HOMEWORK #1 (DUE FRIDAY, SEPT. 23).

9/11/2011

Note: Turn in only the “starred” problems; out of these, only selected problems will be graded.

1.* Find all subgroups of the additive group \( \mathbb{Z} \times \mathbb{Z} \).

2. Let \( G \) be group and let \( H \) be a subgroup in \( G \).
   (a) Show that there is a bijection between the sets of left and right cosets of \( H \) in \( G \). In particular, one can define the index \((G : H)\) as the cardinality of the set of right cosets.
   (b) Show that a subgroup of index two is always normal.

3.* Let \( G \) be a group, \( H \) a subgroup in \( G \), and let \( N_H \) be the normalizer of \( H \).
   (a) Show that if \( K \subset G \) is a subgroup such that \( H \) is a normal subgroup of \( K \), then \( K \subset N_H \), i.e., \( N_H \) is the largest subgroup of \( G \) in which \( H \) is normal.
   (b) If \( K \) is a subgroup contained in \( N_H \), then \( KH \) is a group and \( H \) is a normal subgroup in \( KH \).
   (c) If \( G \) is finite and \( K \subset N_H \), then
   \[
   |KH| = \frac{|H||K|}{|H \cap K|}.
   \]

4. Determine all (nonisomorphic) finite groups with order at most 8.

5.* Problem 7, page 75 in Lang.

6.* Problem 9, page 75 in Lang.

7.* (Divisible groups) An abelian group \((G, +)\) is said to be divisible if for any \( y \in G \) and \( n \in \mathbb{Z}, n \neq 0 \), there is an \( x \) in \( G \) with \( nx = y \). (The simplest example is \((\mathbb{Q}, +)\).)
   (a) Show that any divisible group \( G \) is infinite, and that \( G \) has no subgroups of finite index other than \( G \) itself.
   (b) Let \( U = \mathbb{Q}/\mathbb{Z} \). Show that every element of \( U \) is a torsion element, that is, every element has finite order (or finite period, in the terminology in Lang’s book). For each \( n \geq 1 \) show that \( U \) has a unique subgroup of order \( n \), and that this subgroup is cyclic.
   (c) For a prime \( p \), let \( U_p \) be the subgroup of \( U \) consisting of all \( p \)-torsion elements, that is, all elements whose order is a power of \( p \). Show that \( U_p \) is a divisible group, and describe all its subgroups.
   (Remark: We’ll revisit divisible groups (in a more general context) towards the end of the Spring semester when we’ll do a bit of Homological Algebra, and we’ll use the results in this problem at that time. We’ll also show then that any divisible group is a direct sum of copies of \( \mathbb{Q} \) and \( U_p \) for various \( p \).)

8.* Let \( G \) be a finite abelian group which is not cyclic. Prove that there is a prime number \( p \) and a subgroup \( H \) of \( G \) with \( H \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \).