Research Statement

My broad research interests are in algebraic geometry and its role in mathematical physics. Currently, my research focuses on supergeometry, super Riemann surfaces, and the punctured moduli spaces $\mathcal{M}_{g,n_{NS}+n_R}$ of super Riemann surfaces.

Background

One of the main goals of superstring perturbation theory is to compute the probabilities, called *scattering amplitudes*, of certain string interactions occurring. Such computations are carried out by integrating over the various moduli spaces of *super Riemann surfaces*. A super Riemann surface is an ordinary Riemann surface endowed with a structure sheaf of $\mathbb{Z}_2$ graded algebras and extra piece of data called a *superconformal structure*, which is an *odd maximally nonintegrable distribution*, i.e., a rank $(0|1)$ subbundle $\mathcal{D}$ of the tangent bundle $\mathcal{T}$, such that the supercommutator of vector fields gives an isomorphism

$$\mathcal{D} \otimes \mathcal{D} \xrightarrow{\sim} \mathcal{T}/\mathcal{D}. \tag{1}$$

If we set $T = \text{Spec} \mathbb{C}$, then $\Sigma$ is locally isomorphic to $\mathbb{C}^{1|1}$, and therefore can be described using coordinates $(z, \theta)$. For a suitable choice of coordinates, the *superconformal structure* $\mathcal{D}$ on $\Sigma$ is generated by the vector field

$$D_\theta = \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial z}. \tag{2}$$

The maximal nonintegrability condition in 1 then rewrites as

$$D_\theta^2 = \frac{1}{2} [D_\theta, D_\theta] = \partial_z. \tag{3}$$

On a super Riemann surface we also need to consider two types of punctures, called *Neveu-Schwarz punctures* ($n_{NS}$) and *Ramond punctures* ($n_R$). Ramond punctures appear on a super Riemann surface when its superconformal structure $\mathcal{D}$ “degenerates” along a divisor $\mathcal{F}$, called the Ramond divisor. A superconformal structure $\mathcal{D}$ on a super Riemann surface with Ramond punctures is a rank $(0|1)$ subbundle $\mathcal{D}$ of the tangent bundle $\mathcal{T}$, such that

$$\mathcal{D} \otimes \mathcal{D} \xrightarrow{\sim} \mathcal{T}/\mathcal{D}(-\mathcal{F}). \tag{4}$$

In particular, $\mathcal{D}$ fails to be nonintegrable along $\mathcal{F}$. As in the unpunctured case, if $T = \text{Spec} \mathbb{C}$, then $\Sigma$ is locally isomorphic to $\mathbb{C}^{1|1}$ and may be described by a set of coordinates $(z, \theta)$. For a suitable choice of such coordinates, the superconformal structure $\mathcal{D}$ is generated by the vector field

$$D_\theta = \frac{\partial}{\partial \theta} + z\theta \frac{\partial}{\partial z}. \tag{5}$$

The integrability condition in 4 then rewrites as

$$D_\theta^2 = \frac{1}{2} [D_\theta, D_\theta] = z\partial_z. \tag{6}$$

The integrability of $\mathcal{D}$ along $\mathcal{F}$ means that a super Riemann surface with Ramond punctures is not an honest super Riemann surface. Even calling Ramond punctures— punctures— is a misnomer: Instead, they are divisors of codimension $(1|0)$. Note, that the natural correspondence
between divisors and punctures on ordinary curves holds only for supercurves equipped with a
(non-degenerate) superconformal structure [Man88]. The failure of this correspondence to hold
in the event of Ramond punctures means that Ramond punctures cannot be added at specified
marked points; instead, their appearance is unique to the theory of super Riemann surfaces, and
thereby their presence results in novel geometric situations.

Neveu-Schwarz (NS) punctures
are entirely analogous to the marked points from the ordinary
moduli theory of curves. If Σ is a super Riemann surface over
T
then NS punctures correspond
to a choice of nNS sections s_i : T → X. This means that, unlike Ramond punctures, NS punctures
can be added to a super Riemann surface by specifying marked points (z_0, θ_0) on X. In particular,
they can be added to any previously constructed supermoduli space by such a specification on the
universal curve. For the sake of brevity, we, therefore, set nNS = 0 in our given construction of
M
0,nR.

Past Work: Construction of M
0,nR

In [Wit12b] and [Wit15], E. Witten suggests an approach to constructing the moduli spaces
M
0,nR. Using this approach he constructs the ordinary manifold (M
0,nR)red underlying M
0,nR as a nR−3
dimensional space, however, he stops short of giving a full construction of M
0,nR as a super space
of dimension nR−3nR/2−2. It now appears that the nR/2−2 remaining odd moduli of M
0,nR were hidden within the supermoduli theory of genus zero supercurves. We found in [CITE] that
genus zero supercurves are quite distinct from their ordinary counterpart, the projective line, as
they are not all isomorphic, as is proved in the following lemma.

Lemma 1. Every compact, genus zero supercurve X over C is isomorphic to the total space of the
rank 0|1 vector bundle ΠO(m), for some m, over P
1, and thereby also isomorphic to the weighted
superprojective space ωP
1|1(1, 1|m).

Lemma 1 implies that the underlying topological space of the moduli space of genus zero supercurves is just the discrete set of integers. This is in striking contrast to the moduli space of ordinary genus zero curves—a single point. What further sets genus zero supercurves apart from ordinary genus zero curves is that they are not all rigid. In fact, we prove the following lemma.

Lemma 2.

\[ \dim H^1(\omega P^1|1(1, 1|m), T) = \begin{cases} 0 | m - 1 & \text{for } m < -1 \\ 0 & \text{for } 0 \leq m \leq 3 \\ 0 | m - 3 & \text{for } m > 3. \end{cases} \]

The computation given in Lemma 2 is crucial to the moduli problem because it shows that
ωP^1|1(1, 1|1−nR/2) has nR/2−2 odd moduli, which is exactly the number of odd moduli missing
from Witten’s earlier investigation of M
0,nR. This hints at the existence of a “versal family
of supercurves” underlying super Riemann surfaces with nR Ramond punctures such that the
superconformal structures supported on this family of supercurves would account for the nR − 3
even moduli of M
0,nR determined by Witten in [Wit12a]. For the remainder of this section, set
A = C[η_1, · · · , η_{nR/2−2}]— where η_i are anti-commuting (odd) coordinates—and set S = Spec A. We
construct a family of genus zero supercurves Z over S, glued from the affine charts U = Spec A[z, ζ]
and $V = \text{Spec } A[w, \xi]$ with transition functions
\begin{align*}
z &\mapsto 1/w \\
\zeta &\mapsto \xi w^{n_R/2-1} + \sum_{j=1}^{n_R/2-2} \eta_j w^{n_R/2-1-j}.
\end{align*}

We prove that the family $Z \to S$ has the following “versal” property:

**Theorem 1.** Let $X \to T$ be a family of genus zero super Riemann surfaces with $n_R \geq 4$ Ramond punctures over a superscheme $T$, and let $\overline{X}$ denote its underlying family of supercurves (forgetting the superconformal structure). Then there exists a morphism $T \to S$ such that $Z \times_S T \cong \overline{X}$

as a family of supercurves.

The proof of Theorem 1 relies on the observation that the supercurve underlying any is isomorphic to $\mathbb{W}[\mathbb{P}^1]^1(1,1|1-n_R/2) \times R$. This result allows us to treat the supercurve $\overline{X}$ underlying any as a deformation of $\mathbb{W}[\mathbb{P}^1]^1(1,1|1-n_R/2) \times T_{red}$. The proof of Theorem 1 then follows from explicitly defining a morphism $T \to S$ such that the Kodaira-Spencer class $\kappa(Z \times_S T)$ of $Z \times_S T$ is equal to the Kodaira-Spencer class $\kappa(\overline{X})$ of $\overline{X}$. We then deduce the following fact about superconformal structures supported on $Z$:

**Corollary 2.** Let $X \to T$ be a family of genus zero super Riemann surfaces with $n_R$ Ramond punctures over a superscheme $T$, and let $\mathcal{D}$ denote its superconformal structure. Then there exists a morphism $f : T \to S$ and a superconformal structure $\mathcal{D}$ on $Z$, such that $\mathcal{D}$ is isomorphic to the pullback of $\mathcal{D}$ by $f$.

The proof of Corollary 2 uses Theorem 1 and the exactness of the pullback functor. To determine the superconformal structures on $Z$, we utilize a computation done previously by Witten in [Wit12b], in which he determined that superconformal structures on $\mathbb{W}[\mathbb{P}^1]^1(1,1|1-n_R/2)$ correspond to certain global meromorphic 1-forms on $\mathbb{W}[\mathbb{P}^1]^1(1,1|1-n_R/2)$. We generalize his computation to include families over arbitrary superschemes and use this result to find that superconformal structures on $Z$ correspond to global meromorphic 1-forms, which can, locally, in a coordinate chart $U(z, \zeta)$, be described concretely by the forms

\begin{equation}
\left(a_0 + \zeta \sum_{i=0}^{n_R/2-1} \alpha_i z^i\right) dz + \left(\sum_{i=0}^{n_R/2+1} \beta_i z^i + \zeta \sum_{i=0}^{n_R} b_i z^i\right) d\zeta
\end{equation}

with the coefficients $a_0, \alpha_i, \beta_i$ and $b_i$ being functions on the base $S$. Given any superscheme $T$ over $S$, the correspondence between superconformal structures on $Z \times_S T$ and the 1-forms in (10) is unique only up to, multiplication by global sections of $\mathcal{O}_{Z \times_S T}$, and automorphisms of $Z \times_S T$ over $T$. Let $\Gamma^*$ denote the functor assigning to a superscheme $T$ over $S$ the group $\Gamma^*(T) = \Gamma(Z \times_S T, \mathcal{O}^*)$. We prove that $\Gamma^*$ is represented by an $S$-supergroup scheme isomorphic to $G_m \times (G_a^{[1]|n_R/2}) \times S$. To determine the representability of $\text{Aut}_{Z/S}$, we first dealt with the automorphism functor of the special fiber $\mathbb{W}[\mathbb{P}^1]^1(1,1|1-n_R/2)$ of $Z \to S$, and found that:
Theorem 3. There is an isomorphism of supergroup schemes
\[ \text{Aut}_{\mathbb{P}^1(1,1)|1-n_R/2} \cong \text{Aut}_R / \mathbb{G}_m \times (\mathbb{C}^{[1,1]}|n_R/2) \]
where \( R = \mathbb{C}[u,v|\theta] \) and where the \( \mathbb{Z} \)-grading on the homogeneous coordinates \( u, v \) and \( \theta \) defined by \( |u| = |v| = 1 \) and \( |\theta| = 1 - n_R/2 \). In particular, \( \text{Aut}_{\mathbb{P}^1(1,1)|1-n_R/2} \) is a supergroup scheme of dimension \( 4|n_R/2 + 2 \).

The proof relies on the Geometric Invariant Theory (GIT) techniques, see [MFK94] and [Amr89], relating to the automorphism group scheme of \( \mathbb{P}^n \) and \( \mathbb{W}/\mathbb{P}^n \), respectively. Using Theorem 3, we then prove the following corollary.

Corollary 4. There is an isomorphism of supergroup schemes over \( S \),
\[ \text{Aut}_{Z/S} \cong \text{Aut}_{\mathbb{P}^1(1,1)|1-n_R/2} \times S. \] (11)

The proof of Corollary 4 is done using deformation theory. The main result of [CITE] is the following theorem.

Theorem 5. The moduli space \( \mathcal{M}_{0,n_R} \) of genus zero super Riemann surfaces with \( n_R \geq 4 \) unlabeled Ramond punctures may be described as follows:
\[ \mathcal{M}_{0,n_R}(n_R) \cong (W/\Gamma^*)/G \] (12)
where \( W \) is an open subset of the superspace \( H^0(Z, \Omega^1_{Z/S}(2)) \setminus 0 \cong (\mathbb{C}^{n_R+2}|n_R+2 \setminus 0 \times S, \Gamma^* \) is as in (1), and \( G = \text{Aut}_{Z/S} \) is identified in Theorem 3 and Corollary 4. In particular, \( \mathcal{M}_{0,n_R}(n_R) \) is a Deligne-Mumford superstack of dimension \( n_R - 3|n_R/2 - 2 \).

Here the open subset \( W \) is given by the conditions that the Ramond divisor does not acquire multiplicities and that the superconformal structure is maximally nonintegrable, see (1).

Further Research

My research goal is to further study the geometric properties of \( \mathcal{M}_{g,n_{\text{NS}}+n_R} \), with a special focus on applying the results toward computations in superstring perturbation theory. I have organized this broad goal into three specific research projects:

- Develop a supergeometric version of geometric invariant theory (GIT), and use it to give explicit constructions of \( \mathcal{M}_{g,n_{\text{NS}}+n_R} \) and \( \mathcal{M}_{g,n_{\text{NS}}+n_R} \) in low genus.
- Use my previous construction of \( \mathcal{M}_{0,n_R} \) to compute the integrals providing scattering amplitudes in the case of genus \( g = 0 \) and \( n_R \geq 4 \) using non-split methods.
- Investigate the splitness of \( \mathcal{M}_{g,n_{\text{NS}}+n_R} \) and \( \mathcal{M}_{g,n_{\text{NS}}+n_R} \) in genus \( g = 3, 4 \) and for punctures.

Merit: String theory is to date the most promising theory for the so-called theory of everything. It has also proven to be one of the most fruitful sources of extra-disciplinary mathematical ideas. The proposed research project aims to develop further the critical role played by algebraic geometry within superstring theory. From the perspective of string theorists, a better understanding of the
various moduli spaces of super Riemann surfaces would solve many of the geometric difficulties encountered in computations of scattering amplitudes. Scattering amplitudes have historically been computed using an integration method reliant on a subtle property of the moduli spaces called \textit{splitness}. However, for many moduli spaces, the question of its splitness remains an open one. Therefore, a further investigation of splitness would represent a significant boon to string theorists, as it would determine the potential of utilizing split integration methods. The explicit constructions of the moduli of curves is an extremely classical subject in ordinary algebraic geometry, dating back to the late 19th century works of Riemann and Hurwitz \cite{Hur91}. Therefore, generalizing the classical theory of curves to include supercurves holds intrinsic mathematical value, as it would represent a modern take on classical geometric theory, such as geometric invariant theory. Finally, explicit constructions of the moduli spaces of super Riemann surfaces in low genus will allow for computations of scattering amplitudes using holomorphic, and thereby non-split integration methods.

Goal 1: Explicit Constructions

In \cite{LR88}, LeBrun and Rothstein gave the first written account of the moduli spaces of super Riemann surfaces as (super) orbifolds. Constructions of $\mathcal{M}_g$ have since been extended to include algebraic superspaces \cite{PRD97}, and, very recently, Deligne-Mumford superstacks in \cite{CV17}; however, these sources do not provide explicit constructions of the punctured moduli spaces expected in the low-genus case. I propose to make such constructions by extending the methods of geometric invariant theory (GIT) to supermoduli theory. GIT methods would represent not only a more modern approach to supermoduli theory but would simplify many geometric computations. For example, many of the computations in \cite{CITE} (e.g., $\text{Pic}(\mathcal{W} \mathcal{P}^{1\dagger}(1,1|1-n_R/2))$ and $\Gamma(\mathcal{W} \mathcal{P}^{1\dagger}(1,1|1-n_R/2), \Omega^1(2))$) were done using Čech cohomology. I have since noticed that these computations could have been significantly simplified and made more conceptual using the methods of GIT. For this reason, I propose to give explicit constructions of low genus moduli spaces with Neveu-Schwarz and Ramond punctures, utilizing a supergeometric version of GIT, as well as extend this approach to prior work on the moduli of super Riemann surfaces in higher genus to include punctures.

A central theme in the classical moduli space theory is the construction of the Deligne-Mumford compactifications of the moduli spaces of curves. P. Deligne first described the compactification $\overline{\mathcal{M}}_g$ of $\mathcal{M}_g$ in a letter to Y. Manin \cite{Del} in 1987. In the letter, P. Deligne outlines a construction of $\overline{\mathcal{M}}_g$ via super Riemann surfaces having at worst nodal singularities, but with two possible types of superconformal structure at the nodes: a true superconformal structure and a degenerate one (corresponding to Ramond punctures). However, no explicit constructions of $\overline{\mathcal{M}}_g$ exist in the literature, nor has Deligne’s method ever been written out in any detail. From the perspective of string theory, this is particularly unfortunate, given the fact that one of the main problems to haunt any unified field theory is the divergence of integrals along the boundary of the integration space. Therefore, I propose to revisit Deligne’s compactification and give explicit constructions of $\overline{\mathcal{M}}_{g,n_R+n_{NS}}$.

Goal 2: Non-Split Methods

The earliest investigations into scattering amplitude problems in genus $g = 0, 1$ are described in \cite{FMS86; AS87; Gro+86}, and \cite{Gro+86}, to name a few. The scattering amplitudes at genus $g = 2$ with \textit{Neveu-Schwarz punctures} (marked points) were computed by D’Hoker and Phong in \cite{DP02}.
Recent efforts by E. Witten in [Wit12a; Wit12c; Wit13] and E. Witten and R. Donagi in [DW15; DW14] have shed new light on the situation of Ramond punctures in higher genus (i.e. genus $g \geq 3$), which are bound to yield exciting new computations of scattering amplitudes. However, computing scattering amplitudes has remained a particularly challenging task in the case of higher genus ($g \geq 3$) and in the presence of punctures. This is primarily due to the tendency of physical questions regarding integration devolving into rather subtle questions concerning the geometry of moduli spaces. In particular, previous computations of scattering amplitudes have been heavily reliant on the splitness of the integration spaces $\mathcal{M}_{g,n\text{NS}+n_R}$. A supermanifold is said to be split if and only if it takes the form of a fiber bundle over its underlying reduced space. The usefulness of this for integration is that it allows one to integrate out the odd fibers and obtain an ordinary integral over the reduction of $\mathcal{M}_g$. Integration techniques reliant on splitness have also tended to be quite ad hoc. For example, in [DP02] D'Hoker and Phong show that the moduli space of super Riemann surfaces is split for genus $g = 2$; however, the projection to the ordinary reduced moduli space they utilize is not a canonical one, but rather one dependent on their discovery of an analytic formula for the period map; a rather confounding object in its own right. It would, therefore, be of significant value to compute scattering amplitudes using non-split methods of integration.

For a start, I plan to compute the integrals producing scattering amplitudes in the case of $g = 0$ and $n_R \geq 4$ Ramond punctures, using our construction of the corresponding supermoduli space in [CITE], outlined in the Past Work section above. I then plan to extend this to an integral over $\mathcal{M}_{0,n_R}$ using the construction I propose in Goal 1. To compute these using non-split methods, I will need to give explicit descriptions of super Mumford forms in low genus and with punctures. A super Mumford form is a trivializing section of the super Mumford isomorphism which can be used to define a measure whose integral over $\mathcal{M}_{g,n\text{NS}+n_R}$ computes scattering amplitudes. Super Mumford forms on $\mathcal{M}_{g,n\text{NS}}$ were first described in [RSV89] and [Vor88] for genus $g \geq 2$, and have been extended in [Dir19] to $\mathcal{M}_{g,n\text{NS}+n_R}$ for genus $g \geq 2$. However, the super Mumford isomorphism is true for all families of super Riemann surfaces and for all genus $g$, and so I believe that the techniques in [Dir19] can be extended to the case of genus zero. A long term project would be to extend this construction to genus $g = 1$, and compute scattering amplitudes using my proposed construction of $\mathcal{M}_{1,n_R+n\text{NS}}$. In particular, it would be interesting to investigate the regularity of the super Mumford form along the boundaries of $\mathcal{M}_{1,n_R+n\text{NS}}$ and $\mathcal{M}_{0,n_R+n\text{NS}}$.

**Goal 3: Splitness**

It was shown by R. Donagi and E. Witten in [DW14] and [DW15] that $\mathcal{M}_g$ is not split, or even projected, for genus $g \geq 5$. However, the question of splitness in the presence of punctures remains an open one, and one I aim to solve as part of the proposed project. The general strategy of Donagi and Witten in [DW15] is to find non-split spaces embedded into the moduli space in question. I believe that their techniques would generalize quite naturally to account for punctures. In particular, I expect that Donagi and Witten’s results for the unpunctured moduli will hold true in the presence of punctures. Finally, Donagi and Witten conjecture that $\mathcal{M}_g$ is not split for genus $g = 3, 4$, but so far, no proof of this has been given. The question of splitness in genus $g = 3, 4$, with or without punctures, is particularly difficult because the techniques used in higher genus do not extend to these cases. This novelty makes resolving the question of splitness in genus $g = 3, 4$ a long-term project I am particularly excited to work on. A natural next step is then to investigate the splitness of the compactifications $\overline{\mathcal{M}}_{g,n_R+n\text{NS}}$. Asking for the compactifications $\overline{\mathcal{M}}_{g,n_R+n\text{NS}}$ to be split is an even stronger condition, and one that has yet to be studied.
References


