Poisson Brackets for the Grassmann Pentagram Map

Nick Ovenhouse

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October, 2019 Notre Dame
Outline

1. Background
2. Grassmann Version
3. Non-Commutative Integrability
4. Combinatorial Models
5. Recovering the Lax Invariants
The Basic Idea

Draw a polygon in the (projective) plane
Label the vertices 1,...,n
Draw diagonals connecting i, i+2
Label intersections of diagonals by 1′,...,n′

Definition [Schwartz]
The pentagram map sends the original polygon to the new polygon.


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\[\text{Definition [Schwartz]} \]

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---

Twisted Polygons

More generally, we consider twisted polygons.

Definition
A twisted $n$-gon is a bi-infinite sequence $(p_i)_{i \in \mathbb{Z}}$ of points in $\mathbb{P}^2$, such that $p_{i+n} = Mp_i$ for some $M \in \text{PGL}_3$. The matrix $M$ is called the monodromy of the polygon.

Definition
$\text{PGL}_3$ acts on the set of twisted polygons by $A \cdot (p_i) = (Ap_i)$. Two polygons in the same orbit are called projectively equivalent. Denote by $P_n$ the set of projective equivalence classes of twisted $n$-gons.
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Integrability

Definition
A Poisson mapping $T : M \to M$ on a manifold $M$ is called completely integrable if there are sufficiently many independent conserved quantities $f_1, \ldots, f_N$ so that all
$\{f_i, f_j\} = 0$.

Theorem [OST] [GSTV]
The pentagram map on $P_n$ is completely integrable.


A Poisson mapping $T: M \to M$ on a manifold $M$ is called \textit{completely integrable} if there are sufficiently many independent conserved quantities $f_1, \ldots, f_N$ so that all
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The Basic Idea

The Grassmann pentagram map sends $P_1, \ldots, P_n$ to $Q_1, \ldots, Q_n$.

Theorem

The Grassmann pentagram map has a Lax representation. That is, there is a matrix whose spectral invariants are conserved quantities.
The Basic Idea

- “Draw a polygon” (choose $n$ points $P_1, \ldots, P_n \in \text{Gr}_N(3N)$)
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Twisted Grassmann Polygons

Again, we consider more generally twisted polygons:

Definition
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Definition
Let \(\text{GP}_n, N\) denote the moduli space of twisted n-gons in \(\text{Gr}_N(3^N)\), up to the action of \(\text{PGL}_{3^N}\).
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**Definition**

Let \(\mathcal{GP}_{n,N}\) denote the moduli space of twisted \(n\)-gons in \(\text{Gr}_N(3N)\), up to the action of \(\text{PGL}_{3N}\).
If $P_1, \ldots, P_n \in Gr_N(3N)$ are vertices, choose a lift $V_1, \ldots, V_n \in \text{Mat}_{3N \times N}$. If the polygon is “generic”, then the combined columns of $V_i, V_i+1, V_i+2$ are a basis of $\mathbb{R}^{3N}$. Then there are $A_i, B_i, C_i \in \text{GL}_N$ so that $V_i+3 = V_i+1 A_i + V_i B_i + V_i+2 C_i$.

Lemma
The lift can be chosen so that $C_i = \text{Id}_N$ for all $i$. So then for all $i$, $V_i+3 = V_i+1 X_i + V_i Y_i + V_i+2 C_i$, and there is a $Z \in \text{GL}_N$ so that $X_i+\pi = ZX_i Z^{-1}$ and $Y_i+\pi = ZY_i Z^{-1}$.
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$$V_{i+3} = V_i + X_i V_{i+1} + Y_i V_{i+2},$$

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Pentagram Map  

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Expression for the Map

The Grassmann pentagram map transforms the matrices $X_i, Y_i$ by

$$X_i \mapsto X_i + Y_i + 1 - X_i + Y_i + 1 = X_i + 2Y_i + 3.$$  

$$Y_i \mapsto X_i + Y_i + 1 - Y_i + 1 = X_i + 2Y_i + 3.$$  

In what follows, we will consider this as a transformation on a set of formal noncommutative variables.
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If \( A \) is any associative algebra, then the commutator bracket
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[a, b] = ab - ba
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is always a Poisson bracket.

Theorem [Farkas, Letzter]

Let \( A \) be an associative, but non-commutative, prime ring. Then the only Poisson bracket on \( A \) (up to scalar multiple) is the commutator bracket \([a, b]\). In the commutative case, this reduces to the trivial Poisson bracket.

Question: What is the right notion of Poisson structure for a non-commutative algebra?


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Let $A$ be an associative algebra. Denote $A^\natural := A/\mathbf{[}A, A\mathbf{]}$, the cyclic space. Elements are cyclic words in $A$, since $x_1 \ldots x_n = x_n x_1 \ldots x_{n-1} \pmod{\mathbf{[}A, A\mathbf{]}}$.

Each $a^\natural \in A^\natural$ gives a $\text{GL}_n$-invariant function $\text{tr}(a^\natural)$ on $\text{Hom}(A, \text{Mat}_n)$, and hence a function on $\text{Rep}_n(A) := \text{Hom}(A, \text{Mat}_n)/\text{GL}_n$.

**Definition [Crawley-Boevey]**

An $H_0$-Poisson structure on $A$ is a Lie bracket $\mathbf{[}-,-\mathbf{]}$ on $A^\natural$ such that each $\mathbf{[}a^\natural,-\mathbf{]}$ is induced by a derivation of $A$.

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An $H_0$-Poisson structure on $A$ induces a Poisson bracket on $\text{Rep}_n(A)$ given by

$$\{\text{tr}(a^\natural), \text{tr}(b^\natural)\} = \text{tr}\mathbf{[}a^\natural, b^\natural\mathbf{]}.$$
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$H_0$-Poisson Structures

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- Denote $A^\bullet := A/[A, A]$, the *cyclic space*.


[10.1016/j.jalgebra.2010.09.033]
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$H_0$-Poisson Structures

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Let $A$ be an associative algebra.

- Denote $A^\sharp := A/[A, A]$, the *cyclic space*.
- Elements are cyclic words in $A$, since $x_1 \cdots x_n = x_n x_1 \cdots x_{n-1} \mod [A, A]$.
- Each $a^\sharp \in A^\sharp$ gives a $\text{GL}_n$-invariant function $\text{tr}(a)$ on $\text{Hom}(A, \text{Mat}_n)$, and hence a function on $\text{Rep}_n(A) := \text{Hom}(A, \text{Mat}_n)/\text{GL}_n$.

**Definition [Crawley-Boevey]¹**

An $H_0$-Poisson structure on $A$ is a Lie bracket $[\cdot, \cdot]$ on $A^\sharp$ such that each $[a^\sharp, \cdot]$ is induced by a derivation of $A$.

**Theorem [Crawley-Boevey]¹**

An $H_0$-Poisson structure on $A$ induces a Poisson bracket on $\text{Rep}_n(A)$ given by

$$\{\text{tr}(a), \text{tr}(b)\} = \text{tr}[a^\sharp, b^\sharp]$$

By a non-commutative integrable system in $A$, we mean a map $T: A \to A$ such that there is an infinite family of invariants $t^\#_1, t^\#_2, \ldots, t^\#_i, \ldots \in A$ (i.e. $t^\#_i = T(t^\#_i) \mod [A, A]$).

There is an $H_0$-Poisson structure so that $[t^\#_i, t^\#_j] = 0$ for all $i, j$.

Consider the expression for the Grassmann pentagram map in the $X, Y$ matrices formally as a map on the free skew field in the indeterminates $X_i, Y_i$.

Theorem \cite{Ovenhouse}

The Grassmann pentagram map is a non-commutative integrable system in the free skew field.
By a *non-commutative integrable system* in $A$, we mean a map $T: A \to A$ such that

1. There is an infinite family of invariants $t_{\blacklozenge 1}, t_{\blacklozenge 2}, ..., t_{\blacklozenge i}, ... \in A$ (i.e. $t_i = T(t_i) \mod [A, A]$).
2. There is an $H_0$-Poisson structure so that $[t_{\blacklozenge i}, t_{\blacklozenge j}] = 0$ for all $i, j$.

Consider the expression for the Grassmann pentagram map in the $X, Y$ matrices formally as a map on the free skew field in the indeterminates $X_i, Y_i$.

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By a *non-commutative integrable system* in $A$, we mean a map $T: A \to A$ such that

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By a *non-commutative integrable system* in $A$, we mean a map $T: A \rightarrow A$ such that

- There is an infinite family of invariants $t_1^\triangledown, t_2^\triangledown, \ldots, t_i^\triangledown, \ldots \in A^\triangledown$ (i.e. $t_i = T(t_i) \mod [A, A]$)
- There is an $H_0$-Poisson structure so that $[t_i^\triangledown, t_j^\triangledown] = 0$ for all $i, j$. 

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Non-Commutative Integrability

By a *non-commutative integrable system* in $A$, we mean a map $T: A \to A$ such that

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Theorem [Ovenhouse]

The Grassmann pentagram map is a non-commutative integrable system in the free skew field.

By a *non-commutative integrable system* in $A$, we mean a map $T: A \to A$ such that

- There is an infinite family of invariants $t_1^\mathbb{h}, t_2^\mathbb{h}, \ldots, t_i^\mathbb{h}, \ldots \in A^\mathbb{h}$  
  (i.e. $t_i = T(t_i) \mod [A, A]$)
- There is an $H_0$-Poisson structure so that $[t_i^\mathbb{h}, t_j^\mathbb{h}] = 0$ for all $i, j$.

Consider the expression for the Grassmann pentagram map in the $X, Y$ matrices formally as a map on the free skew field in the indeterminates $X_i, Y_i$. 

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Consider the expression for the Grassmann pentagram map in the $X, Y$ matrices formally as a map on the free skew field in the indeterminates $X_i, Y_i$.

**Theorem [Ovenhouse] ¹**

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Consider the following graph embedded on a torus:

\[
\begin{array}{c}
V_1 & V_2 & V_3 & V_4 & V_5 \\
A_1 & A_3 & B_1 & B_2 & B_4 \\
C_1 & C_2 & C_3 & C_4 & C_5 \\
G_{-1} & G_1 & G_{-1} & G_1 & G_{-1} \\
X_1 & X_2 & X_3 & X_4 & X_5 \\
Y_1 & Y_2 & Y_3 & Y_4 & Y_5 \\
Z_1 & Z_2 & Z_3 & Z_4 & Z_5 \\
\end{array}
\]

Recall the relations
\[
V_i + 3 \equiv V_i + 1 \pmod{G_i},
\]

If we change the lift $V_i \mapsto V_i G_{-1}$...

Do this repeatedly to cancel the $C_i$'s...
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Recall the relations $V_{i+3} = V_{i+1}A_i + V_iB_i + V_{i+2}C_i$. 
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Let $A$ be an associative algebra. A double bracket on $A$ is a bilinear operation $\{\cdot, \cdot\}$:

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\{a, bc\} = \{a, b\} (1 \otimes c) + (b \otimes 1) \{a, c\}
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\[
\tau(x \otimes y) \tau = y \otimes x
\]

\[
\{a, b\} = -\{b, a\}
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Nick Ovenhouse (UMN)  
October, 2019 Notre Dame
Definition [Van den Bergh]¹

Let $A$ be an associative algebra. A *double bracket* on $A$ is a bilinear operation

$$\{\cdot, \cdot\} : A \otimes A \rightarrow A \otimes A$$

such that:

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---

Double Brackets from Networks

Let $Q$ be a network drawn on a cylinder, so that Boundary vertices are univalent. One boundary component has only sources, the other only sinks. All internal vertices are trivalent, and neither sources nor sinks:

Let $A$ be the algebra generated by the arrows. Define a double bracket on $A$ by

$$\{\{y, z\}\} = y \otimes z$$

and

$$\{\{b, c\}\} = c \otimes b$$

By composing with multiplication and the quotient map, we get an operation on $A^{\#} = A / [A, A]$

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Nick Ovenhouse (UMN)

Pentagram Map

October, 2019 Notre Dame
Double Brackets from Networks

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![Network Diagram](image)

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Choosing a vertex of the graph $\bullet$, let $\mathcal{L}_{\bullet}$ be the space of loops based at $\bullet$. It is a subalgebra.
If $f, g \in \mathcal{L}$ intersect at a point $p$, then $f_p g_p$ represents the loop which follows $f$, then $g$, based at $p$. 

Theorem
The induced bracket $\langle -,- \rangle$ on $\mathcal{L}_{\bullet}$ is a Lie bracket, and it is given by:

$$\langle f, g \rangle = \sum_{p \in f \cap g} \varepsilon_p(f, g) f_p g_p$$

The coefficients $\varepsilon_p(f, g)$ are given by:

$$\varepsilon_p(f, g) = \begin{cases} 1 & \text{if } p \text{ is a common point of } f \text{ and } g \\ 0 & \text{otherwise} \end{cases}$$
An $H_0$ Poisson Bracket

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$$\varepsilon_p(f, g) = 1$$

$$\varepsilon_p(f, g) = 0$$
The X, Y Coordinates

The $X_i$ and $Y_i$ variables represent the pictured cycles. Their brackets are given by

\[
\langle X_i + 1, X_i \rangle = X_i + 1 X_i
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\[
\langle Y_i + k, Y_i \rangle = Y_i + k Y_i \quad \text{for } k = 1, 2
\]
\[
\langle Y_i + k, X_i \rangle = Y_i + k X_i \quad \text{for } k = 0, 1
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\end{align*}
\]
Given a path $p$ in the network, let $w_t(p)$ be the product of the edge weights (in order). Let $W_t(p) := w_t(p)^{\lambda d_p}$, where $d_p$ is the "winding number" of the path.

The boundary measurement matrix $B(\lambda) = (b_{ij}(\lambda))$ is given by $b_{ij}(\lambda) = \sum_{p : i \rightarrow j} W_t(p)$.
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Boundary Measurements

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We perform the following sequence of local transformations ("Postnikov moves") of the network, considered on a torus:

**"Square Move"**

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After this sequence, we end up with the same network, but with weights
\[
\tilde{X}_i = (X_i + Y_i) - 1 \quad X_i (X_i + 2 + Y_i + 2)
\]
\[
\tilde{Y}_i = (X_i + 1 + Y_i + 1) - 1 \quad Y_i + 1 (X_i + 3 + Y_i + 3)
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These are the expressions for the pentagram map (up to a shift of indices $Y_i \mapsto Y_i + 1$).
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These are the expressions for the pentagram map (up to a shift of indices \( Y_i \mapsto Y_{i+1} \)).
Integrability

The sequence of moves we performed do not change the boundary measurements. But moving the edges/vertices around on the torus can change the matrix up to conjugation. So although the entries of $B(\lambda)$ are not invariants, the spectral invariants are.

Let $\text{tr}(B(\lambda) i) = \sum t_{ij} \lambda^j$.

Theorem [Ovenhouse] $t_{ik}^\flat, t_{j\ell}^\flat \rangle = 0$ for all $i, j, k, \ell$.

The proof is combinatorial and topological in nature. It is done by enumerating the paths with given homology classes (on the torus), and examining their intersections.

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Outline

1. Background
2. Grassmann Version
3. Non-Commutative Integrability
4. Combinatorial Models
5. Recovering the Lax Invariants
Another View of the Moduli Space

A polygon, and a choice of lift $V_i$, determines the $2^{n+1}$ matrices $X_i, Y_i, Z$. However, simultaneously changing all $V_i$ by $V_i \mapsto V_i G$ for some fixed $G \in \text{GL}_N$, induces $X_i \mapsto G^{-1} X_i G$, $Y_i \mapsto G^{-1} Y_i G$, and $Z \mapsto G^{-1} Z G$. So the matrices $X_i, Y_i, Z$ are only well-defined up to simultaneous conjugation.

The moduli space is identified with $\text{GP}_n, \sim = \text{GL}_{2^{n+1}} \text{N} / \text{GL}_N$. If $A$ is the group algebra of the free group on $2^{n+1}$ generators, then this is $\text{Rep}_N(A)$. So by Crawley-Boevey's theorem, the $H_0$-Poisson bracket induces a Poisson bracket on $\text{GP}_n, \sim$ so that

$$\{\text{tr}(a), \text{tr}(b)\} = \text{tr}(\langle a, b \rangle)$$
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$$\{ \text{tr}(a), \text{tr}(b) \} = \text{tr}(\langle a, b \rangle)$$
Interpreting the Invariants

Since the invariants $t_{ij}$ are (noncommutative) polynomials in the $X_i$'s and $Y_i$'s, we can interpret the traces $\text{tr}(t_{ij})$ as functions on $\text{GP}_n$, $\text{N}_n$, and by Crawley-Boevey’s theorem, \{\text{tr}(t_{ij}), \text{tr}(t_{k\ell})\} = 0.

Marí-Beffa and Felipe’s Lax matrix was constructed as follows. Form the matrix $V_i$ whose columns are $V_i$, $V_i + 1$, and $V_i + 2$.

Consider the “shift” matrix $L_i = \begin{pmatrix} 0 & 0 & Y_i & \text{Id} \\ X_i & 0 & \text{Id} & N \\ \text{Id} & N & \text{Id} & 0 \end{pmatrix}$.

Then $V_i + 1 = V_i L_i$, and the action of the monodromy matrix is given by $M V_i = V_i L_1 L_2 \cdots L_n$. 

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Pentagram Map

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$$M V_i = V_i L_1 L_2 \cdots L_n$$
Scaling Parameter

Consider the modified matrices, with a scaling parameter:

\[ L_i(\lambda) = \begin{bmatrix} 0 & 0 & \lambda Y_i & 0 \\ 0 & 0 & 0 & \lambda X_i \\ Id & N & 0 & Id \\ 0 & Id & N & 0 \end{bmatrix} \]

The product \( L(\lambda) = L_1(\lambda) \cdots L_n(\lambda) \) is the Lax matrix of Marí-Beffa and Felipe.

Proposition

The product \( L(\lambda) \) is conjugate to the boundary measurement matrix \( B(\lambda) \).
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Thank You!