Laurent Polynomials from the Super Ptolemy Relation

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Ptolemy’s Theorem

Take a quadrilateral inscribed in a circle, with lengths labelled as in the picture.

Then $xy = ac + bd$. 
Triangulated Polygons

For a polygon (inscribed in a circle), let

\[ x_{ij} = \text{length of diagonal (i, j)} \]

Fix a triangulation.
By repeatedly applying Ptolemy relations, we can express any \( x_{ij} \) in terms of the variables from the triangulation.

**Example:**

\[
x_{25} = \frac{x_{15}x_{23} + x_{12}x_{35}}{x_{13}}
\]

\[
= x_{15}x_{23} + x_{12} \left( \frac{x_{15}x_{34} + x_{13}x_{45}}{x_{14}} \right)
\]

\[
= \frac{x_{15}x_{23}}{x_{13}} + \frac{x_{12}x_{15}x_{34}}{x_{13}x_{14}} + \frac{x_{12}x_{45}}{x_{14}}
\]
Consider the graph $\Gamma$ coming from a triangulated polygon.

A “$T$-path”\(^1\) from $i$ to $j$ is a path in $\Gamma$ starting at vertex $i$, ending at $j$, such that

- (T1) the path does not use any edge twice
- (T2) the path has an odd number of edges
- (T3) the even-numbered edges cross the digaonal $(i, j)$
- (T4) the intersections of the path with the diagonal $(i, j)$ get closer to $j$ and farther from $i$ along the path

Let $T_{ij}$ denote the set of $T$-paths from $i$ to $j$.

Example

Here are all the $T$-paths in $T_{25}$

(odd steps are blue and even steps are red)

For a $T$-path $\alpha$, using edges $(i_1, i_2), (i_2, i_3), \ldots$, define the Laurent monomial

$$x_\alpha := \prod_{k} x_{i_k i_{k+1}}$$

($\varepsilon = (-1)^{k+1}$)
The Laurent Formula

**Theorem [Schiffler]¹**

\[ x_{ij} = \sum_{\alpha \in T_{ij}} x_{\alpha} \]

**Corollary**

Each \( x_{ij} \) is a Laurent polynomial in the lengths of the diagonals from any fixed triangulation.

The “Laurent phenomenon” also follows from the cluster algebra structure².

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A “super algebra” is a $\mathbb{Z}_2$-graded algebra.

i.e. $A = A_0 \oplus A_1$, (the “even” and “odd” parts) and

$$A_iA_j \subseteq A_{i+j}$$

A basic example is the algebra generated by $x_1, \ldots, x_n, \theta_1, \ldots, \theta_m$, subject to the relations

$$x_i x_j = x_j x_i, \quad x_i \theta_j = \theta_j x_i, \quad \theta_i \theta_j = -\theta_j \theta_i$$

in particular, $\theta_i^2 = 0$

The $x$’s are the “even generators” and the $\theta$’s are the “odd generators”.

In this example, $A_0$ is spanned by monomials with either no $\theta$’s, or an even number of $\theta$’s, and $A_1$ is spanned by monomials containing an odd number of $\theta$’s.
Given an \( n \)-gon, choose:

- a triangulation \( T \)
- an orientation of each edge in \( T \)
  (We usually do not draw the boundary orientations)

Consider the super algebra with one even generator \( x_{ij} \) for each diagonal in \( T \), and one odd generator \( \theta_{ijk} \) for each triangle in \( T \).

The example pictured above would have 7 even generators \( x_{ij} \), and 3 odd generators \( \theta_{ijk} \).
The Super Ptolemy Relation

Given a quadrilateral, which is part of some triangulated polygon, we get a new triangulation by "flipping" the diagonal:

We define the new variables via the relations\(^1\):

\[
e_{f} = ac + bd + \sqrt{abcd} \sigma \theta
\]

\[
\theta' = \frac{\sqrt{bd} \theta + \sqrt{ac} \sigma}{\sqrt{ac + bd}}
\]

\[
\sigma' = \frac{\sqrt{bd} \sigma - \sqrt{ac} \theta}{\sqrt{ac + bd}}
\]

The Odd Variables

Unlike the ordinary Ptolemy relation, this one is not an involution.

\[ \theta \sigma \theta' \sigma' \sigma'' \theta'' \]

Using the super Ptolemy relation twice, one gets that \( \theta'' = \sigma \) and \( \sigma'' = -\theta \).

Thus reversing the orientations around a triangle corresponds to negating the odd variable.
The Main Question

Starting with a fixed triangulation, we can reach any diagonal by a sequence of flips. Using the super Ptolemy relation, we will get some algebraic expression attached to this diagonal.

**Question:** Can we explicitly describe this algebraic expression?

**Question:** Does it have a nice combinatorial description (analogous to $T$-paths)?
Fix a triangulation $T$ of a polygon. We only consider triangulations that have a “longest edge” (an edge which crosses all internal diagonals of $T$). Call the endpoints of the longest edge $i$ and $j$.

The longest edge splits the triangles in $T$ into triangular and quadrilateral regions:

The vertices incident to the triangular regions will be called “fan centers”, and we will label them $c_1, c_2, \ldots$, 

![Diagram of a triangulation with fan centers labeled $c_1, c_2, c_3$]
The Auxiliary Graph

Given a triangulated polygon, we construct the “auxiliary graph” $\Gamma$:

The vertices and edges of the triangulation are in $\Gamma$.

There is a vertex within each triangle (labelled by odd variables $\theta_i$), connected by an edge to the adjacent fan center. These edges are labelled $\sigma_i$.

For each pair of triangles, there is an edge (labelled $\tau_{ij}$) connecting $\theta_i$ and $\theta_j$. 
A “super T-path” from $i$ to $j$ is a path in $\Gamma$ which satisfies:

(T1) the path does not use any edge twice  
(T2) the path has an odd number of edges  
(T3) the even-numbered edges cross $(i, j)$  
(T4) $\sigma$-edges can only be even steps ($\sigma$-edges are considered to be crossing $(i, j)$), and $\tau$-edges can only be odd steps  
(T5) The points where the path crosses $(i, j)$ progressively move from $i$ to $j$

**Examples:**

![Diagram of Super T-Paths](image)
Weights

If a super $T$-path uses edges $t_1, t_2, \ldots$, we define weights, with values in our super algebra:

- If $t = (k, \ell)$ is a diagonal in the triangulation, then:
  \[ \text{wt}(t) = x_{k\ell} \text{ if } t \text{ is an odd step, and } \text{wt}(t) = x_{k\ell}^{-1} \text{ if } t \text{ is an even step} \]
- If $t = \tau_{a,b}$, then $\text{wt}(t) = 1$
- If $t = \sigma_k$, then $\text{wt}(t) = \mu_k := \sqrt{\frac{c}{ab}} \theta_k$

If $\alpha$ is a super $T$-path with edges $t_1, t_2, \ldots$, define $\text{wt}(\alpha) = \prod_i \text{wt}(t_i)$. 
First, we define a “default orientation” on the diagonals of the triangulation:

Edges connecting fan centers are oriented $c_i \rightarrow c_{i+1}$.

Within each fan segment, orient edges away from the fan center.

**Another Example:**
Label the triangles $\theta_1, \theta_2, \ldots$ from $i$ to $j$.

Look at the oriented edge separating $\theta_k$ and $\theta_{k+1}$. If $\theta_k$ is on the right, then define $\theta_k > \theta_i$ for all $i > k$. If $\theta_k$ is on the left, define $\theta_k < \theta_i$ for all $i > k$.

$$\theta_3 > \theta_4 > \theta_6 > \theta_5 > \theta_2 > \theta_1$$

When we write $\text{wt}(\alpha) = \prod_i \text{wt}(t_i)$, this product is taken with respect to this ordering.
Examples

\[ \theta_3 > \theta_4 > \theta_6 > \theta_5 > \theta_2 > \theta_1 \]

\[
\begin{array}{c}
\frac{x_{12}x_{67}x_{45}}{x_{46}} \mu_4 \mu_2 \\
\frac{x_{18}x_{27}x_{45}}{x_{28}} \mu_3 \mu_6 \\
\frac{x_{12}x_{27}x_{47}x_{56}}{x_{46}} \mu_4 \mu_2 \mu_1
\end{array}
\]
Laurent Formula

Theorem\[Musiker, O., Zhang\]¹

Given a fixed triangulation (with the default orientation),

\[ x_{ij} = \sum_{\alpha \in T_{ij}} \text{wt}(\alpha) \]

Corollary

Each term of \( x_{ij} \):

- is a Laurent monomial in the \( x \)'s times a monomial in the \( \mu \)'s.
- has a positive coefficient when the \( \mu \)'s are written in the positive order

A Complete Example

$\theta_1 > \theta_2 > \theta_3$

$x_{23}x_{15}$
$x_{13}$

$x_{12}x_{34}x_{15}$
$x_{13}x_{14}$

$x_{12}x_{45}$
$x_{14}$

$x_{12}x_{15}$
$\mu_1\mu_2$

$x_{12}x_{15}$
$\mu_1\mu_3$

$x_{12}x_{15}$
$\mu_2\mu_3$
Step 1: Prove for “fan triangulations”.

Step 2: Prove for “zig-zag triangulations”:
**Step 3:** Perform the following flip sequence.

Flip the diagonals within each fan segment (but *not* the edges \(c_i \rightarrow c_{i+1}\)).

We get a zig-zag triangulation containing the longest edge.

By **Part 2**, we can express the longest edge in terms of this triangulation.

By **Part 1**, we can express everything in this triangulation in terms of the original triangulation.