There are several ways to visualize a 2-dimensional or 3-dimensional region in Mathematica. Some helpful commands are \texttt{Plot(3D)}, \texttt{ContourPlot(3D)}, and \texttt{RegionPlot(3D)}. Let’s first consider how to sketch a 2-dimensional region (Problem 1 and 2).

1. Visualize \( R = \{(x, y) : a \leq x \leq b, \ f(x) \leq y \leq g(x)\} \):

\[
\text{Plot}\left\{\{f(x), g(x)\}, \{x, a, b\}, \text{Filling} \rightarrow \{1 \rightarrow \{2\}\}, \text{PlotLegends} \rightarrow "\text{Expressions}"\right\}
\]

The part “1→\{2\}” means to fill the region from the graph of \( f(x) \) (the \textit{first} function) to the graph of \( g(x) \) (the \textit{second} function). The “PlotLegends” part is optional, specifying on the plot which curve is of which function.

\textbf{Practice}

(a) Visualize \( R = \{(x, y) : 0 \leq x \leq 1, \ x^2 \leq y \leq x\} \).

(b) Visualize \( R = \{(x, y) : 0 \leq x \leq 1, \ 0 \leq y \leq 1\} \).

(c) Visualize \( R = \{(x, y) : 0 \leq y \leq 1, \ 0 \leq x \leq 2 - y\} \).

2. When \( R \) is described not as explicitly as in Part 1, we can use the command \texttt{RegionPlot}.

For example, if \( R \) is a domain inside the circle \( x^2 + y^2 = 9 \) and outside the circle \( (x-1)^2 + (y+1)^2 = 1 \), we can write \( R \) as \( R = \{(x, y) : x^2 + y^2 = 9, \ (x-1)^2 + (y+1)^2 = 1\} \).

\textbf{Command:}

\[
\text{RegionPlot}[x^2 + y^2 \leq 9 \&\& \ (x - 1)^2 + (y + 1)^2 \geq 1, \{x, -3, 3\}, \{y, -3, 3\}]
\]

\textbf{Practice}

(a) Sketch the region in the first quadrant, under the line \( y = 2 - x \).
(b) Sketch the region inside the ellipse \( \frac{x^2}{4} + (y-2)^2 = 1 \), on the right of the line \( x = 1 \).

For visualizing a 3-dimensional region, the color effects become more important. Let’s consider some examples (Problem 3 and 4).

3. Visualize \( R = \{(x, y, z) : a \leq x \leq b, \ f(x) \leq y \leq g(x), \ h(x, y) \leq z \leq k(x, y)\} \):

\[
\text{RegionPlot3D}\left[\begin{array}{l}
a \leq x \leq b \&\& f(x) \leq y \leq g(x) \&\& h(x, y) \leq z \leq k(x, y),
\end{array}\right]
\text{PlotStyle} \rightarrow \text{Opacity}[0.5], \text{Mesh} \rightarrow \text{None}
\]

Practice

(a) Sketch the region in the first octant (meaning \( x, y, z \) are nonnegative), under the paraboloid \( z = 1 - x^2 - y^2 \).

(b) Sketch the region inside the ellipsoid \( \frac{x^2}{4} + y^2 + z^2 = 1 \), above the plane \( x+y+z = 0 \).

(c) Sketch the region for the integral

\[
\int_{0}^{1} \int_{0}^{2x} \int_{x^2+y^2}^{x+y} f(x, y, z) \, dz \, dy \, dx
\]

4. Sometimes a 3D region is best visualized by plotting all surfaces (that form it) at once. For example, when \( R \) is the region cut out of the ball \( x^2 + y^2 + z^2 \leq 4 \) by the cylinder \( x^2 + z^2 = 1 \):
ContourPlot3D[{x^2 + y^2 + z^2 == 4, 2x^2 + z^2 == 1}, {x,-2,2}, {y,-2,2}, {z, -2, 2}, ContourStyle→Opacity[0.5], Mesh→None]

Practice

(a) Visualize the region between the cone \( z = \sqrt{x^2 + y^2} \) and the paraboloid \( z = \frac{x^2 + y^2}{x^2 + y^2} \).

(b) Visualize the region bounded by \( z = 0, \ z = \pi, \ y = 0, \ y = 1, \ x = 0 \) and \( x + y = 1 \).

There are several ways to compute a double/triple integral in Mathematica (Problem 5 and 6).

5. One way is to use do an iterated integral, which involves writing the domain in such a way that we can take integral with respect to each variable, one by one. For example, to compute the integral given by Part (c) of Problem 3, we execute

\[
\text{Integrate}[f(x, y, z), \{x, 0, 1\}, \{y, 0, 2x\}, \{z, x^2 + y^2, x+y\}]
\]

Practice

(a) Compute

\[
\int_0^1 \int_{-x}^x \int_{-x+y}^{x+y} (x^2 + zy)dzdydx
\]

(b) Compute the volume of the region in Part (a) of Problem 4.
6. Another way is to define the region of integration. Then let Mathematica compute
the double/triple integral. For example, the region the region \( R \) at the beginning of
Problem 3 can be defined in Mathematica as

\[
R = \text{ImplicitRegion}[a \leq x \leq b \land f(x) \leq y \leq g(x) \land h(x, y) \leq z \leq k(x, y), \{x, y, z\}]
\]

Then the integral

\[
\iiint_R u(x, y, z) \, dV
\]

is evaluated via the command

\[
\text{Integrate}[u(x, y, z), \{x, y, z\} \in R]
\]

The symbol “\( \in \)” can be typed as ESC el ESC.

**Practice**

(a) Compute the volume of the region in Part (b) of Problem 3.

(b) Compute the area of the region in Part (b) of Problem 2.