Main points in Sections 7.1 and 7.2

TA: Tuan Pham

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1 What to remember ?

In these sections, we learn the inverse functions of the 6 trigonometric functions. They are $\cos^{-1} x$, $\sin^{-1} x$, $\tan^{-1} x$, $\cot^{-1} x$, $\sec^{-1} x$, and $\csc^{-1} x$. All what we need to remember is the domain and range of $\cos^{-1} x$, $\sin^{-1} x$, and $\tan^{-1} x$ in the following chart. The graphs below can give you more intuition about these three functions and help you remember the chart.

<table>
<thead>
<tr>
<th></th>
<th>Domain</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\cos^{-1} x$</td>
<td>$-1 \leq x \leq 1$</td>
<td>$0 \leq y \leq \pi$</td>
</tr>
<tr>
<td>$\sin^{-1} x$</td>
<td>$-1 \leq x \leq 1$</td>
<td>$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$</td>
</tr>
<tr>
<td>$\tan^{-1} x$</td>
<td>All real numbers</td>
<td>$-\frac{\pi}{2} &lt; y &lt; \frac{\pi}{2}$</td>
</tr>
</tbody>
</table>

2 Compute $\cos^{-1} x$, $\sin^{-1} x$ and $\tan^{-1} x$

If you want to find the exact values of these functions, you can follow the following steps :

1) Put $\theta = \cos^{-1} x$ (respectively $\sin^{-1} x$ or $\tan^{-1} x$). And we are finding the angle $\theta$ such that $\cos \theta = x$ (respectively $\sin \theta = x$ or $\tan \theta = x$).

2) Write down the range for $\theta$ by looking at the chart.
3) Find $\theta$ by using the chart of common values of trigonometric functions.

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$\sin \theta$</th>
<th>$\cos \theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$\frac{\pi}{6}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{\sqrt{3}}{2}$</td>
</tr>
<tr>
<td>$\frac{\pi}{4}$</td>
<td>$\frac{\sqrt{2}}{2}$</td>
<td>$\frac{\sqrt{2}}{2}$</td>
</tr>
<tr>
<td>$\frac{\pi}{3}$</td>
<td>$\frac{\sqrt{3}}{2}$</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>$\frac{\pi}{2}$</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

It is useful to remind yourself that $\sin \theta$ and $\tan \theta$ are odd functions, while $\cos \theta$ is even. One more property of $\cos \theta$ that you will learn later is $\cos(\theta) = -\cos(\pi - \theta)$. That is, two supplementary angles have cosines of opposite signs.

Ex 1 (Problem 19, page 446)
We are to find $\sin^{-1} \frac{\sqrt{2}}{2}$.

Step 1. Put $\theta = \sin^{-1} \frac{\sqrt{2}}{2}$. We are going to find the angle $\theta$ such that $\sin \theta = \frac{\sqrt{2}}{2}$.

Step 2. Since we are given the inverse sine, the range of $\theta$ is $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

Step 3. By using the chart of common values, we find $\theta = \frac{\pi}{4}$.

Ex 2 (Problem 23, page 446)
We are to find $\cos^{-1} \left( -\frac{\sqrt{3}}{2} \right)$.
Step 1. Put \( \theta = \cos^{-1} \left( -\frac{\sqrt{3}}{2} \right) \). We are going to find the angle \( \theta \) such that \( \cos \theta = -\frac{\sqrt{3}}{2} \).

Step 2. Since we are given the inverse cosine, the range of \( \theta \) is \( 0 \leq \theta \leq \pi \).

Step 3. Unfortunately, we do not see in the chart of common values any angle whose cosine is \( \frac{\sqrt{3}}{2} \). However, we see that \( \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2} \). Thus, the supplementary angle of \( \frac{\pi}{6} \) will have cosine equal \( -\frac{\sqrt{3}}{2} \). Therefore,

\[
\theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}
\]

Ex 3 (Problem 37, page 446)
We are to find \( \cos^{-1} \left( \cos \frac{4\pi}{5} \right) \).

Step 1. Put \( \theta = \cos^{-1} \left( \cos \frac{4\pi}{5} \right) \). We are going to find the angle \( \theta \) such that \( \cos \theta = \cos \frac{4\pi}{5} \).

Step 2. Since we are given the inverse cosine, the range of \( \theta \) is \( 0 \leq \theta \leq \pi \).

Step 3. Since \( \frac{4\pi}{5} \) is already in the range, we can pick \( \theta = \frac{4\pi}{5} \).

Ex 4 (Problem 41, page 446)
We are to find \( \sin^{-1} \left( \sin \frac{9\pi}{8} \right) \).

Step 1. Put \( \theta = \sin^{-1} \left( \sin \frac{9\pi}{8} \right) \). We are going to find the angle \( \theta \) such that \( \sin \theta = \sin \frac{9\pi}{8} \).

Step 2. Since we are given the inverse sine, the range of \( \theta \) is \( -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \).

Step 3. Here we cannot pick \( \theta = \frac{9\pi}{8} \) because \( \frac{9\pi}{8} \) exceeds \( \frac{\pi}{2} \). Now look at the unit circle, we see that the angle between \( -\frac{\pi}{2} \) and \( \frac{\pi}{2} \) that has the same \( y \) is \( -\frac{\pi}{8} \). Therefore, \( \theta = -\frac{\pi}{8} \).

Ex 5 (Problem 45, page 446)
We are to find \( \sin \left( \sin^{-1} \frac{1}{4} \right) \).

Put \( \theta = \sin^{-1} \frac{1}{4} \). We are going to find \( \theta \). By the definition of \( \theta \), we already have \( \sin \theta = \frac{1}{4} \).

3 Compute \( \sec^{-1} x \), \( \csc^{-1} x \) and \( \cot^{-1} x \)

If you are asked to find these inverse functions, the first step is the same as mentioned above; the second step is to convert everything to cosine, sine or tangent.
Then we return to the problem of finding inverse cosine, sine, and tangent.

Ex 6 (Problem 45, page 453)
We are to find $\sec^{-1} 4$.
Put $\theta = \sec^{-1} 4$. We are going to find the angle $\theta$ such that $\sec \theta = 4$. That is $\frac{1}{\cos \theta} = 4$. Thus, $\cos \theta = \frac{1}{4}$. Since $\frac{1}{4}$ is not a value in the chart of common values, we have to use a calculator. Pressing $\cos^{-1} 4$ gives us 1.32

4 Write a trigonometric expression as an algebraic expression

If you are asked to find these inverse functions, the first step is always to denote the inverse function by $\theta$. Let’s look at an example.

Ex 7 (Problem 61, page 453)
We are to express the expression $\sin(\sec^{-1} u)$ as an algebraic expression in $u$.
Step 1. Put $\theta = \sec^{-1} u$. The given expression is $\sin \theta$. We know that $\sec \theta = u$, or $\frac{1}{\cos \theta} = u$. Thus $\cos \theta = \frac{1}{u}$.
Step 2. Since we are given the cosine of $\theta$, the range of $\theta$ is $0 \leq \theta \leq \pi$.
Step 3. Because $\sin \theta \geq 0$ in the above range, we have

$$\sin \theta = \sqrt{1 - \cos^2 \theta} = \sqrt{1 - \frac{1}{u^2}}$$