Find the fifth root of $3 + 3i$.

**Proof**

We express $3 + 3i$ in polar coordinate:

$$3 + 3i = 3\sqrt{2} \left( \frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \right) = 3\sqrt{2} \left( \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right)$$

Thus,

$$3 + 3i = 3\sqrt{2} \left[ \cos \left( \frac{\pi}{4} + k \cdot 2\pi \right) + i \sin \left( \frac{\pi}{4} + k \cdot 2\pi \right) \right]$$

for any $k \in \mathbb{Z}$.

Thus the fifth roots of $3 + 3i$ are

$$\sqrt[5]{3 + 3i} = \sqrt[5]{3\sqrt{2}} \left[ \cos \left( \frac{\pi}{20} + k \cdot \frac{2\pi}{5} \right) + i \sin \left( \frac{\pi}{20} + k \cdot \frac{2\pi}{5} \right) \right]$$

for any $k \in \mathbb{Z}$.

Because cosine and sine are periodic with period $2\pi$, we actually have only 5 fifth roots corresponding to $k = 0, 1, 2, 3, 4$. 

\[\text{Diagram} \quad \frac{\pi}{20} \quad \frac{\pi}{5}\]
(2) Problem 3, Ahlfors p. 9: prove that
\[ \frac{|a-b|}{|1-\bar{a}b|} = 1 \] (\( \star \))

If either \(|a|=1\) or \(|b|=1\), what exception must be made if \(|a|=|b|=1\)?

Proof. If \(|a|=1\) and \(|b|\neq 1\) then \(|\bar{a}b|=|b|\neq 1\). Thus \(1-\bar{a}b\neq 0\). Similarly, if \(|b|=1\) and \(|a|\neq 1\) then \(1-\bar{a}b\neq 0\). Thus in both cases, the fraction \(\frac{a-b}{1-\bar{a}b}\) is well-defined.

We have \(|a-b|^2 = (a-b)(\bar{a}-\bar{b})\) (definition of modulus)
\[ = (a-\bar{b}) (\bar{a}-b) \]
\[ = a\bar{a} - b\bar{a} - \bar{a}b + b\bar{b} \]
\[ = |a|^2 + |b|^2 - 2\Re(\bar{a}b) \quad \text{(since } b\bar{a} = \bar{a}b \text{)} \]

Now if \(|a|=1\) then
\[ |a-b|^2 = 1 + |b|^2 - 2\Re(\bar{a}b) = 1 + |b|^2 - 2\Re(\bar{a}b) \]

Put \(z = \bar{a}b\). We get
\[ 1 + |z|^2 - 2\Re z = 1 + z\bar{z} - 2 - \bar{z} = (1-\bar{z})(1-z) \]
\[ = (1-\bar{z})(1-z) \]
\[ = 1 - |z|^2 = |1-\bar{a}b|^2 \]

Thus \(|a-b|^2 = |1-\bar{a}b|^2\), or \(|a-b| = |1-\bar{a}b|\).
Similarly, if $|b| = 1$ we get also get $|a-b| = |1-a \bar{b}|$. Thus we have the conclusions $|a| = 1 \Rightarrow |a-b| = |1-a \bar{b}|$ and $|b| = 1 \Rightarrow |a-b| = |1-a \bar{b}|$.

If one of $a$ and $b$ has modulus different from 1, then $1-a \bar{b}$ as already shown; thus $\frac{|a-b|}{|1-a \bar{b}|} = 1$, or equivalently $\left| \frac{a-b}{1-a \bar{b}} \right| = 1$.

In case $|a| = |b| = 1$, we cannot take the quotient if and only if $(a-b) = |1-a \bar{b}| = 0$, which is equivalent to $a = b$. Thus the statement $|a| = |b| = 1 \Rightarrow \left| \frac{a-b}{1-a \bar{b}} \right| = 1$ is only false if $a = b$.

(3) Problem 4, Ahlfors p. 9: Find the conditions under which the equation $az + \bar{b}z + c = 0$ in one complex unknown has exactly one solution, and compute that solution.

Proof: We write $a$, $b$, $c$, $z$ and $\bar{z}$ in familiar form

\[ z = x + iy \]
\[ \bar{z} = x - iy \]
\[ a = a_1 + i a_2 \]
\[ b = b_1 + i b_2 \]
\[ c = c_1 + i c_2 \]

Thus
\[ a z + (a_2 - b_2) y + c_1 \]
\[ + b \overline{z} = (b_1 x + b_2 y) + i (b_2 x - b_1 y) \]
\[ c = c_1 + i c_2 \]

\[ a z + b \overline{z} + c = \left[ (a_1 + b_1) x + (a_2 - b_2) y + c_1 \right] + i \left[ (a_2 + b_2) x + (a_1 - b_1) y + c_2 \right] \]

Thus \( az + b \overline{z} + c = 0 \) if and only if
\[ \begin{cases} 
(a_1 + b_1) x + (a_2 - b_2) y + c_1 = 0 \\
(c_2 + b_2) x + (a_1 - b_1) y + c_2 = 0
\end{cases} \]

By Cramer's rule, the above system of linear equations has unique solution \((x, y)\) if and only if
\[ D = \left| \begin{array}{cc}
(a_1 + b_1) & -(a_2 - b_2) \\
(a_2 + b_2) & (a_1 - b_1)
\end{array} \right| \neq 0 \]

or equivalently \(a_1 - b_1^2 + a_2 - b_2^2 \neq 0\), are equivalently \(|a| \neq |b|\).
Whenever $|a| + |b|$, the equation system of equations has unique solution which is given by

\[
\begin{align*}
\alpha &= \frac{-c_2(c_2 - b_2) c_1}{a_1 - b_1} c_2 = \frac{-c_2(c_2 - b_2) - c_1(a_1 - b_1)}{a_1 b_1^2 - b_1^2 b_2} \\
y &= \frac{a_1 b_1}{a_2 + b_2} \frac{c_1}{c_2} = \frac{a_1}{a_2} \frac{c_2(b_1 + b_2) - c_1(a_2 + b_2)}{a_2 + b_2^2 - b_2^2 b_2},
\end{align*}
\]

you can rewrite all of this to get

\[
\alpha = \frac{b_2 - c_1}{a_1 a_1 - b_1 b_1}
\]

(4) Problem 1, Ahlgors p.11: If $|a| < 1$ and $|b| < 1$, prove that

\[
\left| \frac{a-b}{1-\bar{a}b} \right| < 1.
\]

Proof: If $|a| < 1$ and $|b| < 1$, we know that $|\bar{a}b| = |a||b| = |a||b| < 1$. Thus $1-\bar{a}b \neq 0$. Then we have the equivalence

\[
\left| \frac{a-b}{1-\bar{a}b} \right| < 1 \iff |a-b| < |1-\bar{a}b|
\]

\[
\iff |a-b|^2 < |1-\bar{a}b|^2
\]

\[
\iff |a|^2 - 2Re(\bar{a}b) + |b|^2 < 1 - 2Re(\bar{a}b) + |b|^2
\]

\[
\iff |a|^2 + |b|^2 < 1 + |a|^2 |b|^2
\]

\[
\iff 1 + |a|^2 |b|^2 - |a|^2 - |b|^2 > 0
\]

\[
(1 - |a|^2)(1 - |b|^2) > 0, \text{ which is true.}
\]
Problem 4, Ahlfors p. 11: Show that there are complex numbers \( z \) satisfying 
\[ |z - a| + |z + a| = 2|c| \] if and only if 
\[ |a| \leq |c| . \]
If this condition is fulfilled, what are the smallest and largest values of \( |z| \)?

**Proof.** Suppose that \( z \) satisfies 
\[ 2|c| = |z - a| + |z + a| . \] Then 
\[ 2|c| \geq |(z - a) - (z + a)| = 2|a| \quad \text{(Triangle inequality)} \]
Thus \( |a| \geq |c| \) is a necessary condition for the existence of solutions.

- If \( a = 0 \), then the equation becomes \( 2|z| = 2|c| \). It has solutions, which are all points on the circle centered at the origin with radius \( |c| \). Moreover, 
  \[ \max |z| = \min |z| = |c| . \]

- If \( a \neq 0 \):
  
  Then the set of all \( z \) satisfying the equation is the set of all points \( t \) in the plane such that the sum of the distance from \( z \) to two fixed points 
  \(-a \) and \( a \) is \( 2|c| \). Thus, the set of all such \( z \)'s is an ellipse whose foci are \(-a \) and \( a \), and major
axis $1c|$. We are asked to find the smallest and largest values of $|1c|$, i.e. the smallest and largest distance from the origin to a point on that ellipse. Thus what we are concerned is $|1a|$ and $1c|$ rather than $a$ and $c$. By rotating an angle $-\theta$ where $\theta = \arg(a)$, we can assume $a \in \mathbb{R}, a > 0$.

Put $r = |1c|$. Then $r$ belongs to an ellipse whose semi-major axis is $r$ with foci $-a$ and $a$. Thus the maximum of $|1c|$ is the maximum of $r$,

larger between the semi-minor axis and semi-major axis, and the minimum of $|1c|$ is the smaller one. The two axes are $r$ and $\sqrt{r^2 - a^2}$.

Thus

$$\max |1c| = \max \frac{r}{r - a^2}, \quad r = |1c|$$

$$\min |1c| = \sqrt{r^2 - a^2} = \sqrt{|1c|^2 - |a|^2}$$
Problem 1, Ahlfors p. 17: When does \( az + b \bar{z} + c = 0 \) represent a line?

Proof: We write \( z = x + iy \), \( a = a_1 + ia_2 \), \( b = b_1 + ib_2 \), \( c = c_1 + ic_2 \).

As in (3), the equation \( az + b \bar{z} + c = 0 \) is equivalent to the following system of equations:

\[
\begin{align*}
A_1 x + B_1 y + C_1 &= 0 \\
A_2 x + B_2 y + C_2 &= 0
\end{align*}
\]

where \( A_1 = a_1 + b_1 \), \( B_1 = -a_2 + b_2 \), \( B_2 = a_2 - b_2 \), \( C_1 = a_1 \), \( C_2 = c_2 \).

If (1) represents a line, it must have infinitely many solutions. A necessary condition for that is

\[
D = \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} = A_1 B_2 - A_2 B_1 = 0 \quad (1)
\]

\[
D_x = \begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix} = B_1 C_2 - B_2 C_1 = 0 \quad (2)
\]

\[
D_y = \begin{vmatrix} A_1 & C_1 \\ A_2 & C_2 \end{vmatrix} = A_1 C_2 - A_2 C_1 = 0 \quad (3)
\]

The conditions (1), (2), (3) guarantee that (1) has either no solution or infinitely many solutions.
Given (1), (2) and (3), the system (II) has no solution if and only if either $A_1x + B_1y + C_1 = 0$ or $A_2x + B_2y + C_2 = 0$ has no solution. Indeed, suppose that both of them have solutions; then each one represents a line or the whole plane; if one of them represent the whole plane then the intersection of two solution sets will be at least a line (contradiction!); if both of them represent lines then (1) implies that these lines are either coincide or parallel; if they are coincide then the intersection between them is the line itself (contradiction!); if they are parallel lines, that contradicts $D_2 = D_1y = 0$. Therefore (II) has no solution (given (1), (2), (3)) if $A_1x + B_1y + C_1 = 0$ or $A_2x + B_2y + C_2 = 0$ has no solution, which is equivalent to "$(A_1 = B_1 = 0, C_1 \neq 0)$ or $(A_2 = B_2 = 0, C_2 \neq 0)$".

Now, given (1), (2) and (3), if system (II) have solution set of the entire plane then $C = C = 0$ (by substituting $x = y = 0$); if either one of
the coefficients \( A_1, A_2, B_1, B_2 \) is nonzero (say \( A_1 \) for example) then by substituting \( x = A_1^{-1}, y = 0 \) we see that \( (y, 0) = (A_1^{-1}, 0) \) is not a solution of \( (I) \). Therefore \( (I) \) has the solution set of the entire plane if and only if \( A_1 = A_2 = B_1 = B_2 = C_1 = C_2 = 0 \), which is equivalent to

\[
\begin{align*}
4 + 3y &= 0 \\
2 + y &= 0 \\ 2y &= 0 \\
3 - y &= 0 \\
1 &= 0 \\
0 &= 0
\end{align*}
\]

\( \Rightarrow \) \( a_1 = a_2 = b_1 = b_2 = c_1 = c_2 = 0 \)

\Rightarrow \) \( a = b = c = 0 \)

\( \Rightarrow \) \( a \neq b = c = 0 \)

Therefore, \( (I) \) has solution set of a line if and only if

\[
\begin{align*}
(1) & \quad \text{and not } ("A_1 = a = 0, B_1 \neq 0" \text{ or } "A_2 = b = 0, C_2 \neq 0") \\
(2) & \quad \text{and not } ("a = b = c = 0")
\end{align*}
\]
which is equivalent to

\[
\begin{aligned}
\sum_{(\alpha)} \quad \text{and } kl^2 + 1b_1^2 + 1c_1^2 \neq 0 \quad \text{and } \not\left(\begin{array}{l}
A_1 = b_1 = 0, c_1 \neq 0 \\
\text{and } \not\left(\begin{array}{l}
A_2 = b_2 = 0, c_2 \neq 0
\end{array}\right)\end{array}\right)
\end{aligned}
\]

which is equivalent to

\[
\begin{aligned}
\sum_{(\gamma)} \quad \text{and } |a|^2 + |b|^2 + |c|^2 \neq 0 \quad \text{and } \left(\begin{array}{l}
A_1^2 + b_1^2 \neq 0 \text{ or } c_1 = 0 \\
\text{and } (A_2^2 + b_2^2 \neq 0 \text{ or } c_2 = 0)
\end{array}\right)
\end{aligned}
\]

which is equivalent to

\[
\begin{aligned}
a_1^2 + a_2^2 - b_1^2 - b_2^2 &= 0 \\
-c_2 (a_2 - b_2) - c_1 (a_1 - b_1) &= 0 \\
c_2 (a_1 + b_1) - c_1 (a_2 + b_2) &= 0 \\
a_1^2 + b_1^2 + a_2^2 + b_2^2 + c_1^2 + c_2^2 &= 0 \\
a_1^2 + a_2^2 + b_1^2 + b_2^2 + c_1^2 + c_2^2 \neq 0 \\
a_1^2 + a_2^2 + b_1^2 + b_2^2 + c_1^2 + c_2^2 \neq 0 \text{ or } c_2 = 0 \\
(a_2 + b_2)^2 + (a_1 - b_1)^2 \neq 0 \text{ or } c_2 = 0
\end{aligned}
\]
Problem 5, Ahlfors p. 17: Show that all circles that pass through a and \( \frac{1}{a} \) intersect the circle \( |z| = 1 \) at right angles.

Proof: We will use the following property:

- The line passing through 0 and \( 2z_0 \), and
- the line passing through \( z_1 \) and \( z_2 \), are perpendicular if and only if \( \text{Re} \left( \frac{z_2 - z_1}{z_1} \right) = 0 \) (provided that \( z_1 \neq 0 \)).

Or equivalently \( \text{Re} \left( \frac{z_2}{z_1} \right) = 1 \).

This is a corollary from a more general statement in Ahlfors, p. 17.

We see that \( |a(\frac{1}{a})| = 1 \) thus the modulus of one of \( a \) and \( \frac{1}{a} \) has modulus \( \leq 1 \) and the other has modulus \( > 1 \). That means one is in the unit circle and one is out of the unit circle. Thus any circle passing through them must have a common point with the unit circle. Here we eliminate the case \( |a| = 1 \) because these two circles may not intersect as shown in the figure; and in this case two circles don't meet at right angle.
Thus let $z$ be the center of the circle passing through $a$ and $\frac{1}{a}$.

Let $z$ be an intersection point of the circle and the line from some point on the circle.

Then $|z - \omega| = |a - \omega|$. Thus

$$|z - \omega|^2 = |a - \omega|^2$$

or

$$(z - \omega)(\overline{z} - \overline{\omega}) = (a - \overline{\omega})(\overline{a} - \overline{\omega})$$

or

$$|z|^2 - 2 \text{Re}(\overline{\omega} z) + |\omega|^2 = |a|^2 - 2 \text{Re}(\overline{\omega} a) + |\omega|^2$$

or

$$2 \text{Re}(\overline{\omega} z) = |a|^2 + |z|^2 - 2 \text{Re}(\overline{\omega} a)$$

(*)

We know that $z = \frac{1}{\overline{\omega}} \alpha$ is on this circle. Thus (*) applies for $z = \frac{1}{\overline{\omega}}$:

$$2 \text{Re}(\frac{\overline{\omega}^2}{|\omega|^2}) = |a|^2 + \frac{1}{|\omega|^2} + 2 \text{Re}(\overline{\omega} a)$$

or

$$2 \text{Re}(\frac{1}{|\omega|^2} \overline{\omega} a) = \frac{1}{|\omega|^2} + 2 \text{Re}(\overline{\omega} a) - |a|^2$$

or

$$\frac{1}{|\omega|^2} (2 \text{Re}(\overline{\omega} a) - 1) = 2 \text{Re}(\overline{\omega} a) - |a|^2$$

or

$$\left(\frac{1}{|\omega|^2} - 1\right)(2 \text{Re}(\overline{\omega} a) - 1) = 2 \text{Re}(\overline{\omega} a) - |a|^2$$

Put $\alpha = 2 \text{Re}(\overline{\omega} a)$, $\beta = |a|^2$.

Thus $\alpha = 1 + \beta$. Thus

$$\frac{1}{|\omega|^2} (z - 1) = z - \beta$$

we have $\frac{1}{|\omega|^2} (z - 1) = z - \beta$. Thus

$$\frac{1}{|\omega|^2} (z - 1) = z - \beta$$

Thus

$$\frac{1}{|\omega|^2} (z - 1) = z - \beta$$

Thus
\[ 2 \text{Re}(\overline{z}_0 z) = |z|^2 + 1 \quad (***) \]

Now let \( z \) be an intersection point of the circle centered at \( z_0 \) and the unit circle. Then \( |z| = 1 \), \( \overline{z} = \frac{1}{z} \), and (***) gives

\[ 2 \text{Re}(\overline{z}_0 z) = |z|^2 + 1 + 2 \text{Re}(\overline{z}_0 e) \]

Thus

\[ 1 = \text{Re}(\overline{z}_0 z) = \text{Re}(\frac{\overline{z}_0}{z}) = \text{Re}(\frac{\overline{z}_0}{z}) \]

Thus

\[ \text{Re}(\frac{\overline{z}_0}{z}) \text{Re}(\frac{\overline{z}_0 - z}{z}) = \text{Re}(\frac{\overline{z}_0}{z}) - 1 = 0 \]

Thus \( \overline{z}_0 - z \) is perpendicular to \( z \)

Thus the two circles meet at right angle.

\[ \circlearrowright \]

8) Problem 2, Ahlfors p.28

Verify Cauchy- Riemann's equations for \( f(z) = z^2 \) and \( g(z) = z^3 \).

For \( f(z) = z^2 \)

Put \( z = x + iy \). Then

\[ f(z) = (x + iy)^2 = (x^2 - y^2) + 2xyi \]

\[ = u(x, y) + iv(x, y) \]

where \( u(x, y) = x^2 - y^2 \) and \( v(x, y) = 2xy \).
We see that \( u(x, y) \) and \( v(x, y) \) are continuously differentiable at any order and

\[
\begin{align*}
\frac{\partial u}{\partial x} &= 2x = \frac{\partial v}{\partial y} \\
\frac{\partial u}{\partial y} &= -2y = -\frac{\partial v}{\partial x}
\end{align*}
\]

Thus the CR equations are satisfied. Therefore \( f \) is differentiable on \( \mathbb{C} \).

For \( g(z) = z^3 \)

\[
g(z) = (x + iy)^3 = x^3 + 3x^2iy + 3x(iy)^2 + (iy)^3
\]

\[
= x^3 + 3ix^2y + 3yi^2x + i^3y^3
\]

\[
= (x^3 - 3xy^2) + i(3xy - y^3)
\]

\[
= u(x, y) + iv(x, y)
\]

where \( u(x, y) = x^3 - 3xy^2 \) and \( v(x, y) = 3xy - y^3 \). Thus

\[
\frac{\partial u}{\partial x} = 3x^2 \quad u(x, y) \quad \text{and} \quad v(x, y) \quad \text{are continuously differentiable at any order, and}
\]

\[
\begin{align*}
\frac{\partial u}{\partial x} &= 3x^2 - 3y^2 = \frac{\partial v}{\partial y} \\
\frac{\partial u}{\partial y} &= -6xy = -\frac{\partial v}{\partial x}
\end{align*}
\]

Thus the CR equations are satisfied, and \( g \) is differentiable on \( \mathbb{C} \).
Problem 4, Ahlfors p. 28

Suppose that \( f : \mathbb{C} \rightarrow \mathbb{C} \) is analytic and \( |f(z)| \leq C \text{ (const.)} \).

We will show that \( f \) must be a constant function.

Let \( f(z) = u(x, y) + iv(x, y) \) where \( u \) and \( v \) are real functions.

Then \( C^2 = |f(z)|^2 = (u(x, y))^2 + (v(x, y))^2 \). Suppose that \( f \) is If \( C = 0 \) then \( |f(z)| = 0 \) and \( f(z) = 0 \), which is a constant function. Now we consider the case \( C \neq 0 \). We have \( u(x, y) \) and \( v(x, y) \) are continuously differentiable and they satisfy \( C-R \)'s equations at all points \((x, y) \in \mathbb{R}^2: \)

\[
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}
\]

Taking the derivative with respect to \( x \) at both sides, we get

\[
\frac{\partial u}{\partial x} u + \frac{\partial v}{\partial x} v = 0 \quad \text{(1)}
\]

Similarly, with \( y \), we get

\[
\frac{\partial u}{\partial y} u + \frac{\partial v}{\partial y} v = 0 \quad \text{(2)}
\]

Now applies the \( C-R \) relations, (1) and (2) gives
\[
\begin{align*}
&u \frac{\partial u}{\partial x} - v \frac{\partial u}{\partial y} = 0 \\
&v \frac{\partial u}{\partial x} + u \frac{\partial u}{\partial y} = 0
\end{align*}
\]

We consider \( \frac{\partial u}{\partial x} \) and \( \frac{\partial u}{\partial y} \) above as two unknowns of a linear system of equations. The determinant of this system is

\[
D = \det \begin{pmatrix} u & -v \\ v & u \end{pmatrix} = u^2 + v^2 = c^2 > 0
\]

Thus, the system has unique solution \((\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}) = (0, 0)\).

Now we will show that \( u = u(x, y) \) is a constant function. For each \((x, y) \in \mathbb{R}^2\), we will show that \( u(x, y) = u(0, 0) \). By the Mean Value Theorem, we have

\[
u(x, y) - u(0, y) = \frac{\partial u}{\partial x} (\xi, y) = 0
\]

(\(\xi\) is some point between 0 and \(y\))

Thus \( u(x, y) = u(0, y) \). Moreover,

\[
u(\xi, y) = u(0, 0) = 0
\]

(\(\xi\) is some point between 0 and \(y\))
Thus \( u(x, y) = u(0, 0) + iv(x, y) \in \mathbb{C} \).

Similarly, we have \( \left( \frac{\partial u}{\partial x}, \frac{\partial v}{\partial y} \right) = 0 \) by C-R's equations, and thus \( v(x, y) = u(0, 0) + iv(x, y) \in \mathbb{C} \). Therefore \( f(z) = u(x, y) + iv(x, y) \in \mathbb{C} \).

10 Problem 7, Allfor p.28

Let \( u: \mathbb{R}^2 \rightarrow \mathbb{R} \) be a harmonic function, which is defined to be a function in \( C^\infty(\mathbb{R}^2) \) and \( \Delta u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \).

We will verify that the formal differential equation \( \frac{\partial^2 u}{\partial z \partial \overline{z}} = 0 \).

We have \( \frac{\partial u}{\partial \overline{z}} \) (def) \( \frac{\partial}{\partial t} \left( \frac{\partial u}{\partial \overline{z}} \right) = \frac{\partial}{\partial t} \left( \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} \right) \)

\[ \text{def} \quad \frac{\partial}{\partial t} \left( \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} \right) + i \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} \right) \]

\[ = \frac{\partial^2 u}{\partial x^2} - i \frac{\partial^2 u}{\partial x \partial y} + i \frac{\partial^2 u}{\partial y \partial x} - i^2 \frac{\partial^2 u}{\partial y^2} \]

\[ = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + i \left( \frac{\partial^2 u}{\partial y \partial x} - \frac{\partial^2 u}{\partial x \partial y} \right) = 0 \text{ by Cauchy's Theorem} \]

\[ = 0. \]