I would like to make a little change of notations in the problem so that they are more convenient for me to write. Specifically, I would like to write \( u \) instead of \( \alpha \) in the arguments of the functional \( L \), and reserve the name \( x = (x_4, \ldots, x_d) \) for Cartesian coordinates in \( \mathbb{R}^d \). Denote
\[
 u = (u_1, u_2, \ldots, u_d) \in \mathbb{R}^d, \\
p = (p_1, p_2, \ldots, p_d) \in \mathbb{R}^d, \\
r = (r_1, r_2, \ldots, r_d) \in \mathbb{R}^d.
\]
Let \( L = L(t, u, u, r) \in C^3([a, b] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d) \). For \( u \in C^2([a, b], \mathbb{R}^d) \), consider
\[
 I(u) = \int_a^b L(t, u, u, u) \, dt,
\]
with boundary conditions (BC): \( u(a) = u_0, \quad u'(a) = p_0, \quad u(b) = u_d, \quad u'(b) = p_d \).

We'll derive the Euler-Lagrange equations for \( I(u) \).

We see that if \( u \) satisfies the boundary condition (BC) then so does \( u + s\varphi \) where \( s \in \mathbb{R} \) and \( \varphi \in C^0_c([a, b], \mathbb{R}^d) \). Therefore, if \( u \) is a critical point of \( I \) then
\[
 \frac{d}{ds} \bigg|_{s=0} I(u + s\varphi) = 0. 
\]
By the chain rule of differentiation, we have
\[
 0 = \frac{d}{ds} \bigg|_{s=0} I(u + s\varphi) = \int_a^b \frac{d}{ds} \bigg|_{s=0} L(t, u + s\varphi, u + s\varphi, u + s\varphi) \, dt
\]
\[
 = \int_a^b \left[ L_{u_i}(t, u + s\varphi, u + s\varphi, u + s\varphi) \bigg|_{s=0} \varphi_i + L_{u'}(t, u + s\varphi, u + s\varphi, u + s\varphi) \bigg|_{s=0} \varphi_i + L_{u''}(t, u + s\varphi, u + s\varphi, u + s\varphi) \bigg|_{s=0} \varphi_i \\
  + L_{u_i'}(t, u + s\varphi, u + s\varphi, u + s\varphi) \bigg|_{s=0} \varphi_i + L_{u_i''}(t, u + s\varphi, u + s\varphi, u + s\varphi) \bigg|_{s=0} \varphi_i \right] \, dt
\]
\[ = \int_a^b \left[ L_u(t, u, \dot{u}, \ddot{u}) \dot{\phi}_i + L_v(t, u, \dot{u}, \ddot{u}) \dot{\phi}_j + L_w(t, u, \dot{u}, \ddot{u}) \dot{\phi}_k \right] dt \]

Thus,
\[ \int_a^b L_u(t, u, \dot{u}, \ddot{u}) \dot{\phi}_i dt + \int_a^b L_v(t, u, \dot{u}, \ddot{u}) \dot{\phi}_j dt + \int_a^b L_w(t, u, \dot{u}, \ddot{u}) \dot{\phi}_k dt = 0 \quad (1) \]

In the following, we'll use the integration-by-part formula. Thus, we can assume that \( u \in C^4([a,b], \mathbb{R}^d) \) so that all differentiations in \( t \) are valid.

We have
\[ \int_a^b L_v(t, u, \dot{u}, \ddot{u}) \dot{\phi}_j dt + \int_a^b L_w(t, u, \dot{u}, \ddot{u}) \dot{\phi}_k dt = 0 \quad (2) \]

We have
\[ \int_a^b L_w(t, u, \dot{u}, \ddot{u}) \dot{\phi}_k dt = 0 \quad (3) \]

Now substituting (2) and (3) into (1), we get
\[ \int_a^b \left\{ L_u(t, u, \dot{u}, \ddot{u}) \frac{d}{dt} \left[ L_v(t, u, \dot{u}, \ddot{u}) \right] + \frac{d^2}{dt^2} \left[ L_w(t, u, \dot{u}, \ddot{u}) \right] \right\} \phi_i(t) dt = 0 \]

Because this identity is true for all \( \phi = (\phi_1, \ldots, \phi_d) \in C^2([a,b], \mathbb{R}^d) \), by the Fundamental Lemma of Calculus of Variations, we have
\[ L_i(t, u, \dot{u}, \ddot{u}) - \frac{d}{dt} \left[ L_i(t, u, \dot{u}, \ddot{u}) \right] + \frac{d^2}{dt^2} \left[ L_i(t, u, \dot{u}, \ddot{u}) \right] = 0 \quad \forall t \in (a, b), \quad \forall i \in \mathbb{N}. \]

These are the Euler-Lagrange equations for the functional \( I(u) \).

Every critical point \( u \) which is in \( C^4([a, b], \mathbb{R}^d) \) must satisfy these equations.

2. We restate the Noether's theorem with time-variance (a corrected version of Theorem 15.2, page 29, Jost - Li Jost) as follows.

Consider the variational integral \( I(x) = \int_a^b F(t, x(t), \dot{x}(t)) \, dt \), with \( F \in C^2([a, b] \times \mathbb{R}^d \times \mathbb{R}^d) \). Suppose there is a smooth one-parameter family of differentiable maps \( \overline{h}_s = (h_s^0, h_s^1) : [a, b] \times \mathbb{R}^d \rightarrow \mathbb{R} \times \mathbb{R}^d \), for \( s \in (-\varepsilon, \varepsilon) \), with \( \overline{h}_0(t, x) = (t, x) \quad \forall (t, x) \in [a, b] \times \mathbb{R}^d \)

and satisfying

\[
\int_{h_s^0(t_0)}^{h_s^0(t_1)} F(t_s, x_s(t_s), \frac{dx_s}{dt_s}) \, dt_s = \int_{t_0}^{t_1} F(t, x(t), \dot{x}(t)) \, dt \\
\forall [t_0, t_1] \subset [a, b], \quad \forall x \in C^2([a, b], \mathbb{R}^d), \quad \text{where} \quad t_s = h_s^0(t), \quad x_s(t) = h_s^1(x(t)).
\]

Then for any solution \( x \) to the Euler-Lagrange equations of \( I \),

\[
\frac{d}{ds} \bigg|_{s=0} h_s^1(x(t)) + \left[ F(t, x, \dot{x}) - \dot{x} F_p(t, x, \dot{x}) \right] \frac{d}{ds} \bigg|_{s=0} h_s^0(t)
\]

is constant with respect to \( t \in [a, b] \).

Let \( F \in C^2(\mathbb{R} \times \mathbb{R}^3) \) and a family of maps \( \overline{h}_s : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{R} \),

\[
\overline{h}_s(t, x_1, x_2, x_3) = (t+s, x_1 \cos \theta + x_2 \sin \theta, -x_1 \sin \theta + x_2 \cos \theta, x_3 + s)
\]
such that $t_0 \circ h_0(t, x) = \Phi(t, x)$ in $\mathbb{R} \times \mathbb{R}^3$, for all $s \in (-\varepsilon, \varepsilon)$. Let $f : [0, w) \to [0, w)$ be a $C^2$ function. Consider the variational integral, which is an energy functional of an expanding screw motion,

$$ I(x) = \int_{t_1}^{t_2} \left[ \frac{1}{2} f(r) \| x \|^2 - \Phi(t, x(t)) \right] dt,$$

where $t_1, t_2$ are constants and $r = r(x(t)) = \sqrt{x_1(t)^2 + x_2(t)^2}$. We want to use Noether's theorem to find a first integral of motion. Put

$$ h_0(x_1, x_2, x_3) = (x_1 \cos s + x_2 \sin s, -x_1 \sin s + x_2 \cos s, x_3 + s), $$

$$ h_0^0(t) = t + s. $$

Then $h_0 = (h_0^0, h_0)$ and $h_0(t, x_1, x_2, x_3) = (t, x_1, x_2, x_3)$. We want to apply Noether's theorem for $a = t_1$, $b = t_2$,

$$ F(t, x(t)) = \frac{1}{2} f(r) \| x \|^2 - \Phi(t, x), $$

where $r = r(x(t)) = \sqrt{x_1(t)^2 + x_2(t)^2}$. Since $f \in C^2([0, w))$, $\Phi \in C^2([a, b] \times \mathbb{R}^3 \times \mathbb{R}^3)$. In order to apply Noether's theorem, we need to show that for any function $x = (x_1, x_2, x_3) \in C^2([t_1, t_2] \times \mathbb{R}^3)$ and any subinterval $[x_1, x_3] \subset [t_1, t_2]$,

$$ \int_{h_0^0(t)}^{h_0^0(t)} F(t, x(t), \frac{dx_1}{dt}, \frac{dx_2}{dt}, \frac{dx_3}{dt}) dt = \int_a^b F(t, x(t), \frac{dx_1}{dt}, \frac{dx_2}{dt}, \frac{dx_3}{dt}) dt, \quad (2) $$

where $t_0 = h_0^0(t)$ and $x_0(t) = h_0(x_0(t)).$

We have $x_0(t) = h_0^0(x(t)) = (x_1(t) \cos s + x_2(t) \sin s, -x_1(t) \sin s + x_2(t) \cos s, x_3(t) + s).$
\[
\frac{dx_s}{dt_s} (t_s) = \frac{d}{dt} (x, x(t)) \frac{dt}{dt_s} = \left( \frac{dx_1}{dt} \cos s + \frac{dx_2}{dt} \sin s \right) - \frac{dx_3}{dt} \sin s + \frac{dx_4}{dt} \cos s, \frac{dy}{dt_s} \right).
\]

Thus,
\[
\left| \frac{dx_s}{dt_s} (t_s) \right|^2 = \left( \frac{dx_1}{dt} \cos s + \frac{dx_2}{dt} \sin s \right)^2 + \left( - \frac{dx_3}{dt} \sin s + \frac{dx_4}{dt} \cos s \right)^2 + \left( \frac{dy}{dt_s} \right)^2
\]
\[
= \left( \frac{dx_1}{dt} \right)^2 + \left( \frac{dx_2}{dt} \right)^2 + \left( \frac{dy}{dt_s} \right)^2
\]
\[
= \left| \frac{dx}{dt} \right|^2 = |x|^2 \quad (4)
\]

Also,
\[
r(x_s(t_s)) = \sqrt{(x_1(t) \cos s + x_2(t) \sin s)^2 + (x_3(t) \sin s + x_4(t) \cos s)^2}
\]
\[
= \sqrt{x_1(t)^2 + x_2(t)^2}
\]
\[
= r(x(t)) \quad (5)
\]

We have
\[
\text{LHS}(Q) = \int_{t_s}^{t_s+\delta} \left[ \frac{1}{2} f(r(x_s(t_s))) \left| \frac{dx_s}{dt_s} \right|^2 - \Phi(t_s, x_s(t_s)) \right] dt_s
\]
\[
= \int_{t_s}^{t_s+\delta} \left[ \frac{1}{2} f(r(t)) \left| \frac{dx}{dt} \right|^2 \right] dt - \int_{t_s}^{t_s+\delta} \Phi(t_s, x(t)) dt
\]
\[
= \int_{t_s}^{t_s+\delta} \frac{1}{2} f(r(t)) |x|^2 dt - \int_{t_s}^{t_s+\delta} \Phi(t, x(t)) dt
\]
\[
= \Phi(t, x(t))
\]
\[
= \text{RHS}(Q).
\]

Therefore, (2) is proved. Then by Noether's theorem, we have equation (4):
\[
F_p(t, x, \dot{x}, \ddot{x}) \frac{d}{ds} \int_{x_0(t)} \Phi(t, x(t)) dt + \left[ F(t, x, \dot{x}) - \dot{x} F_p(t, x, \dot{x}) \right] \frac{d}{ds} \int_{x_0(t)} \Phi^0(t) dt = \text{const}
\]
\[
(\text{with respect to } t \in [t_1, t_2])
\]
Because \( F(t, x, p) = \frac{1}{2} f(r) |x|^2 - \Xi(t, x) \), \( F_p(t, x, p) = f(r) p \). Thus,
\[
F_p(t, x, \dot{x}) = f(r) \dot{x} \tag{7}
\]

Moreover,
\[
\frac{d}{ds} \left. \varphi_0^i(t) \right|_{s=0} = \frac{d}{ds} \left. (t+s) \right|_{s=0} = 1, \tag{8}
\]

\[
\frac{d}{ds} \left. \varphi_k(x(t)) \right|_{s=0} = \frac{d}{ds} \left. (x_1(t) \cos s + x_2(t) \sin s, -x_1(t) \sin s + x_2(t) \cos s, x_3(t) + s) \right|_{s=0}
\]
\[
= (-x_1(t) \sin s + x_2(t) \cos s, -x_1(t) \cos s - x_2(t) \sin s, 1) \big|_{s=0}
\]
\[
= (x_2(t), -x_1(t), 1) \tag{9}
\]

Substituting (7), (8), (9) into (6), we get
\[
\text{LHS (6)} = f(r) \dot{x} \cdot (x_2(t), -x_1(t), 1) + \left\{ \left[ \frac{1}{2} f(r) |\dot{x}|^2 - \Xi(t, x) \right] - x f(r) \cdot \dot{x} \right\}
\]
\[
= f(r) (x_1(t) \dot{x}_2(t) - x_2(t) \dot{x}_1(t) + \dot{x}_3(t)) - \left( \frac{1}{2} f(r) |\dot{x}|^2 + \Xi(t, x) \right).
\]

Therefore, the first integral of the Euler–Lagrange equations of \( I(x) \) obtained from Noether’s theorem is
\[
f(r) (x_1 \dot{x}_2 - x_2 \dot{x}_1 + \dot{x}_3) - \frac{1}{2} f(r) |\dot{x}|^2 - \Xi(t, x) = \text{const} \quad \text{in } t \in [t_1, t_2].
\]

(3) For any two functions \( F, G \in C^1(\mathbb{R}^{2n}) \), we define the Poisson bracket
\[
\{ F, G \} := \frac{\partial F}{\partial x_j} \frac{\partial G}{\partial p_i} - \frac{\partial G}{\partial x_j} \frac{\partial F}{\partial p_i},
\]

where \( F = F(x, p), \ G = G(x, p), \ x = (x_1, \ldots, x_n), \ p = (p_1, \ldots, p_n). \)

Consider a local diffeomorphism \( \Psi : U \subseteq \mathbb{R}^{2n} \to \mathbb{R}^{2n}, \ \Psi(x, p) = (\xi, \pi), \)

where \( \xi = (\xi_1, \ldots, \xi_n) \) and \( \pi = (\pi_1, \ldots, \pi_n). \) We’ll show that \( \Psi \) is a
Canonical transformation if and only if \( \mathcal{F}, \mathcal{G} \circ \mathcal{Y} = \mathcal{F} \circ \mathcal{Y}, \mathcal{G} \circ \mathcal{Y} \) for all \( \mathcal{F}, \mathcal{G} \in \mathcal{C}^1(\mathbb{R}^{2n}) \). Recall the definition: \( \mathcal{Y} \) is called a canonical transformation if the following identities hold:

\[
\frac{\partial \mathcal{Y}}{\partial x_i} = \frac{\partial \mathcal{Y}}{\partial x_j}, \quad \frac{\partial \mathcal{Y}}{\partial p_i} = -\frac{\partial \mathcal{Y}}{\partial p_j},
\]

\[
\frac{\partial \mathcal{Y}}{\partial p_j} = -\frac{\partial \mathcal{Y}}{\partial x_i}, \quad \frac{\partial \mathcal{Y}}{\partial x_j} = \frac{\partial \mathcal{Y}}{\partial p_i}.
\]

Equivalently, \( \begin{pmatrix} \frac{\partial \mathcal{Y}}{\partial x} & \frac{\partial \mathcal{Y}}{\partial p} \\ \frac{\partial \mathcal{Y}}{\partial p} & \frac{\partial \mathcal{Y}}{\partial x} \end{pmatrix}^{-1} = \begin{pmatrix} \frac{\partial \mathcal{Y}}{\partial p}^T & -\frac{\partial \mathcal{Y}}{\partial x}^T \\ -\frac{\partial \mathcal{Y}}{\partial x}^T & \frac{\partial \mathcal{Y}}{\partial p}^T \end{pmatrix} \), we get

\[
\begin{pmatrix} \mathcal{I}_n & 0 \\ 0 & \mathcal{I}_n \end{pmatrix} = \begin{pmatrix} \frac{\partial \mathcal{Y}}{\partial x} & \frac{\partial \mathcal{Y}}{\partial p} \\ \frac{\partial \mathcal{Y}}{\partial p} & \frac{\partial \mathcal{Y}}{\partial x} \end{pmatrix} \begin{pmatrix} \frac{\partial \mathcal{Y}}{\partial p}^T & -\frac{\partial \mathcal{Y}}{\partial x}^T \\ -\frac{\partial \mathcal{Y}}{\partial x}^T & \frac{\partial \mathcal{Y}}{\partial p}^T \end{pmatrix}.
\]

Thus, \( \mathcal{Y} \) is a canonical transformation if and only if we have 4 identities:

\[
\frac{\partial \mathcal{Y}}{\partial x} \left( \frac{\partial \mathcal{Y}}{\partial p} \right)^T - \frac{\partial \mathcal{Y}}{\partial p} \left( \frac{\partial \mathcal{Y}}{\partial x} \right)^T = \mathcal{I}_n,
\]

\[
-\frac{\partial \mathcal{Y}}{\partial x} \left( \frac{\partial \mathcal{Y}}{\partial p} \right)^T + \frac{\partial \mathcal{Y}}{\partial p} \left( \frac{\partial \mathcal{Y}}{\partial x} \right)^T = 0,
\]

\[
\frac{\partial \mathcal{Y}}{\partial x} \left( \frac{\partial \mathcal{Y}}{\partial p} \right)^T - \frac{\partial \mathcal{Y}}{\partial p} \left( \frac{\partial \mathcal{Y}}{\partial x} \right)^T = 0,
\]

\[
-\frac{\partial \mathcal{Y}}{\partial x} \left( \frac{\partial \mathcal{Y}}{\partial p} \right)^T + \frac{\partial \mathcal{Y}}{\partial p} \left( \frac{\partial \mathcal{Y}}{\partial x} \right)^T = \mathcal{I}_n.
\]
In terms of indices, those equations are
\[
\frac{\partial\xi_k}{\partial y_j} \frac{\partial y_j}{\partial x_l} - \frac{\partial \xi_k}{\partial y_j} \frac{\partial y_j}{\partial x_l} = \delta_{kl}, \quad (1)
\]
\[-\frac{\partial \xi_k}{\partial y_j} \frac{\partial \xi_l}{\partial y_j} + \frac{\partial \xi_k}{\partial x_l} \frac{\partial \xi_l}{\partial x_j} = 0, \quad (2)
\]
\[\frac{\partial \xi_k}{\partial y_l} \frac{\partial \xi_l}{\partial y_j} - \frac{\partial \xi_k}{\partial x_l} \frac{\partial \xi_l}{\partial x_j} = 0, \quad (3)
\]
\[-\frac{\partial \xi_k}{\partial y_l} \frac{\partial \xi_l}{\partial x_j} + \frac{\partial \xi_k}{\partial y_j} \frac{\partial \xi_l}{\partial x_l} = \delta_{kl}, \quad (4)
\]
for all \(1 \leq k, l \leq n\).

For any local diffeomorphism \(\psi : U \subseteq \mathbb{R}^n \to \mathbb{R}^n\) and \(F, G \in C^1(\mathbb{R}^n)\),
we have
\[
\{F, G\} \circ \psi(x, \xi) = \frac{\partial F}{\partial y_j} (\psi(x, \xi)) \frac{\partial G}{\partial y_l} (\psi(x, \xi)) - \frac{\partial G}{\partial y_j} (\psi(x, \xi)) \frac{\partial F}{\partial y_l} (\psi(x, \xi)).
\]

In other words,
\[
\{F, G\} \circ \psi(x, \xi) = \frac{\partial F}{\partial y_j} (\xi, \pi) \frac{\partial G}{\partial y_l} (\xi, \pi) - \frac{\partial G}{\partial y_j} (\xi, \pi) \frac{\partial F}{\partial y_l} (\xi, \pi) \quad (5)
\]

We have
\[
\{F \circ \psi, G \circ \psi\}(x, \xi) = \frac{\partial (G \circ \psi)}{\partial y_j} (x, \xi) \frac{\partial (F \circ \psi)}{\partial y_l} (x, \xi) - \frac{\partial (G \circ \psi)}{\partial y_j} (x, \xi) \frac{\partial (F \circ \psi)}{\partial y_l} (x, \xi) \quad (6)
\]

By the chain rule,
\[
\frac{\partial (G \circ \psi)}{\partial y_j} (x, \xi) = \frac{\partial G}{\partial x_k} (\psi(x, \xi)) \frac{\partial \xi_k}{\partial y_j} + \frac{\partial G}{\partial x_k} (\psi(x, \xi)) \frac{\partial y_j}{\partial y_l} \frac{\partial \xi_k}{\partial x_l}.
\]
\[
= \frac{\partial G}{\partial x_k} (\xi, \pi) \frac{\partial \xi_k}{\partial y_j} + \frac{\partial G}{\partial x_k} (\xi, \pi) \frac{\partial y_j}{\partial y_l} \frac{\partial \xi_k}{\partial x_l}.
\]
\[
\frac{\partial (F \psi)}{\partial \beta_j} (x_i \rho) = \frac{\partial F}{\partial \lambda} (\psi (x_i \rho)) \frac{\partial \delta_e}{\partial \beta_j} + \frac{\partial F}{\partial \rho} (\psi (x_i \rho)) \frac{\partial \rho e}{\partial \beta_j} \\
= \frac{\partial F}{\partial \lambda} (\xi, \pi) \frac{\partial \delta_e}{\partial \beta_j} + \frac{\partial F}{\partial \rho} (\xi, \pi) \frac{\partial \rho e}{\partial \beta_j} 
\]

Thus,
\[
\frac{\partial (G \psi)}{\partial \gamma_j} (x_i \rho) \frac{\partial (F \psi)}{\partial \beta_j} (x_i \rho) = \underbrace{\frac{\partial G}{\partial \xi_e} (\xi, \pi) \frac{\partial F}{\partial \xi_e} (\xi, \pi) \frac{\partial \xi_e}{\partial \gamma_j} \frac{\partial \xi_e}{\partial \beta_j}}_{\{1\}} + \\
+ \underbrace{\frac{\partial G}{\partial \xi_e} (\xi, \pi) \frac{\partial F}{\partial \rho} (\xi, \pi) \frac{\partial \rho e}{\partial \gamma_j} \frac{\partial \rho e}{\partial \beta_j}}_{\{2\}} + \\
+ \underbrace{\frac{\partial G}{\partial \xi_e} (\xi, \pi) \frac{\partial F}{\partial \rho} (\xi, \pi) \frac{\partial \rho e}{\partial \gamma_j} \frac{\partial \rho e}{\partial \beta_j}}_{\{3\}} 
\]

Similarly,
\[
\frac{\partial (G \psi)}{\partial \beta_i} (x_i \rho) = \frac{\partial F}{\partial \lambda} (\xi, \pi) \frac{\partial \delta_e}{\partial \beta_i} + \frac{\partial F}{\partial \rho} (\xi, \pi) \frac{\partial \rho e}{\partial \beta_i} 
\]

\[
\frac{\partial (F \psi)}{\partial \gamma} (x_i \rho) = \frac{\partial F}{\partial \lambda} (\xi, \pi) \frac{\partial \delta_e}{\partial \gamma} + \frac{\partial F}{\partial \rho} (\xi, \pi) \frac{\partial \rho e}{\partial \gamma} 
\]

Thus,
\[
\frac{\partial (G \psi)}{\partial \gamma_j} (x_i \rho) \frac{\partial (F \psi)}{\partial \gamma} (x_i \rho) = \underbrace{\frac{\partial G}{\partial \xi_e} (\xi, \pi) \frac{\partial F}{\partial \xi_e} (\xi, \pi) \frac{\partial \xi_e}{\partial \gamma} \frac{\partial \xi_e}{\partial \gamma}}_{\{5\}} + \\
+ \underbrace{\frac{\partial G}{\partial \xi_e} (\xi, \pi) \frac{\partial F}{\partial \rho} (\xi, \pi) \frac{\partial \rho e}{\partial \gamma} \frac{\partial \rho e}{\partial \gamma}}_{\{6\}} + \\
+ \underbrace{\frac{\partial G}{\partial \xi_e} (\xi, \pi) \frac{\partial F}{\partial \rho} (\xi, \pi) \frac{\partial \rho e}{\partial \gamma} \frac{\partial \rho e}{\partial \gamma}}_{\{7\}} + \\
+ \underbrace{\frac{\partial G}{\partial \xi_e} (\xi, \pi) \frac{\partial F}{\partial \rho} (\xi, \pi) \frac{\partial \rho e}{\partial \gamma} \frac{\partial \rho e}{\partial \gamma}}_{\{8\}} 
\]
Substituting (7) and (8) into (6), we get

\[
\{ \mathcal{F} \circ \Psi, \mathcal{G} \circ \Psi \}(x, p) = \{ 1 \} + \{ 2 \} + \{ 3 \} + \{ 4 \} - (\{ 5 \} + \{ 6 \} + \{ 7 \} + \{ 8 \})
\]

\[= \{ 1 \} + \{ 2 \} + \{ 3 \} + \{ 4 \} - \{ 5 \} - \{ 6 \} - \{ 7 \} - \{ 8 \} + \{ 9 \}
\]

\[= \frac{\partial G}{\partial x} (\xi, \pi) \frac{\partial F}{\partial \pi} (\xi, \pi) \left( \frac{\partial \pi_k}{\partial x_j} \frac{\partial \pi_k}{\partial \pi_j} - \frac{\partial \pi_k}{\partial x_j} \frac{\partial \pi_k}{\partial \pi_j} \right) + \{ 9 \}
\]

\[+ \frac{\partial G}{\partial x} (\xi, \pi) \frac{\partial F}{\partial \pi} (\xi, \pi) \left( \frac{\partial \pi_k}{\partial x_j} \frac{\partial \pi_k}{\partial \pi_j} - \frac{\partial \pi_k}{\partial x_j} \frac{\partial \pi_k}{\partial \pi_j} \right) + \{ 10 \}
\]

\[+ \frac{\partial G}{\partial x} (\xi, \pi) \frac{\partial F}{\partial \pi} (\xi, \pi) \left( \frac{\partial \pi_k}{\partial x_j} \frac{\partial \pi_k}{\partial \pi_j} - \frac{\partial \pi_k}{\partial x_j} \frac{\partial \pi_k}{\partial \pi_j} \right) + \{ 11 \}
\]

\[+ \frac{\partial G}{\partial x} (\xi, \pi) \frac{\partial F}{\partial \pi} (\xi, \pi) \left( \frac{\partial \pi_k}{\partial x_j} \frac{\partial \pi_k}{\partial \pi_j} - \frac{\partial \pi_k}{\partial x_j} \frac{\partial \pi_k}{\partial \pi_j} \right) + \{ 12 \}
\]

\[\text{(9)}
\]

Assume that \( \Psi \) is a canonical transformation (i.e. Eqs. (1)-(4) hold).

By (2), \( \{ 3 \} = 0 \).

By (3), \( \{ 10 \} = 0 \).

By (1), \( \{ 11 \} = \delta_{kl} \).

By (4), \( \{ 12 \} = \delta_{kl} \).

Therefore, (9) reduces to

\[
\{ \mathcal{F} \circ \Psi, \mathcal{G} \circ \Psi \}(x, p) = \frac{\partial G}{\partial x} (\xi, \pi) \frac{\partial F}{\partial \pi} (\xi, \pi) \delta_{kl} - \frac{\partial G}{\partial x} (\xi, \pi) \frac{\partial F}{\partial \pi} (\xi, \pi) \delta_{kl}
\]

\[
\text{(5)}\quad \{ \mathcal{F}, \mathcal{G} \circ \Psi \}(x, p).
\]
By (5) and (9) we have

$$
\frac{\partial G}{\partial y}(\xi, \eta) \frac{\partial F}{\partial y}(\xi, \eta) - \frac{\partial G}{\partial x}(\xi, \eta) \frac{\partial F}{\partial x}(\xi, \eta) =
$$

$$= \frac{\partial G}{\partial x}(\xi, \eta) \frac{\partial F}{\partial y}(\xi, \eta) \left( \frac{\partial x}{\partial y} \frac{\partial \xi}{\partial y} - \frac{\partial x}{\partial \eta} \frac{\partial \xi}{\partial \eta} \right) + \frac{\partial G}{\partial y}(\xi, \eta) \frac{\partial F}{\partial x}(\xi, \eta) \left( \frac{\partial y}{\partial x} \frac{\partial \eta}{\partial x} - \frac{\partial y}{\partial \eta} \frac{\partial \eta}{\partial \eta} \right)$$

$$+ \frac{\partial G}{\partial \xi}(\xi, \eta) \frac{\partial F}{\partial \eta}(\xi, \eta) \left( \frac{\partial \xi}{\partial \eta} \frac{\partial \xi}{\partial \eta} - \frac{\partial \xi}{\partial \xi} \frac{\partial \xi}{\partial \xi} \right) + \frac{\partial G}{\partial \eta}(\xi, \eta) \frac{\partial F}{\partial \xi}(\xi, \eta) \left( \frac{\partial \eta}{\partial \xi} \frac{\partial \eta}{\partial \xi} - \frac{\partial \eta}{\partial \eta} \frac{\partial \eta}{\partial \eta} \right).$$

$$\forall F, G \in C^1(\mathbb{R}^n)$$  \hspace{1cm} (10).

For any $1 \leq l, l \leq n$, we choose

$$G(\xi, \eta) = G^{(l)}(\xi, \eta) := x_l, \quad F(\xi, \eta) = F^{(l)}(\xi, \eta) := \rho \varepsilon_l.$$

Then (10) reduces to

$$\frac{\partial x_l}{\partial y} \frac{\partial x_l}{\partial \eta} = \frac{\partial x_l}{\partial y} \frac{\partial x_l}{\partial \eta} - \frac{\partial x_l}{\partial \eta} \frac{\partial x_l}{\partial \eta}.$$

Thus, (1) is satisfied.

Next, for any $1 \leq k, l \leq n$, we choose

$$G(\xi, \eta) = G^{(k)}(\xi, \eta) := \rho \varepsilon_k, \quad F(\xi, \eta) = F^{(k)}(\xi, \eta) := \rho \varepsilon_k.$$

Then (10) reduces to

$$\frac{\partial \rho \varepsilon_k}{\partial y} \frac{\partial \rho \varepsilon_k}{\partial \eta} - \frac{\partial \rho \varepsilon_k}{\partial \eta} \frac{\partial \rho \varepsilon_k}{\partial \eta}.$$

Thus, (4) is satisfied.

Next, for any $1 \leq k, l \leq n$, we choose

$$G(\xi, \eta) = G^{(l)}(\xi, \eta) := x_l, \quad F(\xi, \eta) = F^{(l)}(\xi, \eta) := x_l.$$

Then (10) reduces to

$$\frac{\partial x_l}{\partial y} \frac{\partial x_l}{\partial \eta} = 0.$$

Thus, (2) is satisfied.
Next, for any $1 \leq k, l \leq n$, we choose
\[ G(x_l) = G^{(q)}(x_l) = \phi_k, \quad F(x_l) = F^{(q)}(x_l) = \phi_k. \]

Then (16) reduces to
\[ \frac{\partial \phi_k}{\partial y} \frac{\partial \phi_l}{\partial y} - \frac{\partial \phi_k}{\partial y} \frac{\partial \phi_l}{\partial y} = 0. \]
Thus, (3) is satisfied.

Because $\Psi$ satisfies (1), (2), (3), (4), it is a canonical transformation.