, Perline}\]

\[\implies\text{Marí–Beffa, Sanders, Wang}\]

\[\implies\text{Qu, Chou, and many more ...}\]
Euclidean plane curves

\[ G = \text{SE}(2) = \text{SO}(2) \rtimes \mathbb{R}^2 \]

\[ d_Y \kappa = (D^2 + \kappa^2) \vartheta, \quad d_Y \varpi = -\kappa \vartheta \wedge \varpi \]

\[ \implies A = D^2 + \kappa^2, \quad B = -\kappa \]

\[ R = A - \kappa_s D^{-1} B = D^2 + \kappa^2 + \kappa_s D^{-1} \cdot \kappa \]

\[ \kappa_t = R(\kappa_s) = \kappa_{sss} + \frac{3}{2} \kappa^2 \kappa_s \]

\[ \implies \text{modified Korteweg-deVries equation} \]
Equi-affine plane curves

\[ G = \text{SA}(2) = \text{SL}(2) \ltimes \mathbb{R}^2 \]

\[ d_V \kappa = A(\vartheta), \quad d_V \varpi = B(\vartheta) \wedge \varpi \]

\[ A = \mathcal{D}^4 + \frac{5}{3} \kappa \mathcal{D}^2 + \frac{5}{3} \kappa_s \mathcal{D} + \frac{1}{3} \kappa_{ss} + \frac{4}{9} \kappa^2, \]

\[ B = \frac{1}{3} \mathcal{D}^2 - \frac{2}{9} \kappa, \]

\[ \mathcal{R} = A - \kappa_s \mathcal{D}^{-1} B \]

\[ = \mathcal{D}^4 + \frac{5}{3} \kappa \mathcal{D}^2 + \frac{4}{3} \kappa_s \mathcal{D} + \frac{1}{3} \kappa_{ss} + \frac{4}{9} \kappa^2 + \frac{2}{9} \kappa_s \mathcal{D}^{-1} \cdot \kappa \]

\[ \kappa_t = \mathcal{R}(\kappa_s) = \kappa_{5s} + 2 \kappa \kappa_{ss} + \frac{4}{3} \kappa_s^2 + \frac{5}{9} \kappa^2 \kappa_s \]

\[ \implies \text{Sawada–Kotera equation} \]
Euclidean space curves

\[ G = \text{SE}(3) = \text{SO}(3) \ltimes \mathbb{R}^3 \]

\[
\begin{pmatrix}
  d_V \kappa \\
  d_V \tau
\end{pmatrix}
= \mathcal{A}
\begin{pmatrix}
  \vartheta_1 \\
  \vartheta_2
\end{pmatrix}
\quad d_V \varpi = \mathcal{B}
\begin{pmatrix}
  \vartheta_1 \\
  \vartheta_2
\end{pmatrix}
\wedge \varpi
\]

\[
\mathcal{A} = \begin{pmatrix}
  2\tau D_s^2 + \frac{3\kappa \tau_s - 2\kappa_s \tau}{\kappa^2} D_s + \frac{\kappa \tau_{ss} - \kappa_s \tau_s + 2\kappa^3 \tau}{\kappa^2} \\
  -2\tau D_s - \tau_s
\end{pmatrix}
\]

\[
\mathcal{B} = \begin{pmatrix}
  \frac{1}{\kappa} D_s^3 - \frac{\kappa_s}{\kappa^2} D_s^2 + \frac{\kappa^2 - \tau^2}{\kappa} D_s + \frac{\kappa_s \tau^2 - 2\kappa \tau \tau_s}{\kappa^2} \\
  \kappa
\end{pmatrix}
\]
Recursion operator:

\[ \mathcal{R} = A - \left( \begin{array}{c} \kappa_s \\ \tau_s \end{array} \right) D^{-1} B \]

\[ \begin{pmatrix} \kappa_t \\ \tau_t \end{pmatrix} = \mathcal{R} \begin{pmatrix} \kappa_s \\ \tau_s \end{pmatrix} \]

\[ \implies \text{vortex filament flow} \]

\[ \implies \text{nonlinear Schrödinger equation (Hasimoto)} \]