Lie Pseudo-Groups

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Sur la théorie, si importante sans doute, mais pour nous si obscure, des « groupes de Lie infinis », nous ne savons rien que ce qui se trouve dans les mémoires de Cartan, première exploration à travers une jungle presque impénétrable; mais celle-ci menace de se refermer sur les sentiers déjà tracés, si l’on ne procède bientôt à un indispensable travail de défrichement.

— André Weil, 1947
What’s the Difficulty with Infinite–Dimensional Groups?

- Lie invented Lie groups to study symmetry and solution of differential equations.

♢ In Lie’s time, there were no abstract Lie groups. All groups were realized by their action on a space.

♠ Therefore, Lie saw no essential distinction between finite-dimensional and infinite-dimensional group actions.

However, with the advent of abstract Lie groups, the two subjects have gone in radically different directions.

♥ The general theory of finite-dimensional Lie groups has been rigorously formalized and applied.

♣ But there is still no generally accepted abstract object that represents an infinite-dimensional Lie pseudo-group!
Ehresmann’s Trinity

1953:
Ehresmann’s Trinity

1953:

- Lie pseudo-groups
Ehresmann’s Trinity

1953:

• Lie pseudo-groups
• Jets
Ehresmann’s Trinity

1953:

- Lie pseudo-groups
- Jets
- Groupoids
Lie Pseudo-groups — History

- Lie, Medolaghi, Vessiot
- É. Cartan
- Ehresmann, Libermann
- Kuranishi, Spencer, Singer, Sternberg, Guillemin, Kumpera, ...
Lie Pseudo-groups — Applications

- Relativity
- Noether’s (Second) Theorem
- Gauge theory and field theories:
  - Maxwell, Yang–Mills, conformal, string, ...
- Fluid mechanics, meteorology:
  - Navier–Stokes, Euler, boundary layer, quasi-geostrophic, ...
- Linear and linearizable PDEs
- Solitons (in $2 + 1$ dimensions):
  - K–P, Davey-Stewartson, ...
- Kac–Moody
- Morphology and shape recognition
- Control theory
- Geometric numerical integration
- Lie groups!
Lie Pseudo-groups — Moving Frames

◊ Motivation: To develop an algorithmic invariant calculus for Lie group and pseudo-group actions. Classify and construct differential invariants — including their generators and syzygies — invariant differential forms, invariant differential operators, invariant differential equations, invariant variational problems, etc.

♦ Tools: The equivariant approach to moving frames — which can be implemented for arbitrary Lie group and most Lie pseudo-group actions — along with the induced invariant variational bicomplex.

♥ Additional benefits: A new, elementary approach to the structure theory for Lie pseudo-groups, including explicit construction of Maurer–Cartan forms and direct, elementary determination of structure equations from the infinitesimal generators.

⇒ PJO, Fels, Pohjanpelto, Cheh, Itskov, Valiquette
Lie Pseudo-groups — Further applications

- Symmetry groups of differential equations
- Vessiot group splitting; explicit solutions
- Gauge theories
- Calculus of variations
- Invariant geometric flows
- Computer vision and mathematical morphology
- Geometric numerical integration
Pseudo-groups

\(M\) — analytic (smooth) manifold

**Definition.** A pseudo-group is a collection of local analytic diffeomorphisms \(\phi: \text{dom } \phi \subset M \rightarrow M\) such that

- **Identity:** \(1_M \in \mathcal{G}\)
- **Inverses:** \(\phi^{-1} \in \mathcal{G}\)
- **Restriction:** \(U \subset \text{dom } \phi \implies \phi \mid U \in \mathcal{G}\)
- **Continuation:** \(\text{dom } \phi = \bigcup U_\kappa\) and \(\phi \mid U_\kappa \in \mathcal{G} \implies \phi \in \mathcal{G}\)
- **Composition:** \(\text{im } \phi \subset \text{dom } \psi \implies \psi \circ \phi \in \mathcal{G}\)
The Diffeomorphism Pseudo-group

\[ M \quad \text{— } \quad m \text{-dimensional manifold} \]

\[ \mathcal{D} = \mathcal{D}(M) \quad \text{— } \quad \text{pseudo-group of all local analytic diffeomorphisms} \]

\[ Z = \phi(z) \]

\[
\begin{cases}
  z = (z^1, \ldots, z^m) & \text{— source coordinates} \\
  Z = (Z^1, \ldots, Z^m) & \text{— target coordinates}
\end{cases}
\]

\[
\begin{cases}
  L_\psi(\phi) = \psi \circ \phi & \text{— left action} \\
  R_\psi(\phi) = \phi \circ \psi^{-1} & \text{— right action}
\end{cases}
\]
Jets

For $0 \leq n \leq \infty$:

Given a smooth map $\phi : M \to M$, written in local coordinates as $Z = \phi(z)$, let $j_n \phi|_z$ denote its $n$–jet at $z \in M$, i.e., its $n^{\text{th}}$ order Taylor polynomial or series based at $z$.

$J^n(M, M)$ is the $n^{\text{th}}$ order jet bundle, whose points are the jets.

Local coordinates on $J^n(M, M)$:

$$(z, Z^{(n)}) = (\ldots z^a \ldots Z^b \ldots Z^b_A \ldots), \quad Z^b_A = \frac{\partial^k Z^b}{\partial z^{a_1} \ldots \partial z^{a_k}}$$
Diffeomorphism Jets

The $n^{th}$ order diffeomorphism jet bundle is the subbundle

$$\mathcal{D}^{(n)} = \mathcal{D}^{(n)}(M) \subset J^n(M, M)$$

consisting of $n^{th}$ order jets of local diffeomorphisms $\phi: M \to M$.

The Inverse Function Theorem tells us that $\mathcal{D}^{(n)}$ is defined by the non-vanishing of the Jacobian determinant:

$$\det( Z^{a}_b ) = \det( \partial Z^{a}/\partial z^{b} ) \neq 0$$

★ $\mathcal{D}^{(n)}$ forms a groupoid under composition of Taylor polynomials/series.
Groupoid Structure

Double fibration:

\[ \begin{array}{ccc}
\sigma^{(n)} & \xrightarrow{\mathcal{D}^{(n)}} & \tau^{(n)} \\
M & \xleftarrow{\sigma^{(n)}} & M \\
\end{array} \]

\[ \sigma^{(n)}(z, Z^{(n)}) = z \quad \text{— source map} \]
\[ \tau^{(n)}(z, Z^{(n)}) = Z \quad \text{— target map} \]

You are only allowed to multiply \( h^{(n)} \cdot g^{(n)} \) if

\[ \sigma^{(n)}(h^{(n)}) = \tau^{(n)}(g^{(n)}) \]

\[ \diamond \] Composition of Taylor polynomials/series is well-defined only when the source of the second matches the target of the first.
One-dimensional case: \( M = \mathbb{R} \)

Source coordinate: \( x \) \hspace{1cm} Target coordinate: \( X \)

Local coordinates on \( \mathcal{D}^{(n)}(\mathbb{R}) \)

\[
g^{(n)} = (x, X, X_x, X_{xx}, X_{xxx}, \ldots, X_n)
\]

Diffeomorphism jet:

\[
X[h] = X + X_x h + \frac{1}{2} X_{xx} h^2 + \frac{1}{6} X_{xxx} h^3 + \cdots
\]

\( \implies \) Taylor polynomial/series at a source point \( x \)
Groupoid multiplication of diffeomorphism jets:

\[(X, X, X_X, X_{XX}, \ldots) \cdot (x, X, X_x, X_{xx}, \ldots) = (x, X, X_X X_x, X_X X_{xx} + X_{XX} X_x^2, \ldots)\]

\[\Rightarrow\] Composition of Taylor polynomials/series

- The groupoid multiplication (or Taylor composition) is only defined when the source coordinate \(X\) of the first multiplicand matches the target coordinate \(X\) of the second.

- The higher order terms are expressed in terms of Bell polynomials according to the general Fàa–di–Bruno formula.
Any pseudo-group $\mathcal{G} \subset \mathcal{D}$ defines a Lie sub-groupoid $\mathcal{G}^{(n)} \subset \mathcal{D}^{(n)}$.

**Definition.** $\mathcal{G}$ is regular if, for $n \gg 0$, its jets $\sigma : \mathcal{G}^{(n)} \to M$ form an embedded subbundle of $\sigma : \mathcal{D}^{(n)} \to M$ and the projection $\pi_{n+1}^{n} : \mathcal{G}^{(n+1)} \to \mathcal{G}^{(n)}$ is a fibration.

**Definition.** A regular, analytic pseudo-group $\mathcal{G}$ is called a **Lie pseudo-group** of order $n^* \geq 1$ if every local diffeomorphism $\phi \in \mathcal{D}$ satisfying $j_{n^*} \phi \subset \mathcal{G}^{(n^*)}$ belongs it: $\phi \in \mathcal{G}$. 
In local coordinates, \( \mathcal{G}^{(n^*)} \subset \mathcal{D}^{(n^*)} \) forms a system of differential equations

\[
F^{(n^*)}(z, Z^{(n^*)}) = 0
\]
called the determining system of the pseudo-group. The Lie condition requires that every local solution to the determining system belongs to the pseudo-group.

Lemma. In the analytic category, for sufficiently large \( n \gg 0 \) the determining system of a regular pseudo-group is an involutive system of partial differential equations.

Proof: Cartan – Kuranishi + local solvability.
In local coordinates, $G^{(n^*)} \subset D^{(n^*)}$ forms a system of differential equations

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What about integrability/involutivity?

**Lemma.** In the analytic category, for sufficiently large $n \gg 0$ the determining system $G^{(n)} \subset D^{(n)}$ of a regular pseudo-group is an involutive system of partial differential equations.

**Proof:** regularity + Cartan–Kuranishi + local solvability.
Lie Completion of a Pseudo-group

Definition. The Lie completion $\bar{G} \supset G$ of a regular pseudo-group is defined as the space of all analytic diffeomorphisms $\phi$ that solve the determining system $G^{(n^*)}$. 

Theorem. $G$ and $\bar{G}$ have the same differential invariants, the same invariant differential forms, etc. Thus, for local geometry, there is no loss in generality assuming all (regular) pseudo-groups are Lie pseudo-groups!
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**Theorem.** $\mathcal{G}$ and $\overline{\mathcal{G}}$ have the same differential invariants, the same invariant differential forms, etc.

★ Thus, for local geometry, there is no loss in generality assuming all (regular) pseudo-groups are Lie pseudo-groups!
A Non-Lie Pseudo-group

\[ X = \phi(x) \quad Y = \phi(y) \quad \text{where} \quad \phi \in \mathcal{D}(\mathbb{R}) \]

On the off-diagonal set \( M = \{ (x, y) \mid x \neq y \} \), the pseudo-group \( \mathcal{G} \) is regular of order 1, and \( \mathcal{G}^{(1)} \subset \mathcal{D}^{(1)} \) is defined by the first order determining system

\[ X_y = Y_x = 0 \quad X_x, Y_y \neq 0 \]

The general solution to the determining system \( \mathcal{G}^{(1)} \) forms the Lie completion \( \overline{\mathcal{G}} \):

\[ X = \phi(x) \quad Y = \psi(y) \quad \text{where} \quad \phi, \psi \in \mathcal{D}(\mathbb{R}) \]
Structure of Lie Pseudo-groups

Recall:

The structure of a finite-dimensional Lie group $G$ is specified by its Maurer–Cartan forms — a basis $\mu^1, \ldots, \mu^r$ for the right-invariant one-forms:

$$d\mu^k = \sum_{i<j} C^k_{ij} \mu^i \wedge \mu^j$$
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The Maurer–Cartan forms for a Lie group and hence Lie pseudo-group can be identified with the right-invariant one-forms on the jet groupoid $\mathcal{G}(\infty)$. 
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Cartan: Use exterior differential systems and prolongation to determine the structure equations.

I propose a direct approach based on the following observation:

The Maurer–Cartan forms for a Lie group and hence Lie pseudo-group can be identified with the right-invariant one-forms on the jet groupoid $G^{(\infty)}$.

The structure equations can be determined immediately from the infinitesimal determining equations.
The Variational Bicomplex

The differential one-forms on an infinite jet bundle split into two types:

- horizontal forms
- contact forms

Consequently, the exterior derivative on $\mathcal{D}^{(\infty)}$ splits

$$d = d_M + d_G$$

into horizontal (manifold) and contact (group) components, leading to the variational bicomplex structure on the algebra of differential forms on $\mathcal{D}^{(\infty)}$. 
For the diffeomorphism jet bundle
\[ D^{(\infty)} \subset J^\infty(M, M) \]

Local coordinates:
\[
\begin{align*}
&z^1, \ldots, z^m, \\
&Z^1, \ldots, Z^m, \\
&\ldots, Z^b_A, \ldots
\end{align*}
\]
source \hspace{2cm} target \hspace{2cm} jet

Horizontal forms:
\[ dz^1, \ldots, dz^m \]

Basis contact forms:
\[
\Theta^b_A = d_G Z^b_A = dZ^b_A - \sum_{a=1}^m Z^a_{A,a} dz^a
\]
One-dimensional case: $M = \mathbb{R}$

Local coordinates on $\mathcal{D}^{(\infty)}(\mathbb{R})$

$$ (x, X, X_x, X_{xx}, X_{xxx}, \ldots, X_n, \ldots ) $$

Horizontal form:

$$ dx $$

Contact forms:

$$ \Theta = dX - X_x \, dx $$

$$ \Theta_x = dX_x - X_{xx} \, dx $$

$$ \Theta_{xx} = dX_{xx} - X_{xxx} \, dx $$

$$ \vdots $$
Maurer–Cartan Forms

**Definition.** The **Maurer–Cartan forms** for the diffeomorphism pseudo-group are the **right-invariant** one-forms on the diffeomorphism jet groupoid $D^{(\infty)}$. 
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**Key observation:** Since the right action only affects source coordinates, the target coordinate functions $Z^a$ are right-invariant.
Maurer–Cartan Forms

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**Key observation:** Since the right action only affects source coordinates, the target coordinate functions $Z^a$ are right-invariant.

Thus, when we decompose

$$dZ^a = \sigma^a + \mu^a$$

both components $\sigma^a, \mu^a$ are right-invariant one forms.
Invariant horizontal forms:

$$\sigma^a = d_M Z^a = \sum_{b=1}^{m} Z^a_b \, d\bar{z}^b$$

Dual invariant total differentiation operators:

$$\mathbb{D}_{Z^a} = \sum_{b=1}^{m} (Z^a_b)^{-1} \mathbb{D}_{\bar{z}^b}$$

Thus, the invariant contact forms $\mu^b_A$ are obtained by invariant differentiation of the order zero contact forms:

$$\mu^b = d_G Z^b = \Theta^b = dZ^b - \sum_{a=1}^{m} Z^b_a \, d\bar{z}^a$$

$$\mu^b_A = \mathbb{D}^A_{Z^b} \mu^b = \mathbb{D}_{Z^a_1} \cdots \mathbb{D}_{Z^a_n} \mu^b \quad b = 1, \ldots, m, \#A \geq 0$$
One-dimensional case: \( M = \mathbb{R} \)

Contact forms:

\[
\Theta = dX - X_x \, dx
\]

\[
\Theta_x = \mathcal{D}_x \Theta = dX_x - X_{xx} \, dx
\]

\[
\Theta_{xx} = \mathcal{D}_x^2 \Theta = dX_{xx} - X_{xxx} \, dx
\]

Right-invariant horizontal form:

\[
\sigma = d_M X = X_x \, dx
\]

Invariant differentiation:

\[
\mathcal{D}_x = \frac{1}{X_x} \mathcal{D}_x
\]
Invariant contact forms:

\[ \mu = \Theta = dX - X_x \, dx \]

\[ \mu_X = \mathbb{D}_X \mu = \frac{\Theta}{X_x} = \frac{dX_x - X_{xx} \, dx}{X_x} \]

\[ \mu_{XX} = \mathbb{D}_X^2 \mu = \frac{X_x \Theta_{xx} - X_{xx} \Theta_x}{X^3_x} \]

\[ = \frac{X_x \, dX_{xx} - X_{xx} \, dX_x + (X^2_{xx} - X_x X_{xxx}) \, dx}{X^3_x} \]

\[ \vdots \]

\[ \mu_n = \mathbb{D}_X^n \mu \]
The Structure Equations for the Diffeomorphism Pseudo–group

\[ d\mu^b_A = \sum C^{b,B,C}_{A,c,d} \mu^c_B \wedge \mu^d_C \]
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\[ d\mu^b_A = \sum C^{b,B,C}_{A,c,d} \mu^c_B \land \mu^d_C \]

Formal Maurer–Cartan series:

\[ \mu^b[H] = \sum_A \frac{1}{A!} \mu^b_A H^A \]

\[ H = (H^1, \ldots, H^m) \text{ — formal parameters} \]

\[ d\mu[H] = \nabla \mu[H] \land (\mu[H] - dZ) \]

\[ d\sigma = -d\mu[0] = \nabla \mu[0] \land \sigma \]
One-dimensional case: \( M = \mathbb{R} \)

Structure equations:

\[
\begin{align*}
    d\sigma &= \mu_X \wedge \sigma \\
    d\mu[H] &= \frac{d\mu}{dH} \mu[H] \wedge (\mu[H] - dZ)
\end{align*}
\]

where

\[
\begin{align*}
    \sigma &= X_x \, dx = dX - \mu \\
    \mu[H] &= \mu + \mu_X \, H + \frac{1}{2} \mu_{XX} \, H^2 + \cdots \\
    \mu[H] - dZ &= -\sigma + \mu_X \, H + \frac{1}{2} \mu_{XX} \, H^2 + \cdots \\
    \frac{d\mu}{dH} [H] &= \mu_X + \mu_{XX} \, H + \frac{1}{2} \mu_{XXX} \, H^2 + \cdots
\end{align*}
\]
In components:

\[ d\sigma = \mu_1 \wedge \sigma \]

\[ d\mu_n = - \mu_{n+1} \wedge \sigma + \sum_{i=0}^{n-1} \binom{n}{i} \mu_{i+1} \wedge \mu_{n-i} \]

\[ = \sigma \wedge \mu_{n+1} - \sum_{j=1}^{\left\lfloor \frac{n+1}{2} \right\rfloor} \frac{n - 2j + 1}{n + 1} \binom{n + 1}{j} \mu_j \wedge \mu_{n+1-j}. \]

\[ \implies \text{Cartan} \]
The Maurer–Cartan forms for a pseudo-group \( \mathcal{G} \subset \mathcal{D} \) are obtained by restricting the diffeomorphism Maurer–Cartan forms \( \sigma^a, \mu^b_A \) to \( \mathcal{G}^{(\infty)} \subset \mathcal{D}^{(\infty)} \).
The Maurer–Cartan Forms for a Lie Pseudo-group

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Maurer–Cartan forms \( \sigma^a, \mu^b_A \) to \( \mathcal{G}(\infty) \subset \mathcal{D}(\infty) \).

\[ \star \star \] The resulting one-forms are no longer linearly independent, but the dependencies can be determined directly from the infinitesimal generators of \( \mathcal{G} \).
Infinitesimal Generators

\( g \) — Lie algebra of infinitesimal generators of the pseudo-group \( \mathcal{G} \)

\( z = (x, u) \) — local coordinates on \( M \)

Vector field:

\[
\mathbf{v} = \sum_{a=1}^{m} \zeta^a(z) \frac{\partial}{\partial z^a} = \sum_{i=1}^{p} \xi^i \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \varphi^\alpha \frac{\partial}{\partial u^\alpha}
\]

Vector field jet:

\[
\mathbf{j}_n \mathbf{v} \mapsto \zeta^{(n)} = (\ldots \zeta^b_A \ldots )
\]

\[
\zeta_A^b = \frac{\partial^{\#A} \zeta^b}{\partial z^A} = \frac{\partial^k \zeta^b}{\partial z^{a_1} \ldots \partial z^{a_k}}
\]
The infinitesimal generators of \( G \) are the solutions to the \textbf{infinitesimal determining equations}

\[
\mathcal{L}(z, \zeta^{(n)}) = 0
\]

obtained by linearizing the \textit{nonlinear determining equations} at the identity.

\( \star \)

If \( G \) is the symmetry group of a system of differential equations, then \( \star \) is the (involutive completion of) the usual \textit{Lie determining equations} for the symmetry group.
The infinitesimal generators of $\mathcal{G}$ are the solutions to the infinitesimal determining equations

$$\mathcal{L}(z, \zeta^{(n)}) = 0$$

obtained by linearizing the nonlinear determining equations at the identity.

• If $\mathcal{G}$ is the symmetry group of a system of differential equations, then (*) is the (involutive completion of) the usual Lie determining equations for the symmetry group.
Theorem. The Maurer–Cartan forms on $G^{(\infty)}$ satisfy the invariantized infinitesimal determining equations

$$\mathcal{L}( \ldots Z^a \ldots \mu^b_A \ldots ) = 0 \quad (\star\star)$$

obtained from the infinitesimal determining equations

$$\mathcal{L}( \ldots z^a \ldots \zeta^b_A \ldots ) = 0 \quad (*)$$

by replacing

- source variables $z^a$ by target variables $Z^a$
- derivatives of vector field coefficients $\zeta^b_A$ by right-invariant Maurer–Cartan forms $\mu^b_A$
The Structure Equations for a Lie Pseudo-group

**Theorem.** The structure equations for the pseudo-group $\mathcal{G}$ are obtained by restricting the universal diffeomorphism structure equations

$$d\mu[ H ] = \nabla \mu[ H ] \wedge ( \mu[ H ] - dZ )$$

to the solution space of the linear algebraic system

$$\mathcal{L}( \ldots Z^a, \ldots \mu^b_A, \ldots ) = 0.$$
Comparison of Structure Equations

If the action is transitive, then our structure equations are isomorphic to Cartan’s. However, this is not true for intransitive pseudo-groups. Whose structure equations are “correct”? 
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• To find the Cartan structure equations, one first needs to work in an adapted coordinate chart, which requires identification of the invariants on $M$. Ours can be found in any system of local coordinates.
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- Cartan’s procedure for identifying the invariant forms is recursive, and not easy to implement. Ours follow immediately from the structure equations for the diffeomorphism pseudo-group using merely linear algebra.
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- For finite-dimensional intransitive Lie group actions, Cartan’s pseudo-group structure equations do not coincide with the standard Maurer–Cartan equations. Ours do (upon restriction to a source fiber).
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- For finite-dimensional intransitive Lie group actions, Cartan’s pseudo-group structure equations do not coincide with the standard Maurer–Cartan equations. Ours do (upon restriction to a source fiber).

- Cartan’s structure equations for isomorphic pseudo-groups can be non-isomorphic. Ours are always isomorphic.
Lie–Kumpera Example

\[
X = f(x) \quad U = \frac{u}{f'(x)}
\]

Infinitesimal generators:

\[
v = \xi \frac{\partial}{\partial x} + \varphi \frac{\partial}{\partial u} = \xi(x) \frac{\partial}{\partial x} - \xi'(x) u \frac{\partial}{\partial u}
\]

Linearized determining system

\[
\xi_x = -\frac{\varphi}{u} \quad \xi_u = 0 \quad \varphi_u = \frac{\varphi}{u}
\]
Maurer–Cartan forms:

\[
\begin{align*}
\sigma &= \frac{u}{U} \, dx = f_x \, dx, \\
\tau &= U_x \, dx + \frac{U}{u} \, du = -u \frac{f_{xx}}{f_x} \, dx + \frac{f_x}{f_x^2} \, du \\
\mu &= dX - \frac{U}{u} \, dx = df - f_x \, dx, \\
\nu &= dU - U_x \, dx - \frac{U}{u} \, du = -\frac{u}{f_x^2} \left( df - f_{xx} \, dx \right) \\
\mu_X &= \frac{du}{u} - \frac{dU - U_x \, dx}{U} = \frac{df_x - f_{xx} \, dx}{f_x}, \\
\nu_X &= \frac{U}{u} \left( dU_x - U_{xx} \, dx \right) - \frac{U_x}{u} \left( dU - U_x \, dx \right) \\
&= -\frac{u}{f_x^3} \left( df_{xx} - f_{xxx} \, dx \right) + \frac{uf_{xx}}{f_x^4} \left( df_x - f_{xx} \, dx \right) \\
\nu_U &= -\frac{du}{u} + \frac{dU - U_x \, dx}{U} = -\frac{df_x - f_{xx} \, dx}{f_x}
\end{align*}
\]
First order linearized determining system:

\[ \xi_x = -\frac{\varphi}{u} \quad \xi_u = 0 \quad \varphi_u = \frac{\varphi}{u} \]

First order Maurer–Cartan determining system:

\[ \mu_X = -\frac{\nu}{U} \quad \mu_U = 0 \quad \nu_U = \frac{\nu}{U} \]

Substituting into the full diffeomorphism structure equations yields the (first order) structure equations:

\[ d\mu = -d\sigma = \frac{\nu \wedge \sigma}{U}, \quad d\nu = -\nu_X \wedge \sigma - \frac{\nu \wedge \tau}{U} \]

\[ d\nu_X = -\nu_{XX} \wedge \sigma - \frac{\nu_X \wedge (\tau + 2\nu)}{U} \]
Symmetry Groups — Review

System of differential equations:

\[ \Delta_\nu(x, u^{(n)}) = 0, \quad \nu = 1, 2, \ldots, k \]

By a symmetry, we mean a transformation that maps solutions to solutions.

**Lie:** To find the symmetry group of the differential equations, work infinitesimally.

The vector field

\[ \mathbf{v} = \sum_{i=1}^{p} \xi^i(x, u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \varphi_\alpha(x, u) \frac{\partial}{\partial u^\alpha} \]

is an infinitesimal symmetry if its flow \( \exp(t \mathbf{v}) \) is a one-parameter symmetry group of the differential equation.
We prolong \( \mathbf{v} \) to the jet space whose coordinates are the derivatives appearing in the differential equation:

\[
\mathbf{v}^{(n)} = \sum_{i=1}^{p} \xi^i \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_{\#J=0}^{n} \varphi^J_{\alpha} \frac{\partial}{\partial u^\alpha_J}
\]

where

\[
\varphi^J_{\alpha} = D_J \left( \varphi^\alpha - \sum_{i=1}^{p} u^\alpha_i \xi^i \right) + \sum_{i=1}^{p} u^\alpha_{J,i} \xi^i
\]

\( D_J \) — total derivatives

Infinitesimal invariance criterion:

\[
\mathbf{v}^{(n)}(\Delta_{\nu}) = 0 \quad \text{whenever} \quad \Delta = 0.
\]

Infinitesimal determining equations:

\[
\mathcal{L}(x, u; \xi^{(n)}, \varphi^{(n)}) = 0
\]

\( \star \star \star \) We can determine the structure of the symmetry group without solving the determining equations!
The Korteweg–deVries equation

\[ u_t + u_{xxx} + uu_x = 0 \]

Symmetry generator:

\[ v = \tau(t, x, u) \frac{\partial}{\partial t} + \xi(t, x, u) \frac{\partial}{\partial x} + \varphi(t, x, u) \frac{\partial}{\partial u} \]

Prolongation:

\[ v^{(3)} = v + \varphi^t \frac{\partial}{\partial u_t} + \varphi^x \frac{\partial}{\partial u_x} + \cdots + \varphi^{xxx} \frac{\partial}{\partial u_{xxx}} \]

where

\[ \varphi^t = \varphi_t + u_t \varphi_u - u_t \tau_t - u_t^2 \tau_u - u_x \xi_t - u_t u_x \xi_u \]

\[ \varphi^x = \varphi_x + u_x \varphi_u - u_t \tau_x - u_t u_x \tau_u - u_x \xi_x - u_x^2 \xi_u \]

\[ \varphi^{xxx} = \varphi_{xxx} + 3 u_x \varphi_u + \cdots \]
Infinitesimal invariance:

\[ \mathbf{v}^{(3)}(u_t + u_{xxx} + uu_x) = \varphi^t + \varphi^{xxx} + u \varphi^x + u_x \varphi = 0 \]

on solutions

Infinitesimal determining equations:

\[
\begin{align*}
\tau_x &= \tau_u = \xi_u = \varphi_t = \varphi_x = 0 \\
\varphi &= \xi_t - \frac{2}{3} u \tau_t \quad \varphi_u = -\frac{2}{3} \tau_t = -2 \xi_x \\
\tau_{tt} &= \tau_{tx} = \tau_{xx} = \cdots = \varphi_{uu} = 0
\end{align*}
\]

General solution:

\[
\begin{align*}
\tau &= c_1 + 3c_4 t, \\
\xi &= c_2 + c_3 t + c_4 x, \\
\varphi &= c_3 - 2c_4 u.
\end{align*}
\]
Basis for symmetry algebra $\mathfrak{g}_{KdV}$:

$$v_1 = \partial_t,$$

$$v_2 = \partial_x,$$

$$v_3 = t \partial_x + \partial_u,$$

$$v_4 = 3t \partial_t + x \partial_x - 2u \partial_u.$$

The symmetry group $\mathcal{G}_{KdV}$ is four-dimensional

$$(x, t, u) \mapsto (\lambda^3 t + a, \lambda x + c t + b, \lambda^{-2} u + c)$$
\[ v_1 = \partial_t, \quad v_2 = \partial_x, \]
\[ v_3 = t \partial_x + \partial_u, \quad v_4 = 3t \partial_t + x \partial_x - 2u \partial_u. \]

Commutator table:

<table>
<thead>
<tr>
<th></th>
<th>(v_1)</th>
<th>(v_2)</th>
<th>(v_3)</th>
<th>(v_4)</th>
</tr>
</thead>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>(v_1)</td>
</tr>
<tr>
<td>(v_2)</td>
<td>0</td>
<td>0</td>
<td>(v_1)</td>
<td>3 (v_2)</td>
</tr>
<tr>
<td>(v_3)</td>
<td>0</td>
<td>(-v_1)</td>
<td>0</td>
<td>(-2v_3)</td>
</tr>
<tr>
<td>(v_4)</td>
<td>(-v_1)</td>
<td>(-3v_2)</td>
<td>2 (v_3)</td>
<td>0</td>
</tr>
</tbody>
</table>

Entries: \([v_i, v_j] = \sum_k C_{ij}^k v_k\). \(C_{ij}^k\) — structure constants of \(g\).
Diffeomorphism Maurer–Cartan forms:
\[
\mu^t, \mu^x, \mu^u, \mu^t_T, \mu^t_X, \mu^t_U, \mu^x_T, \ldots, \mu^u, \mu^t_T T, \mu^T T X, \ldots
\]

Infinitesimal determining equations:
\[
\tau_x = \tau_u = \xi_u = \varphi_t = \varphi_x = 0
\]
\[
\varphi = \xi_t - \frac{2}{3} u \tau_t \quad \varphi_u = -\frac{2}{3} \tau_t = -2 \xi_x
\]
\[
\tau_{tt} = \tau_{tx} = \tau_{xx} = \cdots = \varphi_{uu} = 0
\]

Maurer–Cartan determining equations:
\[
\mu^t_X = \mu^t_U = \mu^x_U = \mu^u = \mu^u \mu^u = 0,
\]
\[
\mu^u = \mu^x_T - \frac{2}{3} U \mu^t_T, \quad \mu^u = -\frac{2}{3} \mu^t_T = -2 \mu^x_X,
\]
\[
\mu^t_T T T = \mu^t_T X = \mu^t_X X = \cdots = \mu^u U U = \cdots = 0.
\]
Basis ($\dim \mathcal{G}_{KdV} = 4$): 

$$\mu^1 = \mu^t, \quad \mu^2 = \mu^x, \quad \mu^3 = \mu^u, \quad \mu^4 = \mu^t_T.$$ 

Substituting into the full diffeomorphism structure equations yields the structure equations for $\mathfrak{g}_{KdV}$:

$$d\mu^1 = -\mu^1 \wedge \mu^4,$$

$$d\mu^2 = -\mu^1 \wedge \mu^3 - \frac{2}{3} U \mu^1 \wedge \mu^4 - \frac{1}{3} \mu^2 \wedge \mu^4,$$

$$d\mu^3 = \frac{2}{3} \mu^3 \wedge \mu^4,$$

$$d\mu^4 = 0.$$

$$d\mu^i = C^i_{jk} \mu^j \wedge \mu^k$$
Basis \((\dim \mathcal{G}_{KdV} = 4)\):

\[
\mu^1 = \mu^t, \quad \mu^2 = \mu^x, \quad \mu^3 = \mu^u, \quad \mu^4 = \mu^T.
\]

Substituting into the full diffeomorphism structure equations yields the structure equations for \(\mathfrak{g}_{KdV}\):

\[
\begin{align*}
\mathrm{d}\mu^1 &= -\mu^1 \wedge \mu^4, \\
\mathrm{d}\mu^2 &= -\mu^1 \wedge \mu^3 - \frac{2}{3} U \mu^1 \wedge \mu^4 - \frac{1}{3} \mu^2 \wedge \mu^4, \\
\mathrm{d}\mu^3 &= \frac{2}{3} \mu^3 \wedge \mu^4, \\
\mathrm{d}\mu^4 &= 0.
\end{align*}
\]

\[
\mathrm{d}\mu^i = C^{ri}_{jk}(Z) \mu^j \wedge \mu^k
\]
Essential Invariants

- The pseudo-group structure equations live on the bundle \( \tau : G^{(\infty)} \rightarrow M \), and the structure coefficients \( C_{jk}^{i} \) constructed above may vary from point to point.

♥ In the case of a finite-dimensional Lie group action, \( G^{(\infty)} \simeq G \times M \), and this means the basis of Maurer–Cartan forms on each fiber of \( G^{(\infty)} \) is varying with the target point \( Z \in M \). However, we can always make a \( Z \)-dependent change of basis to make the structure coefficients constant.

★ However, for infinite-dimensional pseudo-groups, it may not be possible to find such a change of Maurer–Cartan basis, leading to the concept of essential invariants.
Kadomtsev–Petviashvili (KP) Equation

\[( u_t + \frac{3}{2} u u_x + \frac{1}{4} u_{xxx} )_x \pm \frac{3}{4} u_{yy} = 0 \]

Symmetry generators:
\[
\mathbf{v}_f = f(t) \partial_t + \frac{2}{3} y f'(t) \partial_y + \left( \frac{1}{3} x f'(t) + \frac{2}{9} y^2 f''(t) \right) \partial_x \\
+ \left( -\frac{2}{3} u f'(t) + \frac{2}{9} x f'''(t) \pm \frac{4}{27} y^2 f'''(t) \right) \partial_u,
\]
\[
\mathbf{w}_g = g(t) \partial_y + \frac{2}{3} y g'(t) \partial_x + \frac{4}{9} y g''(t) \partial_u,
\]
\[
\mathbf{z}_h = h(t) \partial_x + \frac{2}{3} h'(t) \partial_u.
\]

\[\implies \text{ Kac–Moody loop algebra } A_4^{(1)}\]
Navier–Stokes Equations

\[ \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \nu \Delta \mathbf{u}, \quad \nabla \cdot \mathbf{u} = 0. \]

Symmetry generators:

\[ \mathbf{v}_\alpha = \alpha(t) \cdot \partial_x + \alpha'(t) \cdot \partial_u - \alpha''(t) \cdot \mathbf{x} \partial_p \]

\[ \mathbf{v}_0 = \partial_t \]

\[ \mathbf{s} = \mathbf{x} \cdot \partial_x + 2t \partial_t - \mathbf{u} \cdot \partial_u - 2p \partial_p \]

\[ \mathbf{r} = \mathbf{x} \wedge \partial_x + \mathbf{u} \wedge \partial_u \]

\[ \mathbf{w}_h = h(t) \partial_p \]
Action of Pseudo-groups on Submanifolds a.k.a. Solutions of Differential Equations

\( \mathcal{G} \) — Lie pseudo-group acting on \( p \)-dimensional submanifolds:

\[
N = \{ u = f(x) \} \subset M
\]

For example, \( \mathcal{G} \) may be the symmetry group of a system of differential equations

\[
\Delta(x, u^{(n)}) = 0
\]

and the submanifolds are the graphs of solutions \( u = f(x) \).

Goal: Understand \( \mathcal{G} \)-invariant objects (moduli spaces)
Prolongation

\( J^n = J^n(M, p) \) — \( n^{th} \) order submanifold jet bundle

Local coordinates:

\( z^{(n)} = (x, u^{(n)}) = (\ldots x^i \ldots u^\alpha_J \ldots) \)

Prolonged action of \( G^{(n)} \) on submanifolds:

\[ (x, u^{(n)}) \mapsto (X, \hat{U}^{(n)}) \]

Coordinate formulae:

\[ \hat{U}^\alpha_J = F^\alpha_J(x, u^{(n)}, g^{(n)}) \]

\( \implies \) Implicit differentiation.
Differential Invariants

A differential invariant is an invariant function $I : J^n \to \mathbb{R}$ for the prolonged pseudo-group action

$$I(g^{(n)} \cdot (x, u^{(n)})) = I(x, u^{(n)})$$

$\implies$ curvature, torsion, ...

Invariant differential operators:

$\mathcal{D}_1, \ldots, \mathcal{D}_p$ $\implies$ arc length derivative

• If $I$ is a differential invariant, so is $\mathcal{D}_j I$.

$\mathcal{I}(\mathcal{G})$ — the algebra of differential invariants
The Basis Theorem

**Theorem.** The differential invariant algebra $\mathcal{I}(G)$ is locally generated by a finite number of differential invariants $I_1, \ldots, I_\ell$

and $p = \dim S$ invariant differential operators $D_1, \ldots, D_p$

meaning that every differential invariant can be locally expressed as a function of the generating invariants and their invariant derivatives:

$$D_J I_\kappa = D_{j_1} D_{j_2} \cdots D_{j_n} I_\kappa.$$
Key Issues

- **Minimal basis** of generating invariants: $I_1, \ldots, I_\ell$

- **Commutation formulae** for the invariant differential operators:

  \[
  [D_j, D_k] = \sum_{i=1}^{p} Y_{jk}^i D_i
  \]

  \[\Rightarrow\] Non-commutative differential algebra

- **Syzygies** (functional relations) among the differentiated invariants:

  \[\Phi(\ldots D_J I_\kappa \ldots ) \equiv 0\]

  \[\Rightarrow\] Codazzi relations
Computing Differential Invariants

♦ The infinitesimal method:
\[ v(I) = 0 \quad \text{for every infinitesimal generator} \quad v \in \mathfrak{g} \]
\[ \implies \text{Requires solving differential equations.} \]

♥ Moving frames.

- Completely algebraic.
- Can be adapted to arbitrary group and pseudo-group actions.
- Describes the complete structure of the differential invariant algebra \( \mathcal{I}(\mathcal{G}) \) — using only linear algebra & differentiation!
- Prescribes differential invariant signatures for equivalence and symmetry detection.
In the finite-dimensional Lie group case, a moving frame is defined as an equivariant map

\[ \rho^{(n)} : J^n \rightarrow G \]
However, we do not have an appropriate abstract object to represent our pseudo-group $\mathcal{G}$.

Consequently, the moving frame will be an equivariant section

$$\rho^{(n)} : J^n \longrightarrow \mathcal{H}^{(n)}$$

of the pulled-back pseudo-group jet groupoid:

$$\begin{array}{c}
\mathcal{G}^{(n)} \\
\downarrow \\
M
\end{array} \quad \begin{array}{c}
\mathcal{H}^{(n)} \\
\downarrow \\
J^n
\end{array}$$
Moving Frames for Pseudo–Groups

Definition. A (right) moving frame of order $n$ is a right-equivariant section $\rho^{(n)}: V^n \rightarrow \mathcal{H}^{(n)}$ defined on an open subset $V^n \subset J^n$. 

$\Rightarrow$ Groupoidoid action.
Moving Frames for Pseudo–Groups

**Definition.** A (right) moving frame of order $n$ is a right-equivariant section $\rho^{(n)} : V^n \to \mathcal{H}^{(n)}$ defined on an open subset $V^n \subset J^n$.

$
\implies \quad \text{Groupoid action.}
$

**Proposition.** A moving frame of order $n$ exists if and only if $\mathcal{G}^{(n)}$ acts freely and regularly.
For Lie group actions, freeness means trivial isotropy:

\[ G_z = \{ g \in G \mid g \cdot z = z \} = \{ e \}. \]

For infinite-dimensional pseudo-groups, this definition cannot work, and one must restrict to the transformation jets of order \( n \), using the \( n^{\text{th}} \) order isotropy subgroup:

\[ G^{(n)}_z = \{ g^{(n)} \in G^{(n)}_z \mid g^{(n)} \cdot z^{(n)} = z^{(n)} \} \]

**Definition.** At a jet \( z^{(n)} \in J^n \), the pseudo-group \( G \) acts

- freely if \( G^{(n)}_{z(n)} = \{ 1^{(n)}_z \} \)
- locally freely if
  - \( G^{(n)}_{z(n)} \) is a discrete subgroup of \( G^{(n)}_z \)
  - the orbits have dimension \( r_n = \dim G^{(n)}_z \)

\[ \implies \text{Kumpera’s growth bounds on Spencer cohomology.} \]
Persistence of Freeness

**Theorem.** If \( n \geq 1 \) and \( \mathcal{G}^{(n)} \) acts (locally) freely at \( z^{(n)} \in J^n \), then it acts (locally) freely at any \( z^{(k)} \in J^k \) with \( \tilde{\pi}_n^{(k)}(z^{(k)}) = z^{(n)} \) for all \( k > n \).
The Normalization Algorithm

To construct a moving frame:

I. Compute the prolonged pseudo-group action

\[ u_\alpha^K \mapsto U_\alpha^K = F_\alpha^K(x, u^{(n)}, g^{(n)}) \]

by implicit differentiation.

II. Choose a cross-section to the pseudo-group orbits:

\[ u_{\alpha \kappa}^J = c_\kappa, \quad \kappa = 1, \ldots, r_n = \text{fiber dim } \mathcal{G}^{(n)} \]
III. Solve the normalization equations

\[ U^{\alpha \kappa}_{J \kappa} = F^{\alpha \kappa}_{J \kappa}(x, u^{(n)}, g^{(n)}) = c_{\kappa} \]

for the \( n^{th} \) order pseudo-group parameters

\[ g^{(n)} = \rho^{(n)}(x, u^{(n)}) \]

IV. Substitute the moving frame formulas into the un-normalized jet coordinates

\[ u^{\alpha}_{K} = F^{\alpha}_{K}(x, u^{(n)}, g^{(n)}) \]

The resulting functions form a complete system of \( n^{th} \) order differential invariants

\[ I^{\alpha}_{K}(x, u^{(n)}) = F^{\alpha}_{K}(x, u^{(n)}, \rho^{(n)}(x, u^{(n)})) \]
Invariantization

A moving frame induces an invariantization process, denoted $\iota$, that projects functions to invariants, differential operators to invariant differential operators; differential forms to invariant differential forms, etc.

Geometrically, the invariantization of an object is the unique invariant version that has the same cross-section values.

Algebraically, invariantization amounts to replacing the group parameters in the transformed object by their moving frame formulas.
Invariantization

In particular, invariantization of the jet coordinates leads to a complete system of functionally independent differential invariants:

\[ \iota(x^i) = H^i \quad \iota(u_{J}^{\alpha}) = I_{J}^{\alpha} \]

- Phantom differential invariants:
  \[ I_{J,\kappa}^{\alpha \kappa} = c_{\kappa} \]
- The non-constant invariants form a complete system of functionally independent differential invariants
- Replacement Theorem

\[ I(\ldots x^i \ldots u_{J}^{\alpha} \ldots) = \iota(I(\ldots x^i \ldots u_{J}^{\alpha} \ldots)) = I(\ldots H^i \ldots I_{J}^{\alpha} \ldots) \]
The Invariant Variational Bicomplex

◊ Differential functions $\implies$ differential invariants

\[ \iota(x^i) = H^i \quad \iota(u^\alpha_j) = I^\alpha_j \]

◊ Differential forms $\implies$ invariant differential forms

\[ \iota(dx^i) = \varpi^i \quad \iota(\theta^\alpha_K) = \vartheta^\alpha_K \]

◊ Differential operators $\implies$ invariant differential operators

\[ \iota(D_{x^i}) = D_i \]
Recurrence Formulae

Invariantization and differentiation do not commute

The recurrence formulae connect the differentiated invariants with their invariantized counterparts:

\[ D_i I_j^\alpha = I_{j,i}^\alpha + M_{j,i}^\alpha \]

\[ \implies M_{j,i}^\alpha \text{ — correction terms} \]
Once established, the recurrence formulae completely prescribe the structure of the differential invariant algebra $I(G)$ — thanks to the functional independence of the non-phantom normalized differential invariants.

The recurrence formulae can be explicitly determined using only the infinitesimal generators and linear differential algebra!
Lie–Tresse–Kumpera Example

\[
X = f(x), \quad Y = y, \quad U = \frac{u}{f'(x)}
\]

Horizontal coframe

\[
d_X X = f_x \, dx, \quad d_X Y = dy,
\]

Implicit differentiations

\[
D_X = \frac{1}{f_x} \, D_x, \quad D_Y = D_y.
\]
Prolonged pseudo-group transformations on surfaces $S \subset \mathbb{R}^3$:

\[ X = f \quad Y = y \quad U = \frac{u}{f_x} \]

\[ U_X = \frac{u_x}{f_x} - \frac{u f_{xx}}{f_x^2} \quad U_Y = \frac{u_y}{f_x} \]

\[ U_{XX} = \frac{u_{xx}}{f_x^2} - \frac{3u_x f_{xx}}{f_x^4} - \frac{u f_{xxx}}{f_x^4} + \frac{3u f_{xx}^2}{f_x^5} \]

\[ U_{XY} = \frac{u_{xy}}{f_x^2} - \frac{u_y f_{xx}}{f_x^3} \quad U_{YY} = \frac{u_{yy}}{f_x} \]

\( f, f_x, f_{xx}, f_{xxx}, \ldots \) — pseudo-group parameters

\[ \rightarrow \text{ action is free at every order.} \]
Coordinate cross-section

\[ X = f = 0, \quad U = \frac{u}{f_x} = 1, \quad U_X = \frac{u_x}{f_x^2} - \frac{u f_{xx}}{f_x^3} = 0, \quad U_{XX} = \cdots = 0. \]

Moving frame

\[ f = 0, \quad f_x = u, \quad f_{xx} = u_x, \quad f_{xxx} = u_{xx}. \]

Differential invariants

\[
\begin{align*}
U_Y &= \frac{u_y}{f_x} \quad \mapsto \quad J = \iota(u_y) = \frac{u_y}{u} \\
U_{XY} &= \cdots \quad \mapsto \quad J_1 = \iota(u_{xy}) = \frac{uu_{xy} - u_x u_y}{u^3} \\
U_{YY} &= \cdots \quad \mapsto \quad J_2 = \iota(u_{xy}) = \frac{u_{yy}}{u} \\
U_{XYY} &\quad \mapsto \quad J_3 = \iota(u_{xxy}) \\
U_{XY} &\quad \mapsto \quad J_4 = \iota(u_{xyy}) \\
U_{YYY} &\quad \mapsto \quad J_5 = \iota(u_{yyy})
\end{align*}
\]
Invariant horizontal forms
\[ d_H X = f_x \, dx \quad \mapsto \quad u \, dx, \quad d_H Y = dy \quad \mapsto \quad dy, \]

Invariant differentiations
\[ D_1 = \frac{1}{u} D_x \quad D_2 = D_y \]

Higher order differential invariants:
\[ D_1^m D_2^n J \]
\[ J_{,1} = D_1 J = \frac{uu_{xy} - u_x u_y}{u^3} = J_1, \]
\[ J_{,2} = D_2 J = \frac{uu_{yy} - u_y^2}{u^2} = J_2 - J^2. \]

Recurrence formulae:
\[ D_1 J = J_1, \quad D_2 J = J_2 - J^2, \]
\[ D_1 J_1 = J_3, \quad D_2 J_1 = J_4 - 3 J J_1, \]
\[ D_1 J_2 = J_4, \quad D_2 J_2 = J_5 - J J_2, \]
Korteweg–deVries Equation

Prolonged Symmetry Group Action:

\[ T = e^{3\lambda_4}(t + \lambda_1) \]

\[ X = e^{\lambda_4}(\lambda_3 t + x + \lambda_1 \lambda_3 + \lambda_2) \]

\[ U = e^{-2\lambda_4}(u + \lambda_3) \]

\[ U_T = e^{-5\lambda_4}(u_t - \lambda_3 u_x) \]

\[ U_X = e^{-3\lambda_4}u_x \]

\[ U_{TT} = e^{-8\lambda_4}(u_{tt} - 2\lambda_3 u_{tx} + \lambda_3^2 u_{xx}) \]

\[ U_{TX} = D_X D_T U = e^{-6\lambda_4}(u_{tx} - \lambda_3 u_{xx}) \]

\[ U_{XX} = e^{-4\lambda_4}u_{xx} \]

\[ \vdots \]
Cross Section:

\[ T = e^{3\lambda_4}(t + \lambda_1) = 0 \]
\[ X = e^{\lambda_4}(\lambda_3 t + x + \lambda_1 \lambda_3 + \lambda_2) = 0 \]
\[ U = e^{-2\lambda_4}(u + \lambda_3) = 0 \]
\[ U_T = e^{-5\lambda_4}(u_t - \lambda_3 u_x) = 1 \]

Moving Frame:

\[ \lambda_1 = -t, \quad \lambda_2 = -x, \quad \lambda_3 = -u, \quad \lambda_4 = \frac{1}{5} \log(u_t + uu_x) \]
Normalized differential invariants:

\[ I_{01} = \nu(u_x) = \frac{u_x}{(u_t + uu_x)^{3/5}} \]

\[ I_{20} = \nu(u_{tt}) = \frac{u_{tt} + 2uu_{tx} + u^2u_{xx}}{(u_t + uu_x)^{8/5}} \]

\[ I_{11} = \nu(u_{tx}) = \frac{u_{tx} + uu_{xx}}{(u_t + uu_x)^{6/5}} \]

\[ I_{02} = \nu(u_{xx}) = \frac{u_{xx}}{(u_t + uu_x)^{4/5}} \]

\[ \vdots \]

Invariant differential operators:

\[ D_1 = \nu(D_t) = (u_t + uu_x)^{-3/5}D_t + u(u_t + uu_x)^{-3/5}D_x, \]

\[ D_2 = \nu(D_x) = (u_t + uu_x)^{-1/5}D_x. \]
Commutation formula:

\[
[D_1, D_2] = I_{01} D_1
\]

Recurrence formulae:

\[
\begin{align*}
D_1 I_{01} & = I_{11} - \frac{3}{5} I_{01}^2 - \frac{3}{5} I_{01} I_{20}, \\
D_1 I_{20} & = I_{30} + 2 I_{11} - \frac{8}{5} I_{01} I_{20} - \frac{8}{5} I_{20}^2, \\
D_1 I_{11} & = I_{21} + I_{02} - \frac{6}{5} I_{01} I_{11} - \frac{6}{5} I_{11} I_{20}, \\
D_1 I_{02} & = I_{12} - \frac{4}{5} I_{01} I_{02} - \frac{4}{5} I_{02} I_{20}, \\
\vdots 
\end{align*}
\]

\[
\begin{align*}
D_2 I_{01} & = I_{02} - \frac{3}{5} I_{01}^3 - \frac{3}{5} I_{01} I_{11}, \\
D_2 I_{20} & = I_{21} + 2 I_{01} I_{11} - \frac{8}{5} I_{01}^2 I_{20} - \frac{8}{5} I_{11} I_{20}, \\
D_2 I_{11} & = I_{12} + I_{01} I_{02} - \frac{6}{5} I_{01}^2 I_{11} - \frac{6}{5} I_{11}^2, \\
D_2 I_{02} & = I_{03} - \frac{4}{5} I_{01}^2 I_{02} - \frac{4}{5} I_{02} I_{11}, \\
\vdots 
\end{align*}
\]
Generating differential invariants:

\[ I_{01} = \iota(u_x) = \frac{u_x}{(u_t + uu_x)^{3/5}}, \quad I_{20} = \iota(u_{tt}) = \frac{u_{tt} + 2uu_{tx} + u^2u_{xx}}{(u_t + uu_x)^{8/5}}. \]

Fundamental syzygy:

\[ \mathcal{D}_1^2 I_{01} + \frac{3}{5} I_{01} \mathcal{D}_1 I_{20} - \mathcal{D}_2 I_{20} + \left(\frac{1}{5} I_{20} + \frac{19}{5} I_{01}\right) \mathcal{D}_1 I_{01} \]
\[ -\mathcal{D}_2 I_{01} - \frac{6}{25} I_{01} I_{20}^2 - \frac{7}{25} I_{01}^2 I_{20} + \frac{24}{25} I_{01}^3 = 0. \]
The Master Recurrence Formula

\[ d_H I^\alpha_J = \sum_{i=1}^{p} \left( D_i I^\alpha_J \right) \omega^i = \sum_{i=1}^{p} I^\alpha_{J,i} \omega^i + \hat{\psi}^\alpha_J \]

where

\[ \hat{\psi}^\alpha_J = \iota( \hat{\varphi}^\alpha_J ) = \Phi^\alpha_J \left( \ldots H^i \ldots I^\alpha_J \ldots ; \ldots \gamma^b_A \ldots \right) \]

are the invariantized prolonged vector field coefficients, which are particular linear combinations of

\[ \gamma^b_A = \iota( \zeta^b_A ) \quad \text{invariantized Maurer–Cartan forms prescribed by the invariantized prolongation map.} \]

• The invariantized Maurer–Cartan forms are subject to the invariantized determining equations:

\[ \mathcal{L}(H^1, \ldots, H^p, I^1, \ldots, I^q, \ldots, \gamma^b_A, \ldots) = 0 \]
\[ d_H I^\alpha_J = \sum_{i=1}^{p} I^\alpha_{j,i} \omega^i + \hat{\psi}_j^\alpha( \ldots \gamma^b_A \ldots ) \]

**Step 1:** Solve the phantom recurrence formulas

\[ 0 = d_H I^\alpha_j = \sum_{i=1}^{p} I^\alpha_{j,i} \omega^i + \hat{\psi}_j^\alpha( \ldots \gamma^b_A \ldots ) \]

for the invariantized Maurer–Cartan forms:

\[ \gamma^b_A = \sum_{i=1}^{p} J^b_{A,i} \omega^i \quad (\ast) \]

**Step 2:** Substitute (\ast) into the non-phantom recurrence formulae to obtain the explicit correction terms.
Only uses linear differential algebra based on the specification of cross-section.

Does not require explicit formulas for the moving frame, the differential invariants, the invariant differential operators, or even the Maurer–Cartan forms!
Lie–Tresse–Kumpera Example (continued)

\[
X = f(x), \quad Y = y, \quad U = \frac{u}{f'(x)}
\]

Phantom recurrence formulae:

\[
0 = dH = \varpi^1 + \gamma, \quad 0 = dI_{10} = J_1 \varpi^2 + \vartheta_1 - \gamma_2, \\
0 = dI_{00} = J \varpi^2 + \vartheta - \gamma_1, \quad 0 = dI_{20} = J_3 \varpi^2 + \vartheta_3 - \gamma_3,
\]

Solve for pulled-back Maurer–Cartan forms:

\[
\gamma = -\varpi^1, \quad \gamma_2 = J_1 \varpi^2 + \vartheta_1, \\
\gamma_1 = J \varpi^2 + \vartheta, \quad \gamma_3 = J_3 \varpi^2 + \vartheta_3,
\]
Recurrence formulae:  

\[ dy = \varpi^2 \]

\[ dJ = J_1 \varpi^1 + (J_2 - J^2) \varpi^2 + \vartheta_2 - J \vartheta, \]

\[ dJ_1 = J_3 \varpi^1 + (J_4 - 3 J J_1) \varpi^2 + \vartheta_4 - J \vartheta_1 - J_1 \vartheta, \]

\[ dJ_2 = J_4 \varpi^1 + (J_5 - J J_2) \varpi^2 + \vartheta_5 - J_2 \vartheta, \]
The Korteweg–deVries Equation (continued)

Recurrence formula:

\[ dI_{jk} = I_{j+1,k} \omega^1 + I_{j,k+1} \omega^2 + \iota(\varphi^{jk}) \]

Invariantized Maurer–Cartan forms:

\[ \iota(\tau) = \lambda, \quad \iota(\xi) = \mu, \quad \iota(\varphi) = \psi = \nu, \quad \iota(\tau_t) = \psi^t = \lambda_t, \quad \ldots \]

Invariantized determining equations:

\[ \lambda_x = \lambda_u = \mu_u = \nu_t = \nu_x = 0 \]
\[ \nu = \mu_t \quad \nu_u = -2\mu_x = -\frac{2}{3}\lambda_t \]
\[ \lambda_{tt} = \lambda_{tx} = \lambda_{xx} = \ldots = \nu_{uu} = \ldots = 0 \]

Invariantizations of prolonged vector field coefficients:

\[ \iota(\tau) = \lambda, \quad \iota(\xi) = \mu, \quad \iota(\varphi) = \nu, \quad \iota(\varphi^t) = -I_{01}\nu - \frac{5}{3}\lambda_t, \]
\[ \iota(\varphi^x) = -I_{01}\lambda_t, \quad \iota(\varphi^{tt}) = -2I_{11}\nu - \frac{8}{3}I_{20}\lambda_t, \quad \ldots \]
Phantom recurrence formulae:

\[ 0 = d_H H^1 = \omega^1 + \lambda, \]

\[ 0 = d_H H^2 = \omega^2 + \mu, \]

\[ 0 = d_H I_{00} = I_{10}\omega^1 + I_{02}\omega^2 + \psi = \omega^1 + I_{01}\omega^2 + \nu, \]

\[ 0 = d_H I_{10} = I_{20}\omega^1 + I_{11}\omega^2 + \psi^t = I_{20}\omega^1 + I_{11}\omega^2 - I_{01}\nu - \frac{5}{3} \lambda_t, \]

\[ \implies \text{Solve for} \quad \lambda = -\omega^1, \quad \mu = -\omega^2, \quad \nu = -\omega^1 - I_{01}\omega^2, \]

\[ \lambda_t = \frac{3}{5} (I_{20} + I_{01})\omega^1 + \frac{3}{5} (I_{11} + I_{01})\omega^2. \]

Non-phantom recurrence formulae:

\[ d_H I_{01} = I_{11}\omega^1 + I_{02}\omega^2 - I_{01}\lambda_t, \]

\[ d_H I_{20} = I_{30}\omega^1 + I_{21}\omega^2 - 2I_{11}\nu - \frac{8}{3} I_{20}\lambda_t, \]

\[ d_H I_{11} = I_{21}\omega^1 + I_{12}\omega^2 - I_{02}\nu - 2I_{11}\lambda_t, \]

\[ d_H I_{02} = I_{12}\omega^1 + I_{03}\omega^2 - \frac{4}{3} I_{02}\lambda_t, \]

\[ \vdots \]
\[ 
\mathcal{D}_1 I_{01} = I_{11} - \frac{3}{5} I^2_{01} - \frac{3}{5} I_{01} I_{20}, \\
\mathcal{D}_1 I_{20} = I_{30} + 2I_{11} - \frac{8}{5} I_{01} I_{20} - \frac{8}{5} I^2_{20}, \\
\mathcal{D}_1 I_{11} = I_{21} + I_{02} - \frac{6}{5} I_{01} I_{11} - \frac{6}{5} I_{11} I_{20}, \\
\mathcal{D}_1 I_{02} = I_{12} - \frac{4}{5} I_{01} I_{02} - \frac{4}{5} I_{02} I_{20}, \\
\vdots \\
\mathcal{D}_2 I_{01} = I_{02} - \frac{3}{5} I^3_{01} - \frac{3}{5} I_{01} I_{11}, \\
\mathcal{D}_2 I_{20} = I_{21} + 2I_{01} I_{11} - \frac{8}{5} I^2_{01} I_{20} - \frac{8}{5} I_{11} I_{20}, \\
\mathcal{D}_2 I_{11} = I_{12} + I_{01} I_{02} - \frac{6}{5} I^2_{01} I_{11} - \frac{6}{5} I^2_{11}, \\
\mathcal{D}_2 I_{02} = I_{03} - \frac{4}{5} I^2_{01} I_{02} - \frac{4}{5} I_{02} I_{11}, \\
\vdots 
\]
Gröbner Basis Approach

Suppose $\mathcal{G}$ acts freely at order $n^*$. The differential invariants of order $> n^*$ are naturally identified with polynomials belonging to a certain algebraic module $\mathcal{J}$, called the invariantized prolonged symbol module which is defined as the invariantized pull-back of the symbol module for the infinitesimal determining equations under a certain explicit linear map.
Constructive Basis Theorem

Theorem. A system of generating differential invariants in one-to-one correspondence with the Gröbner basis elements of the invariantized prolonged symbol module $\mathcal{J}^{>n^*}$ plus, possibly, a finite number of differential invariants of order $\leq n^*$. 
Theorem. Every differential syzygy among the generating differential invariants is either a syzygy among those of order \( \leq n^* \), or arises from an algebraic syzygy among the Gröbner basis polynomials in \( J^{>n^*} \), or comes from a commutator syzygy among the invariant differential operators.
The Symbol Module

Linearized determining equations

\[ \mathcal{L}(z, \zeta^{(n)}) = 0 \]

\[ t = (t_1, \ldots, t_m), \quad T = (T_1, \ldots, T_m) \]

\[ \mathcal{T} = \left\{ P(t, T) = \sum_{a=1}^{m} P_a(t) T_a \right\} \cong \mathbb{R}[t] \otimes \mathbb{R}^m \subset \mathbb{R}[t, T] \]

Symbol module:

\[ I \subset \mathcal{T} \]
The Prolonged Symbol Module

\[ s = (s_1, \ldots, s_p), \quad S = (S_1, \ldots, S_q), \]

\[ \hat{S} = \left\{ T(s, S) = \sum_{\alpha = 1}^{q} T_{\alpha}(s) S_{\alpha} \right\} \simeq \mathbb{R}[s] \otimes \mathbb{R}^q \subset \mathbb{R}[s, S] \]

Define the linear map

\[ s_i = \beta_i(t) = t_i + \sum_{\alpha = 1}^{q} u_{i\alpha}^\alpha t_{p+\alpha}, \quad i = 1, \ldots, p, \]

\[ S_{\alpha} = B_{\alpha}(T) = T_{p+\alpha} - \sum_{i=1}^{p} u_{i\alpha}^\alpha T_i, \quad \alpha = 1, \ldots, q. \]

\[ \implies \text{“symbol” of the vector field prolongation operation} \]

Prolonged symbol module:

\[ \mathcal{J} = (\beta^*)^{-1}(\mathcal{I}) \]