Emmy Noether: Symmetry and Conservation; History and Impact

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References

[1] E. Noether, Invariante Variationsprobleme, *Nachr. König. Gesell. Wissen. Göttingen, Math.–Phys. Kl.* (1918) 235–257.

[2] P.J. Olver, *Applications of Lie Groups to Differential Equations*, Second Edition, Graduate Texts in Math., vol. 107, Springer–Verlag, New York, 1993.

[3] Y. Kosmann-Schwarzbach, *The Noether Theorems. Invariance and Conservation Laws in the Twentieth Century*, Springer, New York, 2011.

Includes English translation of [1]

Amalie Emmy Noether
(1882–1935)

Fraulein Noether was the most significant creative mathematician thus far produced since the higher education of women began.

— Albert Einstein, obituary, New York Times

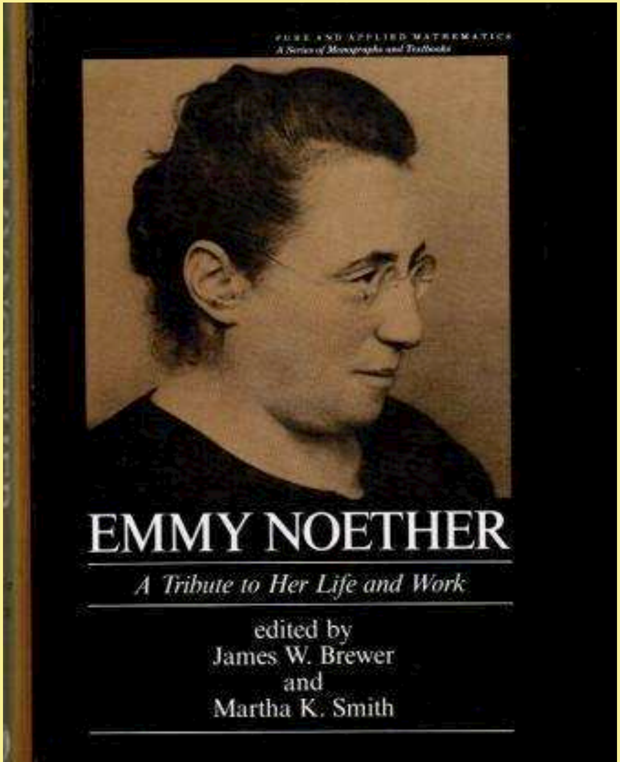
She was a great mathematician, the greatest, I firmly believe that her sex has ever produced and a great woman ... And of all I have known, she was certainly one of the happiest.

— Hermann Weyl

Emmy Noether was one of the most influential mathematicians of the century. The development of abstract algebra, which is one of the most distinctive innovations of twentieth century mathematics, is largely due to her — in published papers, in lectures, and in personal influence on her contemporaries.

— Nathan Jacobson

Noether the Mathematician



III NOETHER'S MATHEMATICS

Emmy Noether — Biography

Born: 1882, Erlangen, Germany

Father: Max Noether (Nöther), German mathematician
— algebraic geometry

1907 Ph.D. under Paul Gordan, Erlangen ("King of invariants")
— calculated all 331 invariants of ternary biquadratic forms
— "Formelgestrüpp", "Mist" (E.N.)

1907–14: Teaches at University of Erlangen without pay

1915–33: Invited to University of Göttingen
by David Hilbert & Felix Klein

1918: Noether's Theorems published

1919: *Habilitation*

1919–35: Established foundations of modern abstract algebra: ideals, rings, noetherian condition, representation theory, etc.

"der Noether" & the Noether boys

van der Waerden: *Moderne Algebra*

1922: Appointed extraordinary professor in Göttingen

1923: Finally paid a small stipend for teaching!

1932: Plenary address at the International Congress of Mathematicians, Zurich

1933: Placed on "leave of absence"; tries to move to Soviet Union

1933: Moves to U.S. — Bryn Mawr College

1935: Dies after surgery, aged 53

Noether's Three Fundamental Contributions to Analysis and Physics

First Theorem. There is a one-to-one correspondence between symmetry groups of a variational problem and conservation laws of its Euler–Lagrange equations.

Second Theorem. An infinite-dimensional variational symmetry group depending upon an arbitrary function corresponds to a nontrivial differential relation among its Euler–Lagrange equations.

⋆ The conservation laws associated with the variational symmetries in the Second Theorem are trivial — this resolved Hilbert's original paradox in relativity that was the reason he and Klein invited Noether to Göttingen.

Noether's Three Fundamental Contributions to Analysis and Physics

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Second Theorem. An infinite-dimensional variational symmetry group depending upon an arbitrary function corresponds to a nontrivial differential relation among its Euler–Lagrange equations.

Introduction of higher order generalized symmetries.

$\Longrightarrow$ later (1960's) to play a fundamental role in the discovery and classification of integrable systems and solitons.

Symmetries $\Longrightarrow$ Conservation Laws

- symmetry under space translations
  $\Longrightarrow$ conservation of linear momentum
- symmetry under time translations
  $\Longrightarrow$ conservation of energy
- symmetry under rotations
  $\Longrightarrow$ conservation of angular momentum
- symmetry under boosts (moving coordinates)
  $\Longrightarrow$ linear motion of the center of mass

Precursors

Lagrange (1788) Lagrangian mechanics & energy conservation

Jacobi (1842–43 publ. 1866) Euclidean invariance
— linear and angular momentum

Schütz (1897) time translation — conservation of energy

Herglotz (1911) Poincaré invariance in relativity
— 10 conservation laws

Engel (1916) non-relativistic limit: Galilean invariance
— linear motion of center of mass

A Curious History

- ⋆ Bessel–Hagen (1922) — divergence symmetries
- ♣ Hill (1951) — a very special case
  (first order Lagrangians, geometrical symmetries)
- ♠ 1951–1980 Over 50 papers rediscover and/or prove
  purported generalizations of Noether's First Theorem
- ♠ 2011 Neuenschwander, *Emmy Noether's Wonderful Theorem*
  — back to special cases again!

Continuum mechanics: Rice, Eshelby (1950's),
Günther (1962), Knowles & Sternberg (1972)

Optics: Baker & Tavel (1974)

The Noether Triumvirate

- ⋆ Variational Principle
- ⋆ Symmetry
- ⋆ Conservation Law

The Calculus of Variations

[Leibniz] conceives God in the creation of the world like a mathematician who is solving a minimum problem, or rather, in our modern phraseology, a problem in the calculus of variations — the question being to determine among an infinite number of possible worlds, that for which the sum of necessary evil is a minimum.

— Paul du Bois-Reymond

The Calculus of Variations

Nature is Leibnizian (Panglossian):

★ A physical system in equilibrium chooses
"the best of all possible worlds" by minimizing some overall cost: energy or force or time or money or . . .

Principle of least action:

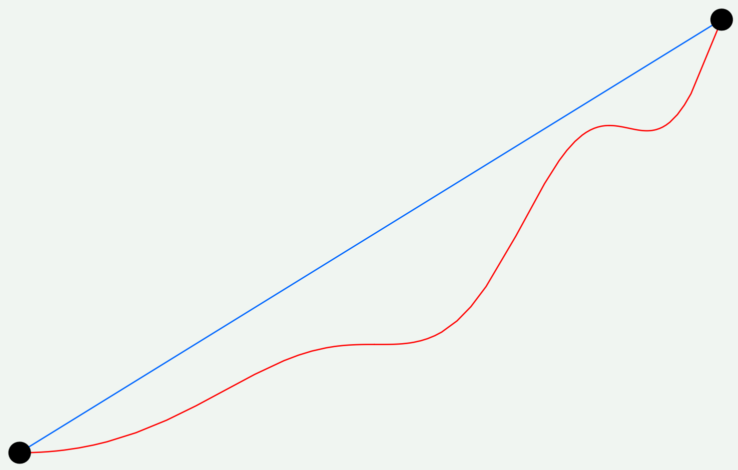
"Nature is thrifty in all its actions."

$\Longrightarrow$ Pierre Louis Maupertuis

★ Analysis developed by various Bernoullis, Euler, Lagrange, Hamilton, Jacobi, Weierstrass, Dirichlet, Hilbert, . . .

Examples of Variational Problems:

The shortest path between two points in space is a straight line.



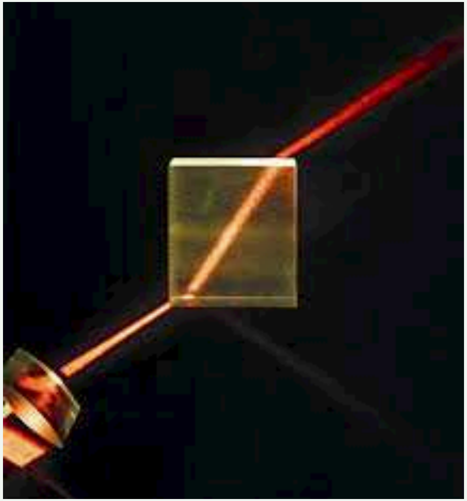
Geodesics

The shortest path between two points on a sphere is a great circular arc.

The shortest path between two points on a curved surface is a geodesic arc.

Fermat's Principle in Optics

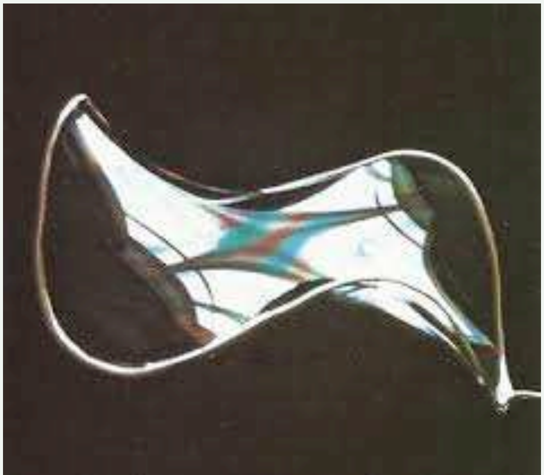
Light travels along the path that takes the least time:



$$\Longrightarrow \quad \text{Snell's Law} \;=\; \text{Loi de Descartes}$$

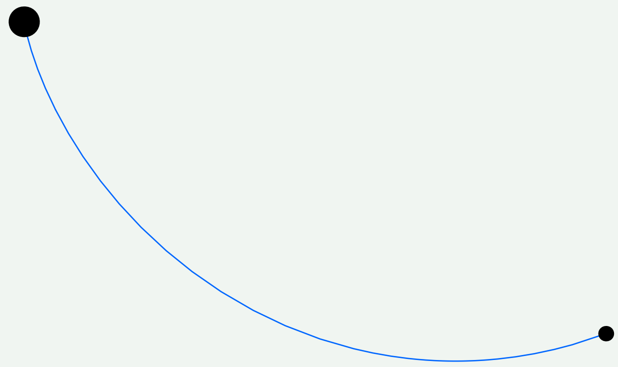
Plateau's Problem

The surface of least area spanning a space curve is a minimal surface. $\Longrightarrow$ soap films

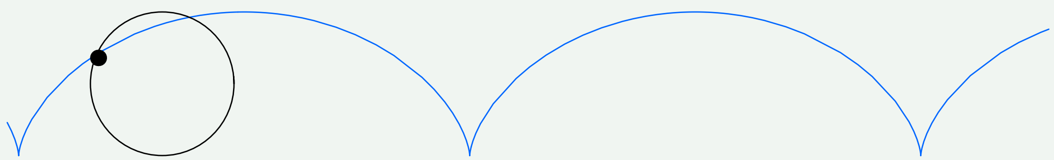


The Brachistochrone

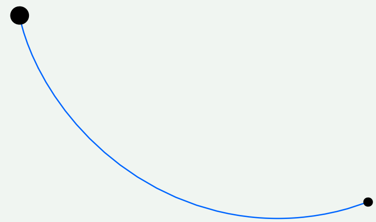
A bead slides down a wire fastest



when it has the shape of a cycloid



The Brachistochrone Cycloid



Galileo (1638) says it is a circular arc

Tautochrone problem — solved by Huygens (1659)

★ produces great increase in accuracy of time-keeping, leading to the solution to the Problem of the Longitude

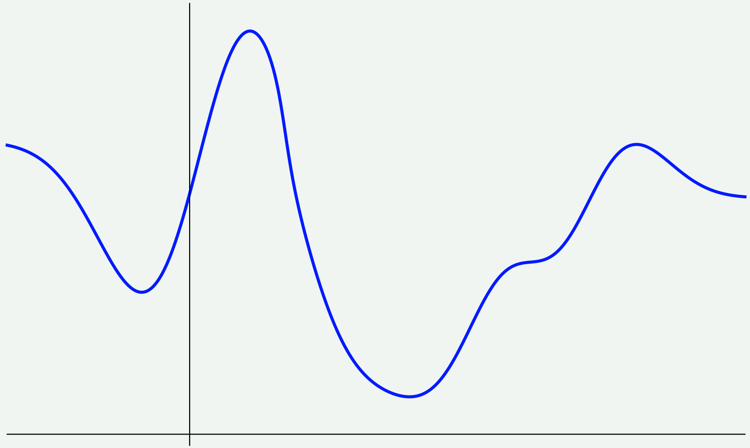
Johann Bernoulli's contest (1696)

$\Longrightarrow$ Correct entries by Newton, Leibniz, Jakob Bernoulli, l'Hôpital, Tschirnhaus

Thus began the calculus of variations!

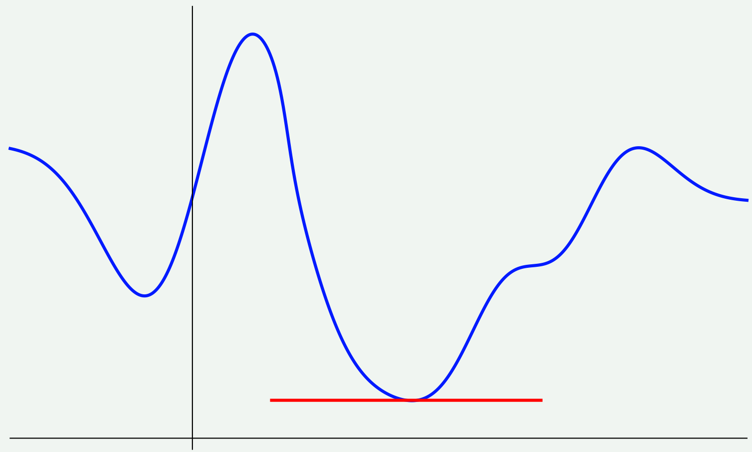
Minimization

How do you minimize a function?



Minimization

- At any minimum of a function the tangent line is horizontal:



The First Derivative Test

A minimum of a (nice) function $f(x)$ of one variable satisfies

$$f'(x) = 0$$

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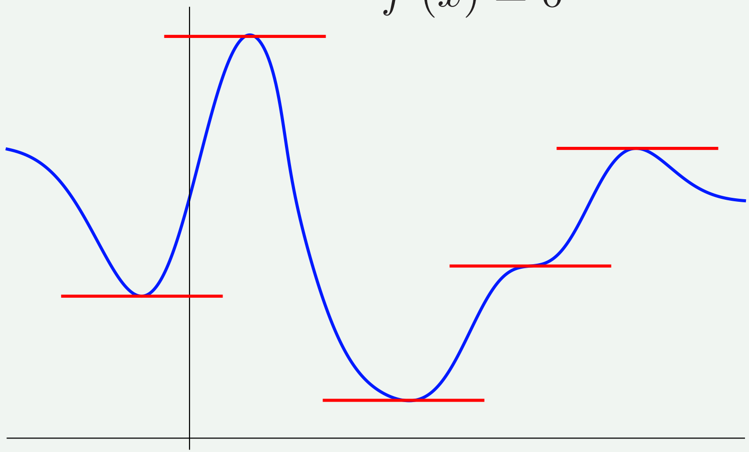
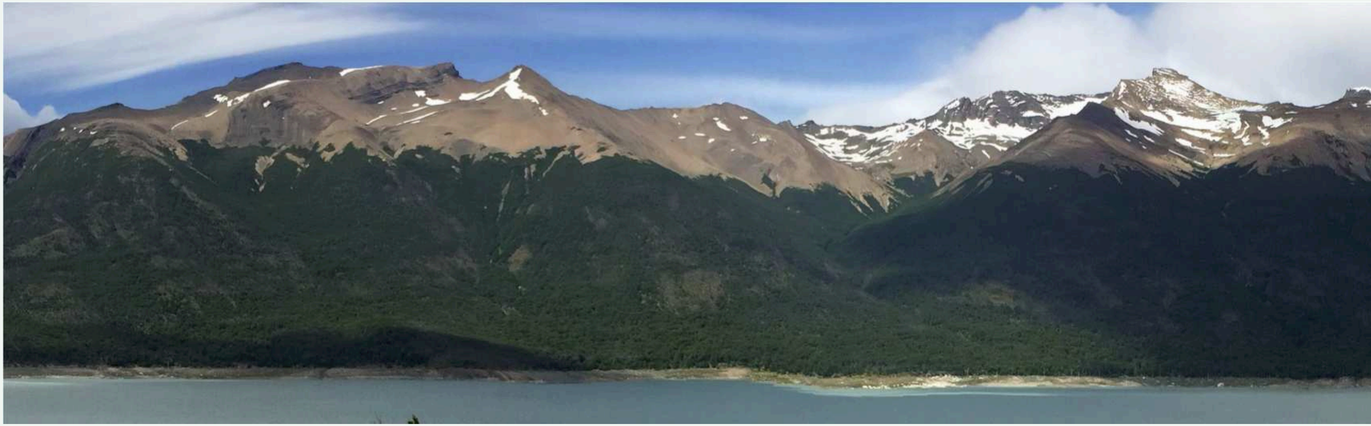


♣ But this also holds at maxima and inflection points!

★ Distinguishing minima from maxima from inflection points requires the second derivative test — not used here!



How do you find the peaks in a mountain range?

$\implies$ maxima of the height function.

- The tangent plane is horizontal.

Similarly at minima — bottom of valleys.

How do you find the peaks in a mountain range?

A better solution:

The gradient of the height function is the vector that points in the direction of steepest increase, i.e., uphill

The gradient ∇F of the height function F is the vector that points in the direction of steepest increase.

Thus, at the summit, you cannot go any further up, and hence the gradient must vanish: $\quad \nabla F = 0.$

Similarly at minima — bottom of valleys.

The Variational Principle

In general, a variational problem requires minimizing a function $\mathcal{F}$ over an infinite-dimensional space, in the form of an action functional, which depends on the space/time coordinates and the physical fields.

$\star$ The functional gradient vanishes at the minima: $\quad \delta\mathcal{F} = 0.$

$\Longrightarrow$ This gives a system of differential equations, whose solutions are the minimizers.

$\diamondsuit$ **Modern Physics:** The action functional should incorporate all of the symmetries of Nature.

A Brief History of Symmetry

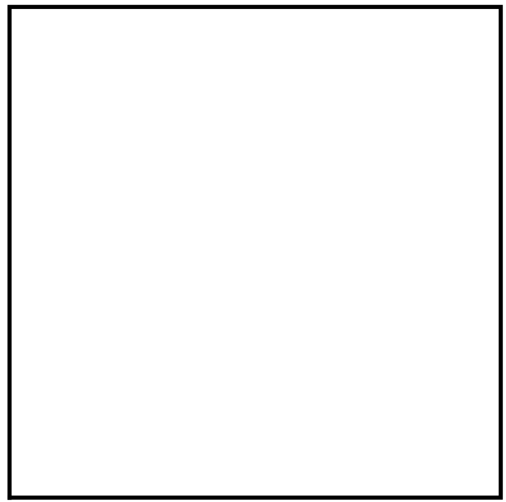
Symmetry $\Longrightarrow$ Group Theory!

- Abel, Galois — polynomials
- Lie — differential equations and variational principles
- Noether — conservation laws and higher order symmetries
- Weyl, Wigner, etc. — quantum mechanics "der Gruppenpest" (J. Slater)

Next to the concept of a function, which is the most important concept pervading the whole of mathematics, the concept of a group is of the greatest significance in the various branches of mathematics and its applications.

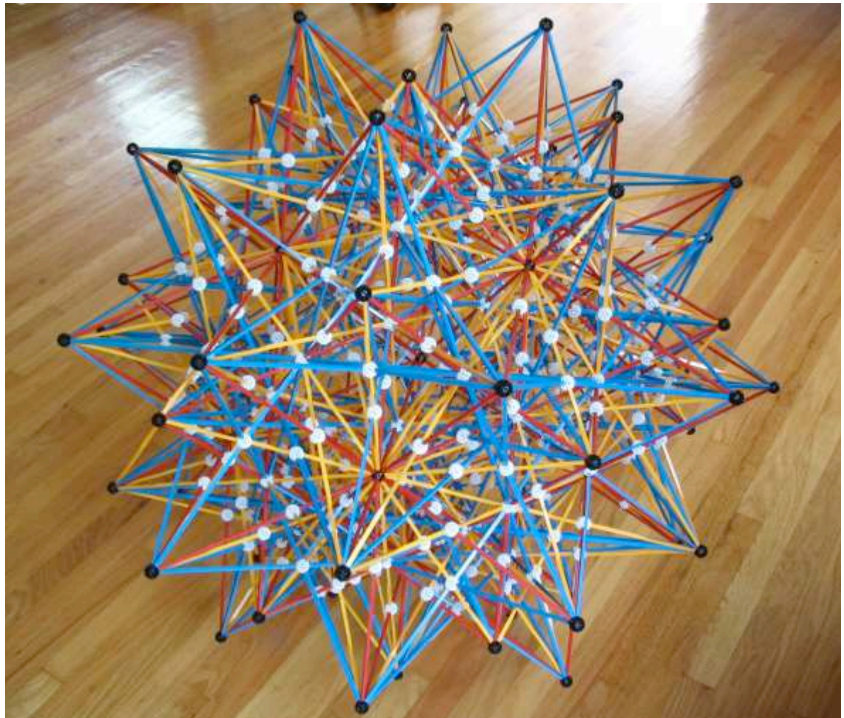
— P.S. Alexandroff

Discrete Symmetry Group

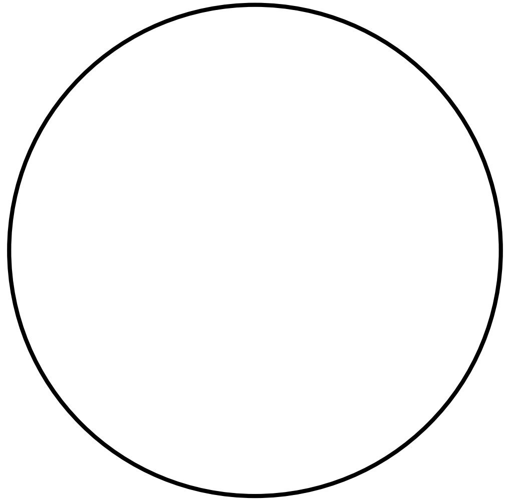


Symmetry group = rotations by $0°, 90°, 180°, 270°$

Discrete Symmetry Group



Continuous Symmetry Group



Symmetry group = *all* rotations

★ A continuous group is known as a Lie group
— in honor of Sophus Lie (1842–1899)

A Brief History of Conservation Laws

In physics, a conservation law asserts that a particular measurable property P of an isolated physical system does not change as the system evolves.

Conservation of momentum: Wallis (1670), Newton (1687)

Conservation of mass: Lomonosov (1748), Lavoisier (1774)

Conservation of energy: Lagrange (1788), Helmholtz (1847), Rankine (1850), also: Mohr, Mayer, Joule, Faraday, . . .

In Summary . . .

Noether's Theorem states that to each continuous symmetry group of the action functional there is a corresponding conservation law of the physical equations and vice versa.

The Modern Manual for Physics

♠ To construct a physical theory:

Step 1: Determine the allowed group of symmetries:

- translations
- rotations
- conformal (angle-preserving) transformations
- Galilean boosts
- Poincaré transformations (relativity)
- gauge transformations
- CPT (charge, parity, time reversal) symmetry
- supersymmetry
- $\mathrm{SU}(3)$, G_2, $\mathrm{E}_8 \times \mathrm{E}_8$, $\mathrm{SO}(32)$, $\ldots$
- etc., etc.

Step 2: Construct a variational principle ("energy") that admits the given symmetry group.

Step 3: Invoke Nature's obsession with minimization to determine the corresponding field equations associated with the variational principle.

Step 4: Use Noether's First and Second Theorems to write down (a) conservation laws, and (b) differential identities satisfied by the field equations.

Step 5: Try to solve the field equations.

Even special solutions are of immense interest

$\Longrightarrow$ black holes.

All Known Physics

Schrödinger
Feynman
Einstein
Maxwell
Kobayashi-
Maskawa
Lagrange

$$\Psi = \int e^{\frac{i}{\hbar}} \int \left(\frac{R}{16\pi G} - \frac{1}{4} F^2 + \overline{\psi} i \not{D} \psi - \lambda \varphi \overline{\psi} \psi + |D\varphi|^2 - V(\varphi) \right)$$

Euler
Planck
Newton
Dirac
Yukawa
Higgs

Amalie Emmy Noether
(1882–1935)