Moving Frames
in Applications

Peter J. Olver
University of Minnesota

http://www.math.umn.edu/~olver

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Moving Frames

Classical contributions:

M. Bartels (∼1800), J. Serret, J. Frénet, G. Darboux, É. Cotton, Élie Cartan

Modern developments: (1970’s)

S.S. Chern, M. Green, P. Griffiths, G. Jensen, . . .

The equivariant approach: (1997 – )

“I did not quite understand how he [Cartan] does this in general, though in the examples he gives the procedure is clear.”

“Nevertheless, I must admit I found the book, like most of Cartan’s papers, hard reading.”

— Hermann Weyl

“Cartan on groups and differential geometry”

Applications of Moving Frames

- Differential geometry
- Equivalence
- Symmetry
- Differential invariants
- Rigidity
- Joint invariants and semi-differential invariants
- Invariant differential forms and tensors
- Identities and syzygies
- Classical invariant theory
• Computer vision — object recognition & symmetry detection
• Invariant numerical methods
• Invariant variational problems
• Invariant submanifold flows
• Poisson geometry & solitons
• Killing tensors in relativity
• Invariants of Lie algebras in quantum mechanics
• Lie pseudo-groups
The Basic Equivalence Problem

$M$ — smooth $m$-dimensional manifold.

$G$ — transformation group acting on $M$

- finite-dimensional Lie group
- infinite-dimensional Lie pseudo-group
Equivalence:

Determine when two $p$-dimensional submanifolds

\[ N \quad \text{and} \quad \overline{N} \subset M \]

are congruent:

\[ \overline{N} = g \cdot N \quad \text{for} \quad g \in G \]

Symmetry:

Find all symmetries,

i.e., self-equivalence or self-congruences:

\[ N = g \cdot N \]
Classical Geometry — F. Klein

- **Euclidean group:**
  \[ G = \begin{cases} 
  \text{SE}(m) = \text{SO}(m) \times \mathbb{R}^m \\
  \text{E}(m) = \text{O}(m) \times \mathbb{R}^m 
  \end{cases} \]
  \[ z \mapsto A \cdot z + b \]
  \[ A \in \text{SO}(m) \text{ or } \text{O}(m), \quad b \in \mathbb{R}^m, \quad z \in \mathbb{R}^m \]
  \[ \Rightarrow \text{isometries: rotations, translations, (reflections)} \]

- **Equi-affine group:**
  \[ G = \text{SA}(m) = \text{SL}(m) \times \mathbb{R}^m \]
  \[ A \in \text{SL}(m) \text{ — volume-preserving} \]

- **Affine group:**
  \[ G = \text{A}(m) = \text{GL}(m) \times \mathbb{R}^m \]
  \[ A \in \text{GL}(m) \]

- **Projective group:**
  \[ G = \text{PSL}(m + 1) \]
  acting on \( \mathbb{R}^m \subset \mathbb{RP}^m \)
  \[ \Rightarrow \text{Applications in computer vision} \]
Tennis, Anyone?
Classical Invariant Theory

Binary form:

\[ Q(x) = \sum_{k=0}^{n} \binom{n}{k} a_k x^k \]

Equivalence of polynomials (binary forms):

\[ Q(x) = (\gamma x + \delta)^n \overline{Q} \left( \frac{\alpha x + \beta}{\gamma x + \delta} \right) \quad g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{GL}(2) \]

- multiplier representation of \text{GL}(2)
- modular forms
\[ Q(x) = (\gamma x + \delta)^n \overline{Q} \left( \frac{\alpha x + \beta}{\gamma x + \delta} \right) \]

Transformation group:
\[ g : (x, u) \mapsto \left( \frac{\alpha x + \beta}{\gamma x + \delta}, \frac{u}{(\gamma x + \delta)^n} \right) \]

Equivalence of functions \( \iff \) equivalence of graphs
\[ \Gamma_Q = \{ (x, u) = (x, Q(x)) \} \subset \mathbb{C}^2 \]
Moving Frames

Definition.

A moving frame is a $G$-equivariant map

$$\rho : M \longrightarrow G$$

Equivariance:

$$\rho(g \cdot z) = \begin{cases} 
  g \cdot \rho(z) & \text{left moving frame} \\
  \rho(z) \cdot g^{-1} & \text{right moving frame}
\end{cases}$$

$$\rho_{left}(z) = \rho_{right}(z)^{-1}$$
The Main Result

Theorem. A moving frame exists in a neighborhood of a point \( z \in M \) if and only if \( G \) acts \textcolor{red}{freely} and \textcolor{blue}{regularly} near \( z \).
**Isotropy & Freeness**

**Isotropy subgroup:** $G_z = \{ g \mid g \cdot z = z \}$ for $z \in M$

- **free** — the only group element $g \in G$ which fixes *one* point $z \in M$ is the identity
  
  \[ \implies G_z = \{e\} \text{ for all } z \in M \]

- **locally free** — the orbits all have the same dimension as $G$
  
  \[ \implies G_z \subset G \text{ is discrete for all } z \in M \]

- **regular** — the orbits form a regular foliation
  
  \[ \not\exists \text{ irrational flow on the torus} \]

- **effective** — the only group element which fixes *every* point in $M$ is the identity: $g \cdot z = z$ for all $z \in M$ iff $g = e$:

  \[ G_M^* = \bigcap_{z \in M} G_z = \{e\} \]
Geometric Construction

Normalization = choice of cross-section to the group orbits
Normalization = choice of cross-section to the group orbits
Geometric Construction

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Geometric Construction

Normalization = choice of cross-section to the group orbits
Algebraic Construction

\[ r = \dim G \leq m = \dim M \]

Coordinate cross-section

\[ K = \{ z_1 = c_1, \ldots, z_r = c_r \} \]

<table>
<thead>
<tr>
<th>left</th>
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<tr>
<td>[ w(g, z) = g^{-1} \cdot z ]</td>
<td>[ w(g, z) = g \cdot z ]</td>
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\[ g = (g_1, \ldots, g_r) \quad \text{— group parameters} \]
\[ z = (z_1, \ldots, z_m) \quad \text{— coordinates on } M \]
Choose $r = \dim G$ components to normalize:

$$w_1(g, z) = c_1 \quad \ldots \quad w_r(g, z) = c_r$$

Solve for the group parameters $g = (g_1, \ldots, g_r)$

$$\implies \text{Implicit Function Theorem}$$

The solution

$$g = \rho(z)$$

is a (local) moving frame.
The Fundamental Invariants

Substituting the moving frame formulae

\[ g = \rho(z) \]

into the unnormalized components of \( w(g, z) \) produces the fundamental invariants

\[ I_1(z) = w_{r+1}(\rho(z), z) \quad \ldots \quad I_{m-r}(z) = w_m(\rho(z), z) \]

\[ \implies \text{These are the coordinates of the canonical form } k \in K. \]
Completeness of Invariants

**Theorem.** Every invariant $I(z)$ can be (locally) uniquely written as a function of the fundamental invariants:

$$I(z) = H(I_1(z), \ldots, I_{m-r}(z))$$
Invariantization

Definition. The invariantization of a function $F : M \to \mathbb{R}$ with respect to a right moving frame $g = \rho(z)$ is the invariant function $I = \iota(F)$ defined by

$$I(z) = F(\rho(z) \cdot z).$$

\[\begin{align*}
\iota(z_1) &= c_1, \ldots \iota(z_r) = c_r, \quad \iota(z_{r+1}) = I_1(z), \ldots \iota(z_m) = I_{m-r}(z).
\end{align*}\]

cross-section variables \quad fundamental invariants

“phantom invariants”

$$\iota \left[ F(z_1, \ldots, z_m) \right] = F(c_1, \ldots, c_r, I_1(z), \ldots, I_{m-r}(z))$$
Invariantization amounts to restricting $F$ to the cross-section

$$I|K = F|K$$

and then requiring that $I = \iota(F)$ be constant along the orbits.

In particular, if $I(z)$ is an invariant, then $\iota(I) = I$.

Invariantization defines a canonical projection

$$\iota : \text{functions} \quad \rightarrow \quad \text{invariants}$$
Prolongation

Most interesting group actions (Euclidean, affine, projective, etc.) are *not* free!

Freeness typically fails because the dimension of the underlying manifold is not large enough, i.e., $m < r = \dim G$.

Thus, to make the action free, we must increase the dimension of the space via some natural prolongation procedure.

- An effective action can usually be made free by:
• Prolonging to derivatives (jet space)

\[ G^{(n)} : J^n(M, p) \longrightarrow J^n(M, p) \]

\[ \Rightarrow \text{differential invariants} \]

• Prolonging to Cartesian product actions

\[ G^{\times n} : M \times \cdots \times M \longrightarrow M \times \cdots \times M \]

\[ \Rightarrow \text{joint invariants} \]

• Prolonging to “multi-space”

\[ G^{(n)} : M^{(n)} \longrightarrow M^{(n)} \]

\[ \Rightarrow \text{joint or semi-differential invariants} \]

\[ \Rightarrow \text{invariant numerical approximations} \]
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\[ \Rightarrow \text{invariant numerical approximations} \]
Euclidean Plane Curves

Special Euclidean group: \( G = \text{SE}(2) = \text{SO}(2) \ltimes \mathbb{R}^2 \)
acts on \( M = \mathbb{R}^2 \) via rigid motions: \( w = Rz + b \)

To obtain the classical (left) moving frame we invert the group transformations:

\[
\begin{align*}
y &= \cos \phi (x - a) + \sin \phi (u - b) \\
v &= -\sin \phi (x - a) + \cos \phi (u - b)
\end{align*}
\]

\( w = R^{-1}(z - b) \)

Assume for simplicity the curve is (locally) a graph:

\[
\mathcal{C} = \{ u = f(x) \}
\]

\( \implies \) extensions to parametrized curves are straightforward
Prolong the action to $J^n$ via implicit differentiation:

$$y = \cos \phi (x - a) + \sin \phi (u - b)$$

$$v = -\sin \phi (x - a) + \cos \phi (u - b)$$

$$v_y = \frac{-\sin \phi + u_x \cos \phi}{\cos \phi + u_x \sin \phi}$$

$$v_{yy} = \frac{u_{xx}}{(\cos \phi + u_x \sin \phi)^3}$$

$$v_{yyy} = \frac{(\cos \phi + u_x \sin \phi) u_{xxx} - 3u_{xx}^2 \sin \phi}{(\cos \phi + u_x \sin \phi)^5}$$

$$\vdots$$
Prolong the action to $J^n$ via implicit differentiation:

\[ y = \cos \phi (x - a) + \sin \phi (u - b) \]
\[ v = -\sin \phi (x - a) + \cos \phi (u - b) \]
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\[ v_{yy} = \frac{u_{xx}}{(\cos \phi + u_x \sin \phi)^3} \]
\[ v_{yyy} = \frac{(\cos \phi + u_x \sin \phi) u_{xxx} - 3u_{xx}^2 \sin \phi}{(\cos \phi + u_x \sin \phi)^5} \]
\[ \vdots \]
Normalization: \( r = \dim G = 3 \)

\[
\begin{align*}
y &= \cos \phi (x - a) + \sin \phi (u - b) = 0 \\
v &= -\sin \phi (x - a) + \cos \phi (u - b) = 0 \\
v_y &= \frac{-\sin \phi + u_x \cos \phi}{\cos \phi + u_x \sin \phi} = 0 \\
v_{yy} &= \frac{u_{xx}}{(\cos \phi + u_x \sin \phi)^3} \\
v_{yyy} &= \frac{(\cos \phi + u_x \sin \phi)u_{xxx} - 3u_{xx}^2 \sin \phi}{(\cos \phi + u_x \sin \phi)^5}
\end{align*}
\]

\[\vdots\]
Solve for the group parameters:

\[ y = \cos \phi (x - a) + \sin \phi (u - b) = 0 \]
\[ v = -\sin \phi (x - a) + \cos \phi (u - b) = 0 \]
\[ v_y = \frac{-\sin \phi + u_x \cos \phi}{\cos \phi + u_x \sin \phi} = 0 \]

\[ \implies \text{Left moving frame } \rho : J^1 \rightarrow \text{SE}(2) \]
\[ a = x \quad b = u \quad \phi = \tan^{-1} u_x \]
\[ a = x \quad b = u \quad \phi = \tan^{-1} u_x \]

Differential invariants

\[ v_{yy} = \frac{u_{xx}}{(\cos \phi + u_x \sin \phi)^3} \quad \mapsto \quad \kappa = \frac{u_{xx}}{(1 + u_x^2)^{3/2}} \]

\[ v_{yyy} = \cdots \quad \mapsto \quad \frac{d\kappa}{ds} = \frac{(1 + u_x^2)u_{xxx} - 3u_x u_{xx}^2}{(1 + u_x^2)^3} \]

\[ v_{yyyy} = \cdots \quad \mapsto \quad \frac{d^2\kappa}{ds^2} - 3\kappa^3 = \cdots \quad \mapsto \quad \text{recurrence formulae} \]

Contact invariant one-form — arc length

\[ dy = (\cos \phi + u_x \sin \phi) \, dx \quad \mapsto \quad ds = \sqrt{1 + u_x^2} \, dx \]
Dual invariant differential operator

\[ \frac{d}{dy} = \frac{1}{\cos \phi + u_x \sin \phi} \frac{d}{dx} \]

\[ \frac{d}{ds} = \frac{1}{\sqrt{1 + u_x^2}} \frac{d}{dx} \]

Theorem. All differential invariants are functions of the derivatives of curvature with respect to arc length:

\[ \kappa, \quad \frac{d\kappa}{ds}, \quad \frac{d^2\kappa}{ds^2}, \quad \cdots \]
The Classical Picture:

Moving frame $\rho : (x, u, u_x) \mapsto (R, a) \in \text{SE}(2)$

$$R = \frac{1}{\sqrt{1 + u_x^2}} \begin{pmatrix} 1 & -u_x \\ u_x & 1 \end{pmatrix} = (t, n) \quad a = \begin{pmatrix} x \\ u \end{pmatrix}$$
Frenet frame
\[ t = \frac{dx}{ds} = \begin{pmatrix} x_s \\ y_s \end{pmatrix}, \quad n = t^\perp = \begin{pmatrix} -y_s \\ x_s \end{pmatrix}. \]

Frenet equations = Pulled-back Maurer–Cartan forms:
\[ \frac{dx}{ds} = t, \quad \frac{dt}{ds} = \kappa n, \quad \frac{dn}{ds} = -\kappa t. \]
Equi-affine Curves \( G = SA(2) \)

\[ z \mapsto A z + b \quad A \in SL(2), \quad b \in \mathbb{R}^2 \]

Invert for left moving frame:

\[
\begin{align*}
y &= \delta (x - a) - \beta (u - b) \\
v &= -\gamma (x - a) + \alpha (u - b) \\
\alpha \delta - \beta \gamma &= 1
\end{align*}
\]

Prolong to \( J^3 \) via implicit differentiation

\[
dy = (\delta - \beta u_x) \, dx \\
D_y = \frac{1}{\delta - \beta u_x} \, D_x
\]
Prolongation:

\[ y = \delta (x - a) - \beta (u - b) \]

\[ v = -\gamma (x - a) + \alpha (u - b) \]

\[ v_y = -\frac{\gamma - \alpha u_x}{\delta - \beta u_x} \]

\[ v_{yy} = -\frac{u_{xx}}{(\delta - \beta u_x)^3} \]

\[ v_{yyy} = -\frac{(\delta - \beta u_x) u_{xxx} + 3 \beta u_{xx}^2}{(\delta - \beta u_x)^5} \]

\[ v_{yyyy} = -\frac{u_{xxxx}(\delta - \beta u_x)^2 + 10 \beta (\delta - \beta u_x) u_{xx} u_{xxx} + 15 \beta^2 u_{xx}^3}{(\delta - \beta u_x)^7} \]

\[ v_{yyyyy} = \ldots \]
Normalization: \( r = \dim G = 5 \)

\[
y = \delta (x - a) - \beta (u - b) = 0
\]

\[
v = -\gamma (x - a) + \alpha (u - b) = 0
\]

\[
v_y = -\frac{\gamma - \alpha u_x}{\delta - \beta u_x} = 0
\]

\[
v_{yy} = -\frac{u_{xx}}{(\delta - \beta u_x)^3} = 1
\]

\[
v_{yyy} = -\frac{(\delta - \beta u_x) u_{xxx} + 3 \beta u_x^2}{(\delta - \beta u_x)^5} = 0
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\]

\[
v_{yyyyy} = \ldots
\]
Equi-affine Moving Frame

\[ \rho : (x, u, u_x, u_{xx}, u_{xxx}) \rightarrow (A, b) \in \text{SA}(2) \]

\[
A = \begin{pmatrix}
\alpha & \beta \\
\gamma & \delta
\end{pmatrix} = \begin{pmatrix}
\sqrt[3]{u_{xx}} & -\frac{1}{3} u_{xx}^{-5/3} u_{xxx} \\
\frac{1}{u_x} \sqrt[3]{u_{xx}} & u_{xx}^{-1/3} - \frac{1}{3} u_{xx}^{-5/3} u_{xxx}
\end{pmatrix}
\]

\[
b = \begin{pmatrix}
a \\
b
\end{pmatrix} = \begin{pmatrix}
x \\
u
\end{pmatrix}
\]

Nondegeneracy condition: \( u_{xx} \neq 0 \).
**Totally Singular Submanifolds**

**Definition.** A $p$-dimensional submanifold $N \subset M$ is **totally singular** if $G^{(n)}$ does not act freely on $j_n N$ for any $n \geq 0$.

**Theorem.** $N$ is totally singular if and only if its symmetry group $G_N = \{ g \mid g \cdot N \subset N \}$ has dimension $> p$, and so $G_N$ does not act freely on $N$ itself.

Thus, the totally singular submanifolds are the only ones that do not admit a moving frame of any order.

In equi-affine geometry, only the straight lines ($u_{xx} \equiv 0$) are totally singular since they admit a three-dimensional equi-affine symmetry group.
Equi-affine arc length

\[ dy = (\delta - \beta u_x) \, dx \quad \mapsto \quad ds = \sqrt[3]{u_{xx}} \, dx \]

Equi-affine curvature

\[ v_{yyyy} \quad \mapsto \quad \kappa = \frac{5 u_{xx} u_{xxxx} - 3 u_{xx}^2}{9 u_{xx}^{8/3}} \]

\[ v_{yyyyy} \quad \mapsto \quad \frac{d\kappa}{ds} \]

\[ v_{yyyyyy} \quad \mapsto \quad \frac{d^2\kappa}{ds^2} - 5\kappa^2 \]
The Classical Picture:

\[ A = \begin{pmatrix} \sqrt[3]{u_{xx}} & -\frac{1}{3} u^{-5/3} u_{xxx} \\ u_x \sqrt[3]{u_{xx}} & u_{xx}^{-1/3} - \frac{1}{3} u_{xx}^{-5/3} u_{xxx} \end{pmatrix} = (t, n) \quad b = \begin{pmatrix} x \\ u \end{pmatrix} \]
Frenet frame

\[ t = \frac{dz}{ds}, \quad n = \frac{d^2 z}{ds^2}. \]

Frenet equations = Pulled-back Maurer–Cartan forms:

\[ \frac{dz}{ds} = t, \quad \frac{dt}{ds} = n, \quad \frac{dn}{ds} = \kappa t. \]
Equivalence & Invariants

• Equivalent submanifolds $N \approx \bar{N}$ must have the same invariants: $I = \bar{I}$. 
Equivalence & Invariants

- Equivalent submanifolds \( N \approx \overline{N} \)
must have the same invariants: \( I = \overline{I} \).

Constant invariants provide immediate information:

\[ \text{e.g. } \quad \kappa = 2 \iff \overline{\kappa} = 2 \]
Equivalence & Invariants

- Equivalent submanifolds \( N \approx \overline{N} \)
must have the same invariants: \( I = \overline{I} \).

Constant invariants provide immediate information:

\[
\text{e.g.} \quad \kappa = 2 \iff \overline{\kappa} = 2
\]

Non-constant invariants are not useful in isolation, because an equivalence map can drastically alter the dependence on the submanifold parameters:

\[
\text{e.g.} \quad \kappa = x^3 \quad \text{versus} \quad \overline{\kappa} = \sinh x
\]
However, a functional dependency or *syzygy* among the invariants *is* intrinsic:

\[ \kappa_s = \kappa^3 - 1 \iff \overline{\kappa}_{\overline{s}} = \overline{\kappa}^3 - 1 \]
However, a functional dependency or syzygy among the invariants is intrinsic:

\[ \kappa_s = \kappa^3 - 1 \iff \bar{\kappa}_{\bar{s}} = \bar{\kappa}^3 - 1 \]

- Universal syzygies — Gauss–Codazzi
- Distinguishing syzygies.
However, a functional dependency or syzygy among the invariants is intrinsic:

\[ \kappa_s = \kappa^3 - 1 \iff \overline{\kappa_s} = \overline{\kappa^3 - 1} \]

- Universal syzygies — Gauss–Codazzi
- Distinguishing syzygies.

---

**Theorem.** (Cartan) Two submanifolds are (locally) equivalent if and only if they have identical syzygies among all their differential invariants.
Finiteness of Generators and Syzygies

♠ There are, in general, an infinite number of differential invariants and hence an infinite number of syzygies must be compared to establish equivalence.

♥ But the higher order syzygies are all consequences of a finite number of low order syzygies!
Example — Plane Curves

If non-constant, both $\kappa$ and $\kappa_s$ depend on a single parameter, and so, locally, are subject to a syzygy:

$$\kappa_s = H(\kappa)$$  \hspace{1cm} (*)

But then

$$\kappa_{ss} = \frac{d}{ds} H(\kappa) = H'(\kappa) \kappa_s = H'(\kappa) H(\kappa)$$

and similarly for $\kappa_{sss}$, etc.

Consequently, all the higher order syzygies are generated by the fundamental first order syzygy (*)

Thus, for Euclidean (or equi-affine or projective or ... ) plane curves we need only know a single syzygy between $\kappa$ and $\kappa_s$ in order to establish equivalence!
The Basis Theorem

**Theorem.** The differential invariant algebra $\mathcal{I}$ is generated by a finite number of differential invariants

$$I_1, \ldots, I_\ell$$

and $p = \dim N$ invariant differential operators

$$\mathcal{D}_1, \ldots, \mathcal{D}_p$$

meaning that *every* differential invariant can be locally expressed as a function of the generating invariants and their invariant derivatives:

$$\mathcal{D}_j I_\kappa = \mathcal{D}_{j_1} \mathcal{D}_{j_2} \cdots \mathcal{D}_{j_n} I_\kappa.$$

$$\implies \text{Lie, Tresse, Ovsiannikov, Kumpera}$$

★ Moving frames provides a constructive proof.
Signature Curves

**Definition.** The *signature curve* $S \subset \mathbb{R}^2$ of a curve $C \subset \mathbb{R}^2$ is parametrized by the two lowest order differential invariants

$$S = \left\{ \left( \kappa, \frac{d\kappa}{ds} \right) \right\} \subset \mathbb{R}^2$$
Definition. The signature curve $S \subset \mathbb{R}^2$ of a curve $C \subset \mathbb{R}^2$ is parametrized by the two lowest order differential invariants

$$S = \left\{ \left( \kappa, \frac{d\kappa}{ds} \right) \right\} \subset \mathbb{R}^2$$

Theorem. Two regular curves $C$ and $\overline{C}$ are equivalent:

$$\overline{C} = g \cdot C$$

if and only if their signature curves are identical:

$$\overline{S} = S$$
Symmetry and Signature

Theorem. The dimension of the symmetry group

$$G_N = \{ g \mid g \cdot N \subset N \}$$

of a nonsingular submanifold $N \subset M$ equals the codimension of its signature:

$$\dim G_N = \dim N - \dim S$$

Corollary. For a nonsingular submanifold $N \subset M$,

$$0 \leq \dim G_N \leq \dim N$$

$$\implies$$ Only totally singular submanifolds can have larger symmetry groups!
Maximally Symmetric Submanifolds

**Theorem.** The following are equivalent:

- The submanifold $N$ has a $p$-dimensional symmetry group
- The signature $S$ degenerates to a point: $\dim S = 0$
- The submanifold has all constant differential invariants
- $N = H \cdot \{ z_0 \}$ is the orbit of a $p$-dimensional subgroup $H \subset G$

$\implies \quad$ **Euclidean geometry:** circles, lines, helices, spheres, cylinders, planes, ...

$\implies \quad$ **Equi-affine plane geometry:** conic sections.

$\implies \quad$ **Projective plane geometry:** $W$ curves ($\text{Lie} \ & \text{Klein}$)
Discrete Symmetries

Definition. The index of a submanifold $N$ equals the number of points in $N$ which map to a generic point of its signature:

$$i_N = \min \left\{ \# \Sigma^{-1}\{w\} \mid w \in S \right\}$$

$\Rightarrow$ Self-intersections

Theorem. The cardinality of the symmetry group of a submanifold $N$ equals its index $i_N$.

$\Rightarrow$ Approximate symmetries
The Index

$\Sigma \rightarrow \mathcal{N} \rightarrow \mathcal{S}$
The Curve \( x = \cos t + \frac{1}{5} \cos^2 t, \quad y = \sin t + \frac{1}{10} \sin^2 t \)

The Original Curve  

Euclidean Signature  

Affine Signature
The Curve \[ x = \cos t + \frac{1}{5} \cos^2 t, \quad y = \frac{1}{2} x + \sin t + \frac{1}{10} \sin^2 t \]

The Original Curve  
Euclidean Signature  
Affine Signature
Canine Left Ventricle Signature

Original Canine Heart MRI Image

Boundary of Left Ventricle
Smoothed Ventricle Signature
Evolution of Invariants and Signatures

Basic question: If the submanifold evolves according to an invariant evolution equation, how do its differential invariants & signatures evolve?

Theorem. Under the curve shortening flow $C_t = -\kappa \mathbf{n}$, the signature curve $\kappa_s = H(t, \kappa)$ evolves according to the parabolic equation

$$\frac{\partial H}{\partial t} = H^2 H_{\kappa\kappa} - \kappa^3 H_\kappa + 4 \kappa^2 H$$
Signature Metrics

- Hausdorff
- Monge–Kantorovich transport
- Electrostatic repulsion
- Latent semantic analysis
- Histograms
- Gromov–Hausdorff & Gromov–Wasserstein
Signatures

Original curve

Classical Signature

Differential invariant signature
Signatures

Original curve

Classical Signature

Differential invariant signature
Occlusions

Original curve

Classical Signature

Differential invariant signature
The Baffler Solved

⇒ Dan Hoff
Classical Invariant Theory

\[ M = \mathbb{R}^2 \setminus \{ u = 0 \} \]

\[ G = \text{GL}(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mid \Delta = \alpha \delta - \beta \gamma \neq 0 \right\} \]

\[ (x, u) \mapsto \left( \frac{\alpha x + \beta}{\gamma x + \delta}, \frac{u}{(\gamma x + \delta)^n} \right) \quad n \neq 0, 1 \]
Prolongation:

\[ y = \frac{\alpha x + \beta}{\gamma x + \delta} \quad \sigma = \gamma x + \delta \]

\[ v = \sigma^{-n} u \quad \Delta = \alpha \delta - \beta \gamma \]

\[ v_y = \frac{\sigma u_x - n \gamma u}{\Delta \sigma^{n-1}} \]

\[ v_{yy} = \frac{\sigma^2 u_{xx} - 2(n - 1) \gamma \sigma u_x + n(n - 1) \gamma^2 u}{\Delta^2 \sigma^{n-2}} \]

\[ v_{yyy} = \ldots \]
Normalization:

\[ y = \frac{\alpha x + \beta}{\gamma x + \delta} = 0 \]

\[ v = \sigma^{-n} u = 1 \]

\[ v_y = \frac{\sigma u_x - n \gamma u}{\Delta \sigma^{n-1}} = 0 \]

\[ v_{yy} = \frac{\sigma^2 u_{xx} - 2(n-1) \gamma \sigma u_x + n(n-1) \gamma^2 u}{\Delta^2 \sigma^{n-2}} = \frac{1}{n(n-1)} \]

\[ v_{yyy} = \ldots \]
Moving frame:
\[
\alpha = u^{(1-n)/n} \sqrt{H} \quad \beta = -x u^{(1-n)/n} \sqrt{H}
\]
\[
\gamma = \frac{1}{n} u^{(1-n)/n} \quad \delta = u^{1/n} - \frac{1}{n} x u^{(1-n)/n}
\]

Hessian:
\[
H = n(n - 1) u u_{xx} - (n - 1)^2 u_x^2 \neq 0
\]

Note: \( H \equiv 0 \) if and only if \( Q(x) = (a x + b)^n \) \( \implies \) Totally singular forms

Differential invariants:
\[
v_{yyy} \mapsto \frac{J}{n^2(n - 1)} \approx \kappa \quad v_{yyyy} \mapsto \frac{K + 3(n - 2)}{n^3(n - 1)} \approx \frac{d\kappa}{ds}
\]
Absolute rational covariants:

\[ J^2 = \frac{T^2}{H^3} \quad K = \frac{U}{H^2} \]

\[ H = \frac{1}{2}(Q, Q)^{(2)} = n(n - 1)QQ'' - (n - 1)^2Q'^2 \sim Q_{xx}Q_{yy} - Q_{xy}^2 \]

\[ T = (Q, H)^{(1)} = (2n - 4)Q'H - nQH' \sim Q_xH_y - Q_yH_x \]

\[ U = (Q, T)^{(1)} = (3n - 6)Q'T - nQT' \sim Q_xT_y - Q_yT_x \]

\[ \text{deg } Q = n \quad \text{deg } H = 2n - 4 \quad \text{deg } T = 3n - 6 \quad \text{deg } U = 4n - 8 \]
Signatures of Binary Forms

Signature curve of a nonsingular binary form $Q(x)$:

$$S_Q = \left\{ (J(x)^2, K(x)) = \left( \frac{T(x)^2}{H(x)^3}, \frac{U(x)}{H(x)^2} \right) \right\}$$

Nonsingular: $H(x) \neq 0$ and $(J'(x), K'(x)) \neq 0$.

Theorem. Two nonsingular binary forms are equivalent if and only if their signature curves are identical.
Maximally Symmetric Binary Forms

Theorem. If \( u = Q(x) \) is a polynomial, then the following are equivalent:

- \( Q(x) \) admits a one-parameter symmetry group
- \( T^2 \) is a constant multiple of \( H^3 \)
- \( Q(x) \simeq x^k \) is complex-equivalent to a monomial
- the signature curve degenerates to a single point
- all the (absolute) differential invariants of \( Q \) are constant
- the graph of \( Q \) coincides with the orbit of a one-parameter subgroup
**Symmetries of Binary Forms**

**Theorem.** The symmetry group of a nonzero binary form $Q(x) \neq 0$ of degree $n$ is:

- A two-parameter group if and only if $H \equiv 0$ if and only if $Q$ is equivalent to a constant. $\Rightarrow$ totally singular
- A one-parameter group if and only if $H \neq 0$ and $T^2 = cH^3$ if and only if $Q$ is complex-equivalent to a monomial $x^k$, with $k \neq 0, n$. $\Rightarrow$ maximally symmetric
- In all other cases, a finite group whose cardinality equals the index of the signature curve, and is bounded by

$$\nu_Q \leq \begin{cases} 
6n - 12 & \quad U = cH^2 \\
4n - 8 & \quad \text{otherwise}
\end{cases}$$
Symmetry–Preserving Numerical Methods

• Invariant numerical approximations to differential invariants.

• Invariantization of numerical integration methods.

⇒ Structure-preserving algorithms
Numerical approximation to curvature

Heron’s formula

$$\tilde{\kappa}(A, B, C) = 4 \frac{\Delta}{abc} = 4 \frac{\sqrt{s(s-a)(s-b)(s-c)}}{abc}$$

$$s = \frac{a + b + c}{2} \quad \text{— semi-perimeter}$$
Invariantization of Numerical Schemes

Pilwon Kim

Suppose we are given a numerical scheme for integrating a differential equation, e.g., a Runge–Kutta Method for ordinary differential equations, or the Crank–Nicolson method for parabolic partial differential equations.

If $G$ is a symmetry group of the differential equation, then one can use an appropriately chosen moving frame to invariantize the numerical scheme, leading to an invariant numerical scheme that preserves the symmetry group. In challenging regimes, the resulting invariantized numerical scheme can, with an inspired choice of moving frame, perform significantly better than its progenitor.
Invariant Runge–Kutta schemes

\[ u_{xx} + x u_x - (x + 1)u = \sin x, \quad u(0) = u_x(0) = 1. \]
Comparison of symmetry reduction and invariantization for

\[ u_{xx} + xu_x - (x + 1)u = \sin x, \quad u(0) = u_x(0) = 1. \]
Invariantization of Crank–Nicolson for Burgers’ Equation

\[ u_t = \varepsilon u_{xx} + u u_x \]
The Calculus of Variations

\[ \mathcal{I}[u] = \int L(x, u^{(n)}) \, dx \quad \text{— variational problem} \]

\[ L(x, u^{(n)}) \quad \text{— Lagrangian} \]

To construct the Euler-Lagrange equations: \( \mathbf{E}(L) = 0 \)

- Take the first variation:

\[ \delta(L \, dx) = \sum_{\alpha, J} \frac{\partial L}{\partial u_\alpha^J} \delta u_\alpha^J \, dx \]

- Integrate by parts:

\[ \delta(L \, dx) = \sum_{\alpha, J} \frac{\partial L}{\partial u_\alpha^J} D_J(\delta u^\alpha) \, dx \]

\[ \equiv \sum_{\alpha, J} (-D)^J \frac{\partial L}{\partial u_\alpha^J} \delta u^\alpha \, dx = \sum_{\alpha=1}^{q} \mathbf{E}_\alpha(L) \delta u^\alpha \, dx \]
Invariant Variational Problems

According to Lie, any $G$–invariant variational problem can be written in terms of the differential invariants:

\[
I[u] = \int L(x, u^{(n)}) \, dx = \int P( \ldots D_K I^\alpha \ldots ) \, \omega
\]

\( I^1, \ldots, I^\ell \) — fundamental differential invariants

\( D_1, \ldots, D_p \) — invariant differential operators

\( D_K I^\alpha \) — differentiated invariants

\( \omega = \omega^1 \wedge \cdots \wedge \omega^p \) — invariant volume form
If the variational problem is $G$-invariant, so

$$ \mathcal{I}[u] = \int L(x, u^{(n)}) \, dx = \int P( \ldots \mathcal{D}_K I^\alpha \ldots ) \, \omega $$

then its Euler–Lagrange equations admit $G$ as a symmetry group, and hence can also be expressed in terms of the differential invariants:

$$ \mathbf{E}(L) \simeq F( \ldots \mathcal{D}_K I^\alpha \ldots ) = 0 $$

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**Main Problem:**

Construct $F$ directly from $P$.

(P. Griffiths, I. Anderson)
Planar Euclidean group $G = \text{SE}(2)$

\[
\kappa = \frac{u_{xx}}{(1 + u_x^2)^{3/2}} \quad \text{— curvature (differential invariant)}
\]

\[
ds = \sqrt{1 + u_x^2} \, dx \quad \text{— arc length}
\]

\[
\mathcal{D} = \frac{d}{ds} = \frac{1}{\sqrt{1 + u_x^2}} \frac{d}{dx} \quad \text{— arc length derivative}
\]

Euclidean–invariant variational problem

\[
\mathcal{I}[u] = \int L(x, u^{(n)}) \, dx = \int P(\kappa, \kappa_s, \kappa_{ss}, \ldots) \, ds
\]

Euler-Lagrange equations

\[
\mathbf{E}(L) \simeq F(\kappa, \kappa_s, \kappa_{ss}, \ldots) = 0
\]
Euclidean Curve Examples

Minimal curves (geodesics):

\[ I[u] = \int ds = \int \sqrt{1 + u_x^2} \, dx \]

\[ E(L) = -\kappa = 0 \]

\[ \implies \text{straight lines} \]

The Elastica (Euler):

\[ I[u] = \int \frac{1}{2} \kappa^2 \, ds = \int \frac{u_{xx}^2 \, dx}{(1 + u_x^2)^{5/2}} \]

\[ E(L) = \kappa_{ss} + \frac{1}{2} \kappa^3 = 0 \]

\[ \implies \text{elliptic functions} \]
General Euclidean–invariant variational problem

\[ \mathcal{I}[u] = \int L(x, u^{(n)}) \, dx = \int P(\kappa, \kappa_s, \kappa_{ss}, \ldots) \, ds \]

To construct the invariant Euler-Lagrange equations:

Take the first variation:

\[ \delta(P \, ds) = \sum_j \left( \frac{\partial P}{\partial \kappa_j} \delta \kappa_j \, ds + P \delta(ds) \right) \]

Invariant variation of curvature:

\[ \delta \kappa = A_\kappa(\delta u) \quad A_\kappa = D^2 + \kappa^2 \]

Invariant variation of arc length:

\[ \delta(ds) = B(\delta u) \, ds \quad B = -\kappa \]

\[ \implies \text{moving frame recurrence formulae} \]
The Recurrence Formula

For any function or differential form $\Omega$:

$$d \iota(\Omega) = \iota(d \Omega) + \sum_{k=1}^{r} \nu^k \wedge \iota[v_k(\Omega)]$$

$\mathbf{v}_1, \ldots, \mathbf{v}_r$ — basis for $\mathfrak{g}$ — infinitesimal generators

$\nu^1, \ldots, \nu^r$ — dual invariantized Maurer–Cartan forms

★★ The $\nu^k$ are uniquely determined by the recurrence formulae for the phantom differential invariants
\[
d ι(Ω) = ι(dΩ) + \sum_{k=1}^{r} \nu^k \wedge ι[v_k(Ω)]
\]

★★★ All identities, commutation formulae, syzygies, etc., among differential invariants and, more generally, the invariant variational bicomplex follow from this universal recurrence formula by letting Ω range over the basic functions and differential forms!
\[ d \iota(\Omega) = \iota(d\Omega) + \sum_{k=1}^{r} \nu^k \wedge \iota[v_k(\Omega)] \]

★★★ All identities, commutation formulae, syzygies, etc., among differential invariants and, more generally, the invariant variational bicomplex follow from this universal recurrence formula by letting \( \Omega \) range over the basic functions and differential forms!

★★★ Therefore, the entire structure of the differential invariant algebra and invariant variational bicomplex can be completely determined using only linear differential algebra; this does not require explicit formulas for the moving frame, the differential invariants, the invariant differential forms, or the group transformations!
Integrate by parts:
\[
\delta(P \, ds) \equiv \left[ \mathcal{E}(P) \, A(\delta u) - \mathcal{H}(P) \, B(\delta u) \right] \, ds \\
\equiv \left[ A^* \mathcal{E}(P) - B^* \mathcal{H}(P) \right] \, \delta u \, ds = \mathbf{E}(L) \, \delta u \, ds
\]

Invariantized Euler–Lagrange expression
\[
\mathcal{E}(P) = \sum_{n=0}^{\infty} (-\mathcal{D})^n \frac{\partial P}{\partial \kappa_n} \quad \quad \mathcal{D} = \frac{d}{ds}
\]

Invariantized Hamiltonian
\[
\mathcal{H}(P) = \sum_{i>j} \kappa_{i-j} \, (-\mathcal{D})^j \frac{\partial P}{\partial \kappa_i} - P
\]

Euclidean–invariant Euler-Lagrange formula
\[
\mathbf{E}(L) = A^* \mathcal{E}(P) - B^* \mathcal{H}(P) = (\mathcal{D}^2 + \kappa^2) \, \mathcal{E}(P) + \kappa \, \mathcal{H}(P) = 0.
\]
The Elastica:

\[ \mathcal{I}[u] = \int \frac{1}{2} \kappa^2 \, ds \quad P = \frac{1}{2} \kappa^2 \]

\[ \mathcal{E}(P) = \kappa \quad \mathcal{H}(P) = -P = -\frac{1}{2} \kappa^2 \]

\[ \mathbf{E}(L) = (D^2 + \kappa^2) \kappa + \kappa \left( -\frac{1}{2} \kappa^2 \right) = \kappa_{ss} + \frac{1}{2} \kappa^3 = 0 \]
The Möbius strip, obtained by taking a rectangular strip of plastic or paper, twisting one end through $180^\circ$, and then joining the ends, is the canonical example of a one-sided surface. Finding its characteristic developable shape has been an open problem ever since its first formulation in refs 1,2. Here we use the invariant variational bicomplex formalism to derive the first equilibrium equations for a wide developable strip undergoing large deformations, thereby giving the first non-trivial demonstration of the potential of this approach. We then formulate the boundary-value problem for the Möbius strip and solve it numerically. Solutions for increasing width show the formation of creases bounding nearly flat triangular regions, a feature also familiar from fabric draping and paper crumpling. This could give new insight into energy localization phenomena in unstretchable sheets, which might help to predict points of onset of tearing. It could also aid our understanding of the relationship between geometry and physical properties of nano- and microscopic Möbius strip structures.

It is fair to say that the Möbius strip is one of the few icons of mathematics that have been absorbed into wider culture. It has mathematical beauty and inspired artists such as Escher. In engineering, pulley belts are often used in the form of Möbius strips to wear 'both' sides equally. At a much smaller scale, Möbius strips have recently been formed in ribbon-shaped NbSe$_2$ crystals under certain growth conditions involving a large temperature gradient.

Figure 1 Photo of a paper Möbius strip of aspect ratio 2π. The strip adopts a characteristic shape. Inextensibility of the material causes the surface to be developable. Its straight generators are drawn and the colouring varies according to the bending energy density.
Figure 2 Computed Möbius strips. The left panel shows their three-dimensional shapes for $w = 0.1$ (a), 0.2 (b), 0.5 (c), 0.8 (d), 1.0 (e) and 1.5 (f), and the right panel the corresponding developments on the plane. The colouring changes according to the local bending energy density, from violet for regions of low bending to red for regions of high bending (scales are individually adjusted). Solution c may be compared with the paper model in Fig. 1 on which the generator field and density colouring have been printed.
Evolution of Invariants and Signatures

$G$ — Lie group acting on $\mathbb{R}^2$

$C(t)$ — parametrized family of plane curves

$G$–invariant curve flow:

$$\frac{dC}{dt} = \mathbf{V} = I \mathbf{t} + J \mathbf{n}$$

- $I, J$ — differential invariants
- $t$ — “unit tangent”
- $n$ — “unit normal”

- The tangential component $I \mathbf{t}$ only affects the underlying parametrization of the curve. Thus, we can set $I$ to be anything we like without affecting the curve evolution.
Normal Curve Flows

$C_t = J \mathbf{n}$

Examples — Euclidean–invariant curve flows

• $C_t = \mathbf{n}$ — geometric optics or grassfire flow;
• $C_t = \kappa \mathbf{n}$ — curve shortening flow;
• $C_t = \kappa^{1/3} \mathbf{n}$ — equi-affine invariant curve shortening flow:
  $C_t = \mathbf{n}_{\text{equi-affine}}$;
• $C_t = \kappa_s \mathbf{n}$ — modified Korteweg–deVries flow;
• $C_t = \kappa_{ss} \mathbf{n}$ — thermal grooving of metals.
Intrinsic Curve Flows

**Theorem.** The curve flow generated by

$$\mathbf{v} = I \mathbf{t} + J \mathbf{n}$$

preserves arc length if and only if

$$\mathcal{B}(J) + \mathcal{D} I = 0.$$  \[1\]

**Terms:**

- $\mathcal{D}$ — invariant arc length derivative
- $\mathcal{B}$ — invariant arc length variation

They satisfy

$$\delta(ds) = \mathcal{B}(\delta u) \, ds$$
Normal Evolution of Differential Invariants

**Theorem.** Under a normal flow $C_t = J n$,

$$
\frac{\partial \kappa}{\partial t} = A_\kappa(J), \quad \frac{\partial \kappa_s}{\partial t} = A_{\kappa_s}(J).
$$

Invariant variations:

$$
\delta \kappa = A_\kappa(\delta u), \quad \delta \kappa_s = A_{\kappa_s}(\delta u).
$$

$A_\kappa = A$ — invariant variation of curvature;

$A_{\kappa_s} = \mathcal{D}A + \kappa \kappa_s$ — invariant variation of $\kappa_s$. 
Euclidean–invariant Curve Evolution

Normal flow: \( C_t = J \mathbf{n} \)

\[
\frac{\partial \kappa}{\partial t} = A_\kappa(J) = (D^2 + \kappa^2)J,
\]

\[
\frac{\partial \kappa_s}{\partial t} = A_{\kappa_s}(J) = (D^3 + \kappa^2D + 3\kappa\kappa_s)J.
\]

**Warning:** For non-intrinsic flows, \( \partial_t \) and \( \partial_s \) do not commute!

**Theorem.** Under the curve shortening flow \( C_t = -\kappa \mathbf{n} \), the signature curve \( \kappa_s = H(t, \kappa) \) evolves according to the parabolic equation

\[
\frac{\partial H}{\partial t} = H^2 H_{\kappa\kappa} - \kappa^3 H_\kappa + 4\kappa^2 H
\]
Smoothed Ventricle Signature
Intrinsic Evolution of Differential Invariants

Theorem.

Under an arc-length preserving flow,

\[ \kappa_t = \mathcal{R}(J) \quad \text{where} \quad \mathcal{R} = A - \kappa_s D^{-1} B \quad (\ast) \]

In surprisingly many situations, (\ast) is a well-known integrable evolution equation, and \( \mathcal{R} \) is its recursion operator!

\[ \implies \quad \text{Hasimoto} \]
\[ \implies \quad \text{Langer, Singer, Perline} \]
\[ \implies \quad \text{Marí–Beffa, Sanders, Wang} \]
\[ \implies \quad \text{Qu, Chou, Anco, and many more ...} \]
Euclidean plane curves

\[ G = \text{SE}(2) = \text{SO}(2) \rtimes \mathbb{R}^2 \]

\[ \mathcal{A} = D^2 + \kappa^2 \quad \mathcal{B} = -\kappa \]

\[ \mathcal{R} = \mathcal{A} - \kappa_s D^{-1} \mathcal{B} = D^2 + \kappa^2 + \kappa_s D^{-1} \cdot \kappa \]

\[ \kappa_t = \mathcal{R}(\kappa_s) = \kappa_{sss} + \frac{3}{2} \kappa^2 \kappa_s \]

\[ \implies \text{modified Korteweg-deVries equation} \]
Equi-affine plane curves

\[ G = SA(2) = SL(2) \times \mathbb{R}^2 \]

\[ A = D^4 + \frac{5}{3} \kappa D^2 + \frac{5}{3} \kappa_s D + \frac{1}{3} \kappa_{ss} + \frac{4}{9} \kappa^2 \]

\[ B = \frac{1}{3} D^2 - \frac{2}{9} \kappa \]

\[ R = A - \kappa_s D^{-1} B \]

\[ = D^4 + \frac{5}{3} \kappa D^2 + \frac{4}{3} \kappa_s D + \frac{1}{3} \kappa_{ss} + \frac{4}{9} \kappa^2 + \frac{2}{9} \kappa_s D^{-1} \cdot \kappa \]

\[ \kappa_t = R(\kappa_s) = \kappa_{5s} + \frac{5}{3} \kappa \kappa_{sss} + \frac{5}{3} \kappa_s \kappa_{ss} + \frac{5}{9} \kappa^2 \kappa_s \]

\[ \implies \text{Sawada–Kotera equation} \]

Recursion operator:

\[ \widehat{R} = R \cdot (D^2 + \frac{1}{3} \kappa + \frac{1}{3} \kappa_s D^{-1}) \]
Euclidean space curves

\[ G = \text{SE}(3) = \text{SO}(3) \ltimes \mathbb{R}^3 \]

\[ A = \begin{pmatrix}
D_s^2 + (\kappa^2 - \tau^2) \\
\frac{2\tau}{\kappa} D_s^2 + \frac{3\kappa \tau s - 2\kappa s \tau}{\kappa^2} D_s + \frac{\kappa \tau s s - \kappa s \tau s + 2\kappa^3 \tau}{\kappa^2} \\
-2\tau D_s - \tau_s
\end{pmatrix} \]

\[ B = \begin{pmatrix}
\kappa \\
0
\end{pmatrix} \]

\[ \mathcal{R} = A - \begin{pmatrix}
\kappa_s \\
\tau_s
\end{pmatrix} \mathcal{D}^{-1} B \]

\[ \begin{pmatrix}
\kappa_t \\
\tau_t
\end{pmatrix} = \mathcal{R} \begin{pmatrix}
\kappa_s \\
\tau_s
\end{pmatrix} \]

\[ \implies \text{vortex filament flow (Hasimoto)} \]