1. Introduction.

The original Lamb system was introduced by Horace Lamb in 1900, [13], as a simple model for the phenomenon of radiation damping experienced by a vibrating body in an energy conducting medium. Examples include vibrations of an elastic sphere in a gaseous medium, electrical oscillations of a spherical conductor, a dielectric sphere with large inductance, relativistic radiation of energy from a concentrated mass via gravity waves, and quantum resonance of nuclei. To understand this phenomenon, Lamb proposed a simpler one-dimensional model consisting of an oscillatory point mass–spring system directly coupled to an infinite string modeled by the usual second order wave equation. The oscillator transfers energy to the string, generating waves that propagate outwards at the intrinsic wave speed. Meanwhile, the string induces a damping force on the oscillator, with the effect that the propagating waves are of progressively smaller and smaller amplitudes. It is worth pointing out a potential source of confusion: Lamb refers to the point mass as a “nucleus”, although he clearly does not have atomic nuclei in mind as these were not discovered by Rutherford until 1911.

More recently, Hagerty, Bloch, and Weinstein, [11], constructed gyroscopically coupled Lamb systems, also known as Chetaev systems, including the spherical pendulum and rigid bodies with internal rotors, that can induce instabilities through the effects of Rayleigh dissipation. They also investigate the effects of dissipation as modeled by the Klein–Gordon equation, although as we will see, this does not cover the range of phenomena exhibited by dispersive Lamb models.

This paper analyzes the both the full line and periodic Lamb problems when the mass is coupled to more general one-dimensional dispersive media. Our motivation is to see whether the remarkable phenomenon of dispersive quantization, [15], also known as the
Talbot effect, [1], in honor of an 1836 optical experiment of William Henry Fox Talbot, [20], manifests itself in the Lamb problem. This phenomenon is exhibited by both linear and nonlinear dispersive wave equations on (at least) periodic domains. Briefly, a solution to the \( n \)-th order periodic initial-boundary value problem on the interval \([0, L]\) with rough initial data, e.g., a step function, is “quantized” as a discontinuous but piecewise constant profile or, more generally, piecewise smooth or even piecewise fractal, at times which are a rational multiple of \( L^n/\pi^{n-1} \), but exhibits a continuous but non-differentiable fractal profile at all other times. This phenomenon is a consequence of the slow, conditional convergence of the Fourier series solution, and requires the dispersion relation to be asymptotically polynomial at large wave number. Systems with sub-quadratic dispersion asymptotics exhibit a different range of fascinating, but as yet poorly understood solution dynamics. The key references include the 1990’s discovery of Michael Berry and collaborators, [1], in the context of optics and quantum mechanics, and the subsequent analytical work of Oskolkov, [17]. In particular, this effect underlies the experimentally observed phenomenon of quantum revival, in which an electron that is initially concentrated at a single location of its orbital shell is, at rational times, re-concentrated at a finite number of orbital locations. The effect’s subsequent rediscovery by the first author, [3, 15], showed that the phenomenon extended to linear dispersive wave models arising in fluid mechanics, plasma dynamics, elasticity and elsewhere. In these models the large wave number asymptotics prescribes the overall qualitative features of the subsequent dynamics, which also extends into the nonlinear regime, both integrable, such as the cubic nonlinear Schrödinger, Korteweg–deVries, and modified Korteweg–deVries equations, as well as non-integrable generalizations with higher degree nonlinearities. Some of these numerical observations have been rigorously confirmed in recent papers of Chousionis, Erdoğan, and Tzirakis, [4, 7, 8], but much more work is required. The full range of this fascinating phenomenon, including other systems, other types of boundary conditions, and higher dimensional problems, remains to be explored.

In the case of the original Lamb system of an oscillator coupled to the wave equation on a periodic domain, the solution is readily seen to remain piecewise smooth, the only derivative discontinuities occurring at the wave front and the location of the oscillator. Extending the analysis to a higher order linearly elastic string model shows that the solution continues to exhibit a piecewise smooth profile. Indeed, smoothness will hold for any linearly dispersive equation with linear or super-linear dispersion asymptotics at large wave numbers, including the Klein–Gordon equation considered in [11]. However, our numerical experiments indicate that the periodic Lamb problem with sub-linear dispersion does exhibit a fractal solution profile. We are able to establish the slow decay of the Fourier series representation of the solution, but do not have a rigorous justification that it becomes fractal, nor do we have an estimate of its possible fractal dimension. Indeed, the detailed analysis of functions represented by slowly decaying, conditionally convergent Fourier series remains, by and large, mysterious, and the range of phenomena exhibited by such dispersive systems remains a considerable challenge. These observations demonstrate that the Mandelbrot’s heuristics, [14; p. 360], concerning the fractal dimension of the graphs of slowly convergent Fourier series are certainly not valid in general. See [2] for some rigorous results on the fractal dimensions of the graphs of certain types of Fourier
series, although these results do not extend in any obvious manner to the class of series that arise here.

The main conclusion is that, surprisingly, for a Lamb system starting at rest, fractalization is induced by sub-linear dispersion, whereas for the unforced linearly dispersive system with rough initial data, dispersive quantization/fractalization requires dispersion relations that are asymptotically quadratic or higher order. It would be of great interest to extend these observations into the nonlinear regime.

Our analysis relies on classical Fourier transform and Fourier series methods. One can envision applying the more general and powerful Unified Transform Method (UTM), due to Fokas and collaborators, \cite{9, 10}, to the Lamb problem. However, our formulation of the problem does not belong to the class of equations currently solvable by the UTM since it involves a space-dependent coefficient. A second way to approach such problems is by viewing the original formulation as an interface problem, as in \cite{6, 19}, and combining this with recent work using the UTM for systems of equations, \cite{5}. However, the Lamb condition is a more complicated interface condition than those previously considered and it is not clear how the current work on interface problems can be extended. This remains an interesting challenge.

*Remark:* The paper includes still shots of a variety of solutions at selected times. MATHEMATICA code for generating the movies, which are even more enlightening, can be found on the first author’s web site:

http://www.math.umn.edu/~olver/lamb

2. **The Bidirectional Lamb Model.**

The original Lamb model, \cite{13}, consists of an oscillating point mass that is connected to an infinite elastic string and constrained to move in the transverse direction. With the system starting at rest, the mass is subject to a sudden blow, and the continuous medium serves to dampen the ensuing vibrations of the mass. In the underdamped regime, the oscillation mass induces a traveling wave displacement, of gradually decreasing amplitude at the source, that propagates along the string at its innate wave speed.

Thus, in the small amplitude regime, the string displacement \( u(t, x) \) satisfies the usual bidirectional wave equation away from the fixed location of the mass, which we take to be at the origin \( x = 0 \):

\[
 u_{tt} = c^2 u_{xx}, \quad x \neq 0,
\]

where \( c^2 = T/\rho \) is the squared wave speed, with \( \rho \) representing the string density and \( T \) its tension, which are both assumed constant for simplicity. As in \cite{13}, force balance on the mass displacement \( h(t) = u(t, 0) \) at the origin yields

\[
 M (h'' + \sigma^2 h) = -2T [u_x]_0,
\]

where \( M \) is the mass and \( \sigma \) its uncoupled oscillatory frequency. The forcing term on the right hand side is the jump in the spatial derivative \( u_x \) at the location of the mass multiplied by \(-2\) times the string tension \( T \).
The first observation is that the Lamb model can be rewritten as a forced wave equation of the form
\[ u_{tt} = c^2 u_{xx} + 2c h'(t) \delta(x), \]  
(2.2)
where \( \delta(x) \) is the usual Dirac delta function and \( h(t) \) satisfies the damped oscillator equation
\[ h'' + 2 \beta h' + \sigma^2 h = 0, \quad h(0) = 0, \]  
(2.3)
whose damping coefficient is
\[ \beta = \frac{1}{M} \sqrt{\frac{T}{\rho}}. \]
To prove this, according to [16; Theorem 2.18], the solution to (2.2) with zero initial conditions
\[ u(0, x) = u_t(0, x) = 0, \]  
(2.4)
(i.e., at rest initially) is
\[ u(t, x) = \int_0^t \int_{x - c(t-s)}^{x + c(t-s)} h'(s) \delta(y) dy \, ds, \quad t > 0. \]  
(2.5)
Integrating twice, we deduce that
\[ u(t, x) = \tilde{h}(t - |x|/c), \quad t > 0, \]  
(2.6)
where
\[ \tilde{h}(t) = \begin{cases} h(t), & t > 0, \\ 0 & t < 0. \end{cases} \]  
(2.7)
On the other hand, the rescaled function \( f(t) = h(t/c) \) satisfies Lamb’s oscillator equation [13; equation (8)]:
\[ f'' + \frac{1}{b} f' + \frac{\sigma^2}{c^2} f = 0, \quad f(0) = 0, \]  
(2.8)
where \( b = c/(2 \beta) \). Thus,
\[ f(t) = C e^{-t/(2b)} \sin \kappa t, \]  
(2.9)
where \( C \) is the integration constant, while
\[ \kappa = \sqrt{\frac{\sigma^2}{c^2} - \frac{1}{4b^2}} = \frac{\varsigma}{c}, \quad \varsigma = \sqrt{\sigma^2 - \beta^2}, \]  
(2.10)
where we assume, as does Lamb, that the mass oscillator (2.8) is underdamped, so
\[ \sigma > \beta > 0, \quad \text{or, equivalently,} \quad c > 2b \sigma. \]
Thus, setting
\[ h(t) = f(ct) = C e^{-\beta t} \sin \varsigma t, \]  
(2.11)
reduces (2.6–7) to Lamb’s solution [13; equation (11)]:

\[
    u(t, x) = \begin{cases} 
        Ce^{-(ct-|x|)/(2b)}\sin \kappa (ct - |x|), & |x| < ct, \\
        0, & |x| > ct. 
    \end{cases}
\]  

(2.12)

The resulting solution profile is continuous and piecewise smooth, and consists of a symmetric pair of successively damped oscillatory disturbances generated by the mass that propagate into the undisturbed region at velocities ±c. The only discontinuities in its derivative occur at the front of the disturbance and at the origin as shown in the following graphs, plotted on the interval \(-6\pi \leq x \leq 6\pi\) with \(-.45 \leq y \leq .45\) for the following values: \(c = 1, C = 1/2, b = 5, \kappa = \sqrt{.99}\). The damping effect of the Lamb coupling is already evident in the waves subsequently generated by the oscillator.

As time progresses, the initial disturbance continues to propagate outwards, while the damping becomes completely dominant on increasingly large interval around the origin, as shown in the following graph, which is plotted with the same vertical scale on the much larger interval \(-50\pi \leq x \leq 50\pi\).

3. The Periodic Lamb Model.

Let us now focus our attention on the periodic Lamb problem, that is equations (2.2–3) on the interval \(-\pi < x < \pi\) with periodic boundary conditions. (Equivalently, we regard \(x\) as an angular coordinate on the circle.) As before, we assume the system is initially at rest, and so the initial conditions are (2.4). Thus, if we view the forcing term in (2.2) as a \(2\pi\) periodically extended delta function, the solution can be written in d’Alembert form as a formally infinite sum

\[
    u(t, x) = \sum_{n = -\infty}^{\infty} \tilde{h}(t - |x - 2n\pi|/c), \quad t > 0.
\]  

(3.1)

Since \(\tilde{h}(t) = 0\) whenever \(t \leq 0\), at any given point in space-time, only finitely many summands in (3.1) are non-zero, and hence, by the preceding remarks, the solution is continuous and piecewise smooth.
Alternatively, one can solve the periodic Lamb problem via Fourier methods. Since the solution is even in \( x \), we can expand it into a Fourier cosine series:

\[
u(t, x) = \frac{1}{2} a_0(t) + \sum_{k=1}^{\infty} a_k(t) \cos k x. \tag{3.2}\]

Substituting (3.2) into (2.2), (2.4), and using the fact that the delta function can be written formally as

\[
\delta(x) \sim \frac{1}{2\pi} + \frac{1}{\pi} \sum_{k=0}^{\infty} \cos k x,
\]

we see that the Fourier coefficients \( a_k(t) \) must satisfy the following initial value problems:

\[
a_k'' + c^2 k^2 a_k = \frac{2c}{\pi} h(t), \quad a_k(0) = a_k'(0) = 0. \tag{3.3}\]

Given the particular Lamb forcing function (2.11), after some tedious algebra, the solution is found:

\[
a_k(t) = \frac{2cC}{\pi} \left[ p_k \cos ck t + q_k \sin ck t - e^{-\beta t} (p_k \cos \varsigma t + r_k \sin \varsigma t) \right], \tag{3.4}\]

where

\[
p_k = \frac{\varsigma (\sigma^2 - c^2 k^2)}{(\sigma^2 + c^2 k^2)^2 + 4 c^2 k^2 \beta^2}, \quad q_k = \frac{2ck \beta \varsigma}{(\sigma^2 + c^2 k^2)^2 + 4 c^2 k^2 \beta^2}, \quad r_k = \frac{\beta (\sigma^2 + c^2 k^2)}{(\sigma^2 + c^2 k^2)^2 + 4 c^2 k^2 \beta^2}. \tag{3.5}\]

Recall that, by (2.10), \( \sigma^2 = \varsigma^2 + \beta^2 \).

Some representative solution profiles over one period appear in the following plots. Keep in mind that the solution is both even and periodic.

Let us now generalize the dispersion relation for the linear system being driven by a Lamb oscillator. We replace the second order wave equation by a second order linear dispersive (integro-)differential equation coupled to an oscillator

\[ u_{tt} = L[u] + h'(t) \delta(x). \] (4.1)

The linear operator \( L \) is assumed to have dispersion relation \( \omega = \omega(k) \) relating temporal frequency \( \omega \) and wave number \( k \). As usual, [22], this is found by substituting the exponential ansatz \( e^{i(kx - \omega t)} \) into the unforced equation and which, for a purely dispersive system, is assumed to be a real-valued function. Unless \( \omega(k) \) is a polynomial, the operator \( L \) is of integro-differential form. For simplicity, we have absorbed the prefactor \( 2c \) in (2.2) into the forcing function: \( h(t) \mapsto h(t)/(2c) \).

In particular, the wave equation (2.2) corresponds to a linear dispersion relation \( \omega = ck \). The higher-order correction to an elastic string, [12, 21], is, in the linear regime,

\[ u_{tt} = c^2 u_{xx} + \varepsilon u_{xxxx} + h'(t) \delta(x), \] (4.2)

with dispersion relation

\[ \omega = \sqrt{c^2 k^2 - \varepsilon k^4}, \]

that is asymptotically quadratic for large wave number: \( \omega(k) \sim k^2 \) as \( |k| \to \infty \). The same dispersion relation arises in the Boussinesq approximations in fluid mechanics and water wave theory, [22]. A regularized version

\[ u_{tt} = c^2 u_{xx} + \varepsilon u_{xxtt} + h'(t) \delta(x), \] (4.3)

in which two of the \( x \) derivatives are replace by \( t \) derivatives, which has the same order of approximation to the full physical model, has been proposed as the linearization of a model for DNA dynamics, [18]. Equation (4.3) has dispersion relation

\[ \omega = \frac{c |k|}{\sqrt{1 + \varepsilon k^2}}, \]

which is asymptotically constant at large wave number. In their work on gyroscopic Lamb systems, [11], Hagerty, Bloch, and Weinstein investigate coupling the Lamb oscillator to the linearly dispersive Klein–Gordon equation:

\[ u_{tt} = c^2 u_{xx} - m^2 u + h'(t) \delta(x), \] (4.4)

with “mass” \( m \). In this case, the dispersion relation

\[ \omega = \sqrt{c^2 k^2 + m^2}, \]

is asymptotically linear: \( \omega(k) \sim |k| \).

We begin by investigating the solution to (4.1) with zero initial condition \( u(0, x) = 0 \) on the full line. While there is no longer a d’Alembert style formula for the solution, the Fourier transform will enable us to express the solution in the following form:

\[ u(t, x) = \frac{C}{4c\pi} \int_{-\infty}^{\infty} e^{ikx} \left[ p_k \cos \omega t + q_k \sin \omega t - e^{-\beta t}(p_k \cos \omega t + r_k \sin \omega t) \right] dk, \] (4.5)
where
\[
p_k = \frac{\varsigma (\sigma^2 - \omega^2)}{(\sigma^2 + \omega^2)^2 + 4\omega^2\beta^2}, \quad q_k = \frac{2\omega \beta \varsigma}{(\sigma^2 + \omega^2)^2 + 4\omega^2\beta^2}, \quad r_k = \frac{\beta (\sigma^2 + \omega^2)}{(\sigma^2 + \omega^2)^2 + 4\omega^2\beta^2}. \tag{4.6}
\]

Note that (4.6) coincides with (3.5) upon setting \(\omega = ck\), which is the dispersion relation for the pure wave equation (2.1). Thus, (4.5) and (2.12) are equivalent up to the factor \(2c\) which was absorbed into \(h(t)\). This is easily seen by taking the Fourier transform of (2.12).

Turning to the corresponding periodic problem for a dispersive Lamb system (4.1) on the interval \(-\pi < x < \pi\), we work with the Fourier series representation formula (3.2). The equations (3.3) for the Fourier coefficients become
\[
a''_0 = \frac{h'(t)}{\pi}, \quad a'_0(0) = a_0(0) = 0,
\]
\[
a''_k + \omega(k)^2 a_k = \frac{h'(t)}{\pi}, \quad a'_k(0) = a_k(0) = 0. \tag{4.7}
\]

Thus, the solution to the periodic dispersive Lamb problem has
\[
a_0(t) = C \frac{\varsigma - e^{-\beta t}(\varsigma \cos t + \beta \sin t)}{2c\pi(\beta^2 + \varsigma^2)},
\]
\[
a_k(t) = C \frac{p_k \cos \omega(k)t + q_k \sin \omega(k)t - e^{-\beta t}(p_k \cos \sigma t + r_k \sin \sigma t)}{2c\pi}, \tag{4.8}
\]
for \(k \geq 1\) where \(p_k, q_k, r_k\) are as in (4.6).

In particular, suppose that the dispersion relation has large wave number asymptotics given by a power law:
\[
\omega(k) \sim |k|^m \quad \text{as} \quad |k| \to \infty, \tag{4.9}
\]
for some \(m \in \mathbb{R}\). Then, assuming \(m > 0\), according to (4.8),
\[
a_k(t) \sim \omega(k)^{-2} \sim |k|^{-2m} \quad \text{as} \quad |k| \to \infty. \tag{4.10}
\]
The resulting Fourier coefficients therefore decay as \(|k| \to \infty\), which, depending upon the magnitude of \(m\), implies smoothness of the solution. Namely, when \(m \geq 1\), (4.10) implies that \(u(t, \cdot) \in C^n\) provided \(2m - 1 > n \in \mathbb{N}\), by a standard result in Fourier analysis, [16]. Thus the solution to the periodic dispersive Lamb problem is smooth and there is no fractalization, at least at the level of the solution profile. This observation is borne out by some MATHEMATICA calculations where \(\omega(k) = k^2\). Note: These are computed by explicitly summing the first 1000 terms of the Fourier series (3.2).
On the other hand, according to [3], if \( m > 1 \), given rough nonzero initial conditions, fractalization and, when \( m \in \mathbb{N} \), dispersive quantization will be (eventually) manifested in the solution profile. However, the Lamb forcing does not in and of itself induce or indeed affect the overall dispersive quantization effect.

A similarly smooth evolution is exhibited by the dispersive Klein–Gordon model when coupled to a Lamb oscillator, as in (4.4), as shown in the following plots:

On the other hand, if the dispersion relation is sublinear, then the resulting Lamb solution coefficients exhibit the slow decay that underlies the fractalization effects observed is dispersive quantization. For example, if \( \omega(k) \sim \sqrt{|k|} \), corresponding to the linear (complex) system

\[
u_{tt} = iu_x,
\]

i.e., the linear Schrödinger equation with the roles of space and time reversed, then (4.10) implies that \( a_k(t) \sim |k|^{-1} \). Interestingly, the dispersion relation for the free boundary problem for water waves, \( \omega(k) = \sqrt{k \tanh k} \) (ignoring physical constants), [22], is also asymptotically of this form. Numerical experiments indicate that intermittent fractalization occurs, although we as yet have no proof that the solution profile has nontrivial fractal dimension.
The Lamb oscillator forced linear regularized Boussinesq equation (4.3) exhibits noticeable high frequency oscillations in its solution profile, whose overall amplitude, as indicated by the “width” of the fattened graphs, is time-varying. At initial times, e.g., \( t \leq 20 \), the graph is noticeably thicker, while at others, e.g., \( t = 57 \) and \( t = 95 \), it has thinned out, with barely any noticeable superimposed oscillations. A while later, at \( t = 205 \), the larger scale oscillations thickening the graph have re-emerged. As yet, we have no explanation for this observed phenomenon.
5. Unidirectional Models.

Now let us turn our attention to unidirectional dispersive wave models forced by a Lamb oscillator. We begin with the linear wave equation and factor the differential operator as usual, \[ \partial_t^2 - c^2 \partial_x^2 = (\partial_t - c \partial_x)(\partial_t + c \partial_x), \]
where the individual factors govern the two unidirectional waves \( u(t, x) = f(x \pm ct) \) moving in opposite directions, each of which is constant on the characteristic lines associated with its annihilating differential operator.

The bidirectional Lamb solution (2.12) is a superposition of left and right moving waves, which do not overlap since the forcing only occurs at the origin:

\[
\begin{align*}
    u(t, x) &= \begin{cases} 
        v(t, x), & x > 0, \\
        v(t, -x), & x < 0,
    \end{cases} \\
    & \text{where } v(t, x) = \begin{cases} 
        f(ct - x), & x > 0, \\
        0, & x < 0,
    \end{cases}
\end{align*}
\]  

and \( f(t) \) is as above, (2.9). Thus, \( v(t, x) \) solves the quarter plane initial-boundary value problem

\[
\begin{align*}
    v_t + c v_x &= 0, \\
    v(0, x) &= 0, \\
    v(t, 0) &= h(t) = f(ct), \\
    x, t &> 0,
\end{align*}
\]

or, equivalently, the forced unidirectional wave equation

\[
v_t + c v_x = h(t) \delta(x), \quad v(0, x) = 0, \quad t > 0.
\]

As in Section 4, we have absorbed the factor of \( 2c \) into the forcing function \( h(t) \). More generally, we can replace the spatial transport term to construct a general unidirectional Lamb model

\[
v_t + L[v] = h(t) \delta(x), \quad v(0, x) = 0, \quad t > 0,
\]

in which the linear integro-differential operator \( L[v] \) has real dispersion relation \( \omega(k) \).

On the full line \( x \in \mathbb{R} \), we can solve the forced initial value problem using the Fourier transform to find

\[
v(t, x) = \frac{C}{4c\pi} \int_{-\infty}^{\infty} e^{i k x} \frac{e^{-i \omega(k)t} \varsigma + e^{-\beta t} ((i \omega(k) - \beta) \sin \varsigma t - \varsigma \cos \varsigma t)}{\sigma^2 - 2i \beta \omega(k) - \omega^2(k)} \, dk.
\]

Let us focus on the periodic initial-boundary value problem on \( -\pi < x < \pi \) corresponding to (5.4). We expand the solution in a Fourier series:

\[
v(t, x) = \frac{1}{2} a_0(t) + \sum_{k=1}^{\infty} \left[ a_k(t) \cos k x + b_k(t) \sin k x \right].
\]

Substituting (5.6) into the initial value problem (5.4) produces the Fourier coefficients

\[
\begin{align*}
    a_0(t) &= \frac{C}{2c\pi \sigma^2} \left[ \varsigma - e^{-\beta t} (\varsigma \cos \varsigma t + \beta \sin \varsigma t) \right], \\
    a_k(t) &= \frac{C}{2c\pi} \left[ p_k \cos \omega t + q_k \sin \omega t - e^{-\beta t} (p_k \cos \varsigma t + r_k \sin \varsigma t) \right], \\
    b_k(t) &= \frac{C}{2c\pi} \left[ -q_k \cos \omega t + p_k \sin \omega t + e^{-\beta t} (q_k \cos \varsigma t + s_k \sin \varsigma t) \right],
\end{align*}
\]
for all $k \geq 1$, where $p_k, q_k, r_k$ are given in (4.6) and

$$s_k = \frac{\omega \left( \frac{2}{\beta^2} - \sigma^2 + \omega^2 \right)}{\left( \sigma^2 + \omega^2 \right)^2 + 4 \omega^2 \beta^2}.$$  \hfill (5.8)

In contrast to the bidirectional equations, the Fourier coefficients in (5.6) decay like $1/\omega(k)$ as $k \to \infty$. As above, we deduce smoothness of the solution profiles provided the dispersion relation grows sufficiently rapidly at large wave number; specifically, (4.9) with $m \geq 2$ implies $u(t, \cdot) \in C^m$ provided $m - 1 > n \in \mathbb{N}$.

We now examine some representative solutions by summing the first 1000 terms in the Fourier series (5.6) using MATHEMATICA. First, given the linear dispersion relation $\omega(k) = ck$, corresponding to the unidirectional transport equation with a Lamb oscillator, the solution remains piecewise smooth with some dispersion emanating from the oscillator at $x = 0$.

For the higher order dispersion relation $\omega(k) = k^2$, the solution smooths very quickly:

In the case of asymptotically constant dispersion, $\omega(k) = k^2 / \left( 1 + \frac{1}{3} k^2 \right)$, we still see significant high frequency oscillations. As in the bidirectional case, the graph thickens and thins at various times.
As above, the amplitude of the oscillations increases with a decaying dispersion relation, e.g. $\omega(k) = 1/k$: 
6. Conclusions and Future Research Directions.

The most striking result in this study is that, when subject to periodic boundary conditions, a Lamb oscillator will generate fractal solution profiles in a dispersive medium possessing an asymptotically sub-linear large wave number dispersion relation whereas in an unforced dispersive medium, fractalization and dispersive quantization require that the dispersion relation grow superlinearly. Our numerical results are in need of theoretical justification, although this may be very difficult owing to the less than satisfying state of the art in rigorously analyzing slowly decaying Fourier series.

Beyond this, several additional directions of research are indicated. First of all, the effect of the boundary conditions on both unforced and forced dispersive wave equations is not yet clear, although in the unforced regime some preliminary results are available, showing that the fractal behavior of solutions is highly dependent upon the form of the boundary conditions. The corresponding problems in higher dimensional dispersive media also remain largely unexplored.

As noted above, in [11], it was shown that coupling the wave equation or Klein–Gordon equation to a gyroscopic oscillator or more general Chetaev system can induce instabilities through the effects of Rayleigh dissipation. Preliminary numerical experiments with more general dispersive media on periodic domains indicate that the coupling does not appear to destabilize an otherwise stable gyroscopic system. However, a more thorough analysis must be performed to confirm this result.

References