Invariant Signatures and Histograms for Object Recognition, Symmetry Detection, and Jigsaw Puzzle Assembly

Peter J. Olver
University of Minnesota

http://www.math.umn.edu/~olver

San Diego, January, 2013
The Basic Equivalence Problem

$M$ — smooth $m$-dimensional manifold.

$G$ — transformation group acting on $M$

- finite-dimensional Lie group
- infinite-dimensional Lie pseudo-group
Equivalence:
Determine when two $p$-dimensional submanifolds $N$ and $\overline{N} \subset M$ are congruent:

$$\overline{N} = g \cdot N \quad \text{for} \quad g \in G$$

Symmetry:
Find all symmetries, i.e., self-equivalences or self-congruences:

$$N = g \cdot N$$
Classical Geometry

- **Euclidean group:**
  \[ G = \begin{cases} 
  \text{SE}(m) = \text{SO}(m) \ltimes \mathbb{R}^m \\
  \text{E}(m) = \text{O}(m) \ltimes \mathbb{R}^m 
  \end{cases} \]

  \[ z \mapsto A \cdot z + b \quad A \in \text{SO}(m) \text{ or } \text{O}(m), \quad b \in \mathbb{R}^m, \quad z \in \mathbb{R}^m \]

  \[ \Rightarrow \text{isometries: rotations, translations, (reflections)} \]

- **Equi-affine group:**
  \[ G = \text{SA}(m) = \text{SL}(m) \ltimes \mathbb{R}^m \]

  \[ A \in \text{SL}(m) \quad \text{— volume-preserving} \]

- **Affine group:**
  \[ G = \text{A}(m) = \text{GL}(m) \ltimes \mathbb{R}^m \]

  \[ A \in \text{GL}(m) \]

- **Projective group:**
  \[ G = \text{PSL}(m + 1) \]

  acting on \( \mathbb{R}^m \subset \mathbb{RP}^m \)

  \[ \Rightarrow \quad F. \ Klein \]
Tennis, Anyone?
Euclidean Plane Curves: \( G = \text{SE}(2) = \text{SO}(2) \ltimes \mathbb{R}^2 \)

Curvature differential invariants:

\[
\kappa = \frac{u_{xx}}{(1 + u_x^2)^{3/2}}, \quad \frac{d\kappa}{ds} = \frac{(1 + u_x^2)u_{xxx} - 3u_xu_{xx}^2}{(1 + u_x^2)^3}, \quad \frac{d^2\kappa}{ds^2} = \ldots
\]

Arc length (invariant one-form):

\[
ds = \sqrt{1 + u_x^2} \, dx, \quad \frac{d}{ds} = \frac{1}{\sqrt{1 + u_x^2}} \frac{d}{dx}
\]

**Theorem.** All Euclidean differential invariants are functions of the derivatives of curvature with respect to arc length: \( \kappa, \kappa_s, \kappa_{ss}, \ldots \)
**Equi-affine Plane Curves:** \( G = SA(2) = SL(2) \ltimes \mathbb{R}^2 \)

**Equi-affine curvature:**

\[
\kappa = \frac{5 u_{xx} u_{xxxx} - 3 u_{xxx}^2}{9 u_{xx}^{8/3}} \quad \frac{d\kappa}{ds} = \ldots
\]

**Equi-affine arc length:**

\[
ds = 3 \sqrt[3]{u_{xx}} \, dx \quad \frac{d}{ds} = \frac{1}{3 \sqrt[3]{u_{xx}}} \frac{d}{dx}
\]

**Theorem.** All equi-affine differential invariants are functions of the derivatives of equi-affine curvature with respect to equi-affine arc length:

\[
\kappa, \quad \kappa_s, \quad \kappa_{ss}, \quad \ldots
\]
Equivalence & Invariants

- Equivalent submanifolds $N \approx \overline{N}$ must have the same invariants: $I = \overline{I}$. 
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Constant invariants provide immediate information:

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\text{e.g.} \quad \kappa = 2 \quad \iff \quad \overline{\kappa} = 2
\]
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Constant invariants provide immediate information:

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Non-constant invariants are not useful in isolation, because an equivalence map can drastically alter the dependence on the submanifold parameters:

\[ \text{e.g.} \quad \kappa = x^3 \quad \text{versus} \quad \overline{\kappa} = \sinh x \]
However, a functional dependency or syzygy among the invariants is intrinsic:

\[ \kappa_s = \kappa^3 - 1 \iff \overline{\kappa_s} = \overline{\kappa}^3 - 1 \]
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- Universal syzygies — Gauss–Codazzi
- Distinguishing syzygies.
However, a functional dependency or *syzygy* among the invariants is intrinsic:

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**Theorem.** (Cartan)

Two regular submanifolds are (locally) equivalent if and only if they have identical syzygies among all their differential invariants.
Finiteness of Generators and Syzygies

♠ There are, in general, an infinite number of differential invariants and hence an infinite number of syzygies must be compared to establish equivalence.
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But the higher order differential invariants are always generated by invariant differentiation from a finite collection of basic differential invariants, and the higher order syzygies are all consequences of a finite number of low order syzygies!
Example — Plane Curves

If non-constant, both $\kappa$ and $\kappa_s$ depend on a single parameter, and so, locally, are subject to a syzygy:

$$\kappa_s = H(\kappa)$$ \hspace{1cm} (*)

But then

$$\kappa_{ss} = \frac{d}{ds} H(\kappa) = H'(\kappa) \kappa_s = H'(\kappa) H(\kappa)$$

and similarly for $\kappa_{sss}$, etc.

Consequently, all the higher order syzygies are generated by the fundamental first order syzygy (*).

Thus, for Euclidean (or equi-affine or projective or ...) plane curves we need only know a single syzygy between $\kappa$ and $\kappa_s$ in order to establish equivalence!
Signature Curves

Definition. The signature curve $S \subset \mathbb{R}^2$ of a plane curve $C \subset \mathbb{R}^2$ is parametrized by the two lowest order differential invariants

$$
\Sigma : C \rightarrow S = \left\{ \left( \kappa, \frac{d\kappa}{ds} \right) \right\} \subset \mathbb{R}^2
$$

$\Rightarrow$ Calabi, PJO, Shakiban, Tannenbaum, Haker

Theorem. Two regular curves $C$ and $\overline{C}$ are equivalent:

$$
\overline{C} = g \cdot C
$$

if and only if their signature curves are identical:

$$
\overline{S} = S
$$

$\Rightarrow$ regular: $(\kappa_s, \kappa_{ss}) \neq 0$. 
Theorem. The following are equivalent:

- The curve $C$ has a 1-dimensional symmetry group $H \subset G$
- $C$ is the orbit of a 1-dimensional subgroup $H \subset G$
- The signature $S$ degenerates to a point: $\dim S = 0$
- All the differential invariants are constant: $\kappa = c$, $\kappa_s = 0$, $\ldots$

$\Rightarrow$ Euclidean plane geometry: circles, lines
$\Rightarrow$ Equi-affine plane geometry: conic sections.
$\Rightarrow$ Projective plane geometry: $W$ curves (Lie & Klein)
Symmetry and Signature

Discrete Symmetries

**Definition.** The index of a curve $C$ equals the number of points in $C$ which map to a single generic point of its signature:

$$\iota_C = \min \left\{ \# \Sigma^{-1}\{w\} \mid w \in S \right\}$$

**Theorem.** The number of discrete symmetries of $C$ equals its index $\iota_C$. 
The Index

$N \xrightarrow{\Sigma} S$
The Curve $x = \cos t + \frac{1}{5} \cos^2 t, \quad y = \sin t + \frac{1}{10} \sin^2 t$

The Original Curve  Euclidean Signature  Affine Signature
The Curve \[ x = \cos t + \frac{1}{5} \cos^2 t, \quad y = \frac{1}{2} x + \sin t + \frac{1}{10} \sin^2 t \]

The Original Curve | Euclidean Signature | Affine Signature
Canine Left Ventricle Signature

Original Canine Heart MRI Image

Boundary of Left Ventricle
Smoothed Ventricle Signature

![Smoothed Ventricle Signature](image-url)
Object Recognition
Signature Metrics

• Hausdorff
• Monge–Kantorovich transport
• Electrostatic/gravitational attraction
• Latent semantic analysis
• Histograms
• Gromov–Hausdorff & Gromov–Wasserstein
Signatures

Original curve

Classical Signature

Differential invariant signature
Signatures

Original curve

Classical Signature

Differential invariant signature
Occlusions

Original curve

Classical Signature

Differential invariant signature
Classical Occlusions

\[ \kappa \rightarrow \]
3D Differential Invariant Signatures

**Euclidean space curves:** \( C \subset \mathbb{R}^3 \)

\[ S = \{ (\kappa, \kappa_s, \tau) \} \subset \mathbb{R}^3 \]

- \( \kappa \) — curvature, \( \tau \) — torsion

**Euclidean surfaces:** \( S \subset \mathbb{R}^3 \) (generic)

\[ S = \{ (H, K, H_1, H_2, K_1, K_2) \} \subset \mathbb{R}^6 \]

or \( \hat{S} = \{ (H, H_1, H_2, H_{11}) \} \subset \mathbb{R}^4 \)

- \( H \) — mean curvature, \( K \) — Gauss curvature

**Equi–affine surfaces:** \( S \subset \mathbb{R}^3 \) (generic)

\[ S = \{ (P, P_1, P_2, P_{11}) \} \subset \mathbb{R}^4 \]

- \( P \) — Pick invariant
Advantages of the Signature Curve

• Purely local — no ambiguities
• Symmetries and approximate symmetries
• Extends to surfaces and higher dimensional sub-manifolds
• Occlusions and reconstruction
• Partial matching and puzzles

Main disadvantage: Noise sensitivity due to dependence on high order derivatives.
Generalized Vertices

Ordinary vertex: local extremum of curvature

Generalized vertex: $\kappa_s \equiv 0$

- critical point
- circular arc
- straight line segment

Mukhopadhya’s Four Vertex Theorem:
A simple closed, non-circular plane curve has $n \geq 4$ generalized vertices.
“Counterexamples”

These degenerate curves all have the same signature:

\[
\begin{array}{ccccccc}
\text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet} \\
\text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet} \\
\end{array}
\]

⋆ Replace vertices with circular arcs: Musso–Nicoldi
Bivertex Arcs

Bivertex arc: $\kappa_s \neq 0$ everywhere

except $\kappa_s = 0$ at the two endpoints

The signature $S$ of a bivertex arc is a single arc that starts and ends on the $\kappa$–axis.
Bivertex Decomposition.

A v-regular curve — finitely many generalized vertices

\[ C = \bigcup_{j=1}^{m} B_j \cup \bigcup_{k=1}^{n} V_k \]

- \( B_1, \ldots, B_m \) — bivertex arcs
- \( V_1, \ldots, V_n \) — generalized vertices: \( n \geq 4 \)

**Main Idea:** Compare individual bivertex arcs, and then determine whether the rigid equivalences are (approximately) the same.

Gravitational/Electrostatic Attraction

★ Treat the two (signature) curves as masses or as oppositely charged wires. The higher their mutual attraction, the closer they are together.
Gravitational/Electrostatic Attraction

* Treat the two (signature) curves as masses or as oppositely charged wires. The higher their mutual attraction, the closer they are together.

* In practice, we are dealing with discrete data (pixels) and so treat the curves and signatures as point masses/charges.
Strength of correspondence:

\[ h(\sigma, \tilde{\sigma}) = \begin{cases} 
\frac{1}{d(\sigma, \tilde{\sigma})\gamma + \epsilon}, & d(\sigma, \tilde{\sigma}) < \infty, \\
0, & d(\sigma, \tilde{\sigma}) = \infty. 
\end{cases} \]

Separation:

\[ d(\sigma, \tilde{\sigma}) = \begin{cases} 
\frac{\| \sigma - \tilde{\sigma} \|}{D - \| \sigma - \tilde{\sigma} \|}, & \| \sigma - \tilde{\sigma} \| < D, \\
\infty, & \| \sigma - \tilde{\sigma} \| \geq D. 
\end{cases} \]

Scale of comparison:

\[ D(C, \tilde{C}) = ( D_\kappa(C, \tilde{C}), D_{\kappa_s}(C, \tilde{C}) ) , \]

\[ D_\kappa(C, \tilde{C}) = \max \left\{ \max_{z \in \tilde{C}}(\kappa|_z) - \min_{z \in C}(\kappa|_z), \max_{\tilde{z} \in \tilde{C}}(\kappa|_{\tilde{z}}) - \min_{\tilde{z} \in \tilde{C}}(\kappa|_{\tilde{z}}) \right\} , \]

\[ D_{\kappa_s}(C, \tilde{C}) = \max \left\{ \max_{z \in \tilde{C}}(\kappa_s|_z) - \min_{z \in C}(\kappa_s|_z), \max_{\tilde{z} \in \tilde{C}}(\kappa_s|_{\tilde{z}}) - \min_{\tilde{z} \in \tilde{C}}(\kappa_s|_{\tilde{z}}) \right\} . \]
Minimize force and torque based on gravitational attraction of the two matching edges.
The Baffler Jigsaw Puzzle
The Baffler Solved
The Rain Forest Giant Floor Puzzle
The Rain Forest Puzzle Solved

The Rain Forest Puzzle Solved

The Distance Histogram

Definition. The distance histogram of a finite set of points \( P = \{z_1, \ldots, z_n\} \subset V \) is the function

\[
\eta_P(r) = \# \left\{ (i, j) \mid 1 \leq i < j \leq n, \ d(z_i, z_j) = r \right\}.
\]
The Distance Set

The support of the histogram function,

$$\text{supp} \; \eta_P = \Delta_P \subset \mathbb{R}^+$$

is the distance set of $P$. 
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Erdös’ distinct distances conjecture (1946):

If $P \subset \mathbb{R}^m$, then

$$\# \Delta_P \geq c_{m, \varepsilon} (\# P)^{2/m - \varepsilon}$$
Characterization of Point Sets

Note: If $\tilde{P} = g \cdot P$ is obtained from $P \subset \mathbb{R}^m$ by a rigid motion $g \in E(n)$, then they have the same distance histogram: $\eta_P = \eta_{\tilde{P}}$. 

Question: Can one uniquely characterize, up to rigid motion, a set of points $\{z_1, ..., z_n\} \subset \mathbb{R}^m$ by its distance histogram?
Characterization of Point Sets

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Question: Can one uniquely characterize, up to rigid motion, a set of points $P\{z_1, \ldots, z_n\} \subset \mathbb{R}^m$ by its distance histogram?

$\implies$ Tinkertoy problem.
Yes:

\[ \eta = 1, 1, 1, 1, \sqrt{2}, \sqrt{2}. \]
No:

Kite

$\eta = \sqrt{2}, \sqrt{2}, 2, \sqrt{10}, \sqrt{10}, 4.$

Trapezoid
No:

\[ P = \{0, 1, 4, 10, 12, 17\} \subset \mathbb{R} \]
\[ Q = \{0, 1, 8, 11, 13, 17\} \]
\[ \eta = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 16, 17 \]

\[ \implies \quad \text{G. Bloom, J. Comb. Theory, Ser. A 22 (1977) 378–379} \]
Characterizing Point Sets by their Distance Histograms

**Theorem.** Suppose \( n \leq 3 \) or \( n \geq m + 2 \).

Then there is a Zariski dense open subset in the space of \( n \) point configurations in \( \mathbb{R}^m \) that are uniquely characterized, up to rigid motion, by their distance histograms.

Limiting Curve Histogram
Limiting Curve Histogram
Limiting Curve Histogram
Sample Point Histograms

Cumulative distance histogram: \( n = \# P:\)

\[
\Lambda_P(r) = \frac{1}{n} + \frac{2}{n^2} \sum_{s \leq r} \eta_P(s) = \frac{1}{n^2} \# \left\{ (i, j) \mid d(z_i, z_j) \leq r \right\},
\]

Note:

\[
\eta_P(r) = \frac{1}{2} n^2 \left[ \Lambda_P(r) - \Lambda_P(r - \delta) \right] \quad \delta \ll 1.
\]

Local cumulative distance histogram:

\[
\lambda_P(r, z) = \frac{1}{n} \# \left\{ j \mid d(z, z_j) \leq r \right\} = \frac{1}{n} \# (P \cap B_r(z))
\]

\[
\Lambda_P(r) = \frac{1}{n} \sum_{z \in P} \lambda_P(r, z) = \frac{1}{n^2} \sum_{z \in P} \# (P \cap B_r(z)).
\]

Ball of radius \( r \) centered at \( z \):

\[
B_r(z) = \{ v \in V \mid d(v, z) \leq r \}
\]
Limiting Curve Histogram Functions

Length of a curve

\[ l(C) = \int_C ds < \infty \]

Local curve distance histogram function

\[ h_C(r, z) = \frac{l(C \cap B_r(z))}{l(C)} \]

\[ \Rightarrow \quad \text{The fraction of the curve contained in the ball of radius } r \text{ centered at } z. \]

Global curve distance histogram function:

\[ H_C(r) = \frac{1}{l(C)} \int_C h_C(r, z(s)) \, ds. \]
**Convergence of Histograms**

**Theorem.** Let $C$ be a regular plane curve. Then, for both uniformly spaced and randomly chosen sample points $P \subset C$, the cumulative local and global histograms converge to their continuous counterparts:

$$
\lambda_P(r, z) \rightarrow h_C(r, z), \quad \Lambda_P(r) \rightarrow H_C(r),
$$

as the number of sample points goes to infinity.

D. Brinkman & PJO, Invariant histograms,

Square Curve Histogram with Bounds
Kite and Trapezoid Curve Histograms
**Histogram–Based Shape Recognition**

500 sample points

<table>
<thead>
<tr>
<th>Shape</th>
<th>(a)</th>
<th>(b)</th>
<th>(c)</th>
<th>(d)</th>
<th>(e)</th>
<th>(f)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) triangle</td>
<td>2.3</td>
<td>20.4</td>
<td>66.9</td>
<td>81.0</td>
<td>28.5</td>
<td>76.8</td>
</tr>
<tr>
<td>(b) square</td>
<td>28.2</td>
<td>.5</td>
<td>81.2</td>
<td>73.6</td>
<td>34.8</td>
<td>72.1</td>
</tr>
<tr>
<td>(c) circle</td>
<td>66.9</td>
<td>79.6</td>
<td>.5</td>
<td>137.0</td>
<td>89.2</td>
<td>138.0</td>
</tr>
<tr>
<td>(d) 2 × 3 rectangle</td>
<td>85.8</td>
<td>75.9</td>
<td>141.0</td>
<td>2.2</td>
<td>53.4</td>
<td>9.9</td>
</tr>
<tr>
<td>(e) 1 × 3 rectangle</td>
<td>31.8</td>
<td>36.7</td>
<td>83.7</td>
<td>55.7</td>
<td>4.0</td>
<td>46.5</td>
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<tr>
<td>(f) star</td>
<td>81.0</td>
<td>74.3</td>
<td>139.0</td>
<td>9.3</td>
<td>60.5</td>
<td>.9</td>
</tr>
</tbody>
</table>
Curve Histogram Conjecture

Two sufficiently regular plane curves $C$ and $\tilde{C}$ have identical global distance histogram functions, so $H_C(r) = H_{\tilde{C}}(r)$ for all $r \geq 0$, if and only if they are rigidly equivalent: $C \simeq \tilde{C}$. 
Possible Proof Strategies

• Show that any polygon obtained from (densely) discretizing a curve does not lie in the Boutin–Kemper exceptional set.

• Polygons with obtuse angles: taking \( r \) small, one can recover (i) the set of angles and (ii) the shortest side length from \( H_C(r) \). Further increasing \( r \) leads to further geometric information about the polygon . . .

• Expand \( H_C(r) \) in a Taylor series at \( r = 0 \) and show that the corresponding integral invariants characterize the curve.
Taylor Expansions

Local distance histogram function:

\[ L h_C(r, z) = 2r + \frac{1}{12} \kappa^2 r^3 + \left( \frac{1}{40} \kappa \kappa_{ss} + \frac{1}{45} \kappa_s^2 + \frac{3}{320} \kappa^4 \right) r^5 + \cdots. \]

Global distance histogram function:

\[ H_C(r) = \frac{2r}{L} + \frac{r^3}{12L^2} \oint_C \kappa^2 \, ds + \frac{r^5}{40L^2} \oint_C \left( \frac{3}{8} \kappa^4 - \frac{1}{9} \kappa_s^2 \right) \, ds + \cdots. \]
Space Curves

Saddle curve:

\[ z(t) = (\cos t, \sin t, \cos 2t), \quad 0 \leq t \leq 2\pi. \]

Convergence of global curve distance histogram function:
Surfaces

Local and global surface distance histogram functions:

\[ h_S(r, z) = \frac{\text{area} (S \cap B_r(z))}{\text{area} (S)} \], \quad H_S(r) = \frac{1}{\text{area} (S)} \int \int_S h_S(r, z) \, dS. \]

Convergence for sphere:
Area Histograms

Rewrite global curve distance histogram function:

\[ H_C(r) = \frac{1}{L} \oint_C h_C(r, z(s)) \, ds = \frac{1}{L^2} \oint_C \oint_C \chi_r(d(z(s), z(s'))) \, ds \, ds' \]

where \( \chi_r(t) = \begin{cases} 1, & t \leq r, \\ 0, & t > r, \end{cases} \)

Global curve area histogram function:

\[ A_C(r) = \frac{1}{L^3} \oint_C \oint_C \oint_C \chi_r(\text{area}(z(\hat{s}), z(\hat{s}'), z(\hat{s}''))) \, d\hat{s} \, d\hat{s}' \, d\hat{s}'', \]

\( d\hat{s} \) — equi-affine arc length element \( L = \int_C d\hat{s} \)

Discrete cumulative area histogram

\[ A_P(r) = \frac{1}{n(n-1)(n-2)} \sum_{z \neq z' \neq z'' \in P} \chi_r(\text{area}(z, z', z'')) , \]

*Boutin & Kemper*: The area histogram uniquely determines generic point sets \( P \subset \mathbb{R}^2 \) up to equi-affine motion.
Area Histogram for Circle

⋆⋆ Joint invariant histograms — convergence???
Triangle Distance Histograms

\[ Z = (\ldots z_i \ldots) \subset M \quad — \quad \text{sample points on a subset } M \subset \mathbb{R}^n \ (\text{curve, surface, etc.}) \]

\[ T_{i,j,k} \quad — \quad \text{triangle with vertices } z_i, z_j, z_k. \]

Side lengths:

\[ \sigma(T_{i,j,k}) = (d(z_i, z_j), d(z_i, z_k), d(z_j, z_k)) \]

Discrete triangle histogram:

\[ S = \sigma(T) \subset K \]

Triangle inequality cone:

\[ K = \{ (x, y, z) \mid x, y, z \geq 0, \ x + y \geq z, \ x + z \geq y, \ y + z \geq x \} \subset \mathbb{R}^3. \]
Triangle Histogram Distributions

Circle  Triangle  Square

Convergence to measures ...  

⇒ Madeleine Kotzagiannidis
Practical Object Recognition

• Scale-invariant feature transform (SIFT) (Lowe)
• Shape contexts (Belongie–Malik–Puzicha)
• Integral invariants (Krim, Kogan, Yezzi, Pottman, …)
• Shape distributions (Osada–Funkhouser–Chazelle–Dobkin)
  Surfaces: distances, angles, areas, volumes, etc.
• Gromov–Hausdorff and Gromov-Wasserstein distances (Mémoli)
  ⇒ lower bounds & stability