1. We'll find the residues at all singularities of the following functions:

\[ f(z) = \frac{z}{z^4 + 1} \]

5/5 The equation \( z^4 + 1 = 0 \) gives four distinct roots: \( z_1 = e^{i\pi/4}, z_2 = e^{i3\pi/4}, z_3 = e^{i5\pi/4}, z_4 = e^{i7\pi/4} \), which are the four 4th roots of -1.

Thus \( z_1, z_2, z_3, z_4 \) are simple poles of \( f(z) \), and

\[ f(z) = \frac{z}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)} \]

The residues of \( f \) at these poles are given by

\[ \text{Res}_{z=z_1} f(z) = \lim_{z \to z_1} (z-z_1)f(z) = \frac{z_1}{(z_1-z_2)(z_1-z_3)(z_1-z_4)} \]

The picture above suggests \( z_1 - z_2 = 2 \text{Re}(z_1) = 2 \frac{\sqrt{2}}{2} = \sqrt{2} \),

\[ z_1 - z_3 = 2i \text{Im}(z_1) = 2i \frac{\sqrt{2}}{2} = i\sqrt{2} \]

Thus \( \text{Res}_{z=z_1} f(z) = \frac{z_1}{\sqrt{2} \cdot 2z_1 \cdot i\sqrt{2}} = \frac{1}{4} \cdot \frac{-1}{4} = \frac{-1}{16} \).

Similarly,
\[
\text{Res}_{z=\bar{z}_2} f(z) = \frac{z_2}{(z_2 - \bar{z}_1)(\bar{z}_2 - \bar{z}_3)(\bar{z}_2 - \bar{z}_q)}, \quad \text{where} \quad z_2 - \bar{z}_1 = \sqrt{2}, \\
z_2 - \bar{z}_3 = i\sqrt{2}, \\
z_2 - \bar{z}_q = 2\bar{z}_2.
\]

Thus,
\[
\text{Res}_{z=\bar{z}_2} f(z) = \frac{z_2}{-\sqrt{2}(i\sqrt{2})(2\bar{z}_2)} = \frac{1}{-4i} = \frac{i}{4}.
\]

Similarly,
\[
\text{Res}_{z=\bar{z}_3} f(z) = \frac{z_3}{(z_3 - \bar{z}_1)(\bar{z}_3 - \bar{z}_2)(\bar{z}_3 - \bar{z}_q)}, \quad \text{where} \quad z_3 - \bar{z}_1 = 2\bar{z}_3, \\
z_3 - \bar{z}_2 = -i\sqrt{2}, \\
z_3 - \bar{z}_q = -\sqrt{2}.
\]

Thus,
\[
\text{Res}_{z=\bar{z}_3} f(z) = \frac{z_3}{(2\bar{z}_3)(-i\sqrt{2})(-\sqrt{2})} = \frac{1}{i4} = -\frac{i}{4}.
\]

Similarly,
\[
\text{Res}_{z=\bar{z}_4} f(z) = \frac{z_4}{(z_4 - \bar{z}_1)(\bar{z}_4 - \bar{z}_3)(\bar{z}_4 - \bar{z}_q)}, \quad \text{where} \quad z_4 - \bar{z}_1 = i\sqrt{2}, \\
z_4 - \bar{z}_3 = \sqrt{2}, \\
z_4 - \bar{z}_q = 4\bar{z}_4.
\]

Thus,
\[
\text{Res}_{z=\bar{z}_4} f(z) = \frac{z_4}{(-i\sqrt{2})(+\sqrt{2})(2\bar{z}_4)} = \frac{1}{-4i} = \frac{i}{4}.
\]

(b)
\[
f(z) = \frac{\sin t}{z(t - \pi)}
\]

The candidates for poles are \( z = 0 \) and \( z = \pi \). Because
\[
\lim_{t \to \pi} (t - z)f(z) = \lim_{t \to \pi} \frac{\sin t}{t(t - \pi)} = 0,
\]

\( z = 0 \) is a removable singularity. We have
\[
\lim_{t \to 0} \frac{\sin t}{t} = \lim_{t \to 0} \frac{\sin t}{t} = \frac{1}{\pi}
\]

Thus, \( z = 0 \) is a pole of order one of \( f \). Then
\[
\text{Res}_{z=0} f(z) = \lim_{t \to 0} tf(t) = \frac{1}{\pi}.
\]
(c) \( f(t) = \frac{e^{it}}{(t-\pi)^2} \)

The potential pole is \( t = \pi \). Since \( \lim_{t \to \pi} (t-\pi)^2 f(t) = \lim_{t \to \pi} t e^{it} = \pi e^{i\pi} = -\pi i \), \( \pi \) is a pole of order two of \( f(t) \). Thus,

\[
\text{Res}_{t=\pi} f(t) = \frac{d}{dt} (t-\pi)^2 f(t) \bigg|_{t=\pi} = \frac{d}{dt} (t e^{it}) \bigg|_{t=\pi} = (e^{i\pi} + i\pi e^{i\pi}) \bigg|_{t=\pi} = e^{i\pi} + i\pi e^{i\pi} = -1 - i\pi.
\]

(d) \( f(t) = \frac{i\pi}{(t^2 + 1)} \)

We have

\[
f(t) = \frac{t^4 + 5t}{t^4 - 1} + 1 = \frac{(t^4 - 1) + (5t + 1)}{t^4 - 1} + 1 = \frac{5t + 1}{t^4 - 1} + 1.
\]

The equation \( t^4 - 1 = 0 \) gives four distinct roots: \( z_1 = 1, z_2 = i, z_3 = -1, z_4 = -i \), which are the 4th roots of \(-1\). Since all of them are not zeros of \( 5t + 1 \), they are simple poles of \( f \). We have

\[
f(t) = \frac{5t + 1}{(t - z_1)(t - z_2)(t - z_3)(t - z_4)} + 2.
\]

As the picture suggests, we have the differences

\[
z_1 - z_2 = 1 - i; \quad z_1 - z_3 = 2; \quad z_1 - z_4 = 1 + i;
\]

\[
z_2 - z_1 = -1 + i; \quad z_2 - z_3 = 2 - i; \quad z_2 - z_4 = 2i;
\]

\[
z_3 - z_2 = -2; \quad z_3 - z_4 = -1 - i;
\]

\[
z_4 - z_3 = -2i; \quad z_4 - z_1 = 1 - i.
\]
Thus, 
\[(z_1 - z_2)(z_1 - z_3)(z_1 - z_4) = \left(1 - i\right)2\left(1 + i\right) = \left(1^2 - i^2\right)2 = 4,
(z_2 - z_1)(z_2 - z_3)(z_2 - z_4) = \left(-1 + i\right)\left(1 + i\right)2i = \left(i^2 - 1\right)2i = -4i,
(z_3 - z_1)(z_3 - z_2)(z_3 - z_4) = -2(-1 - i)(1 + i) = 2\left(i^2 - 1\right) = -4,
(z_4 - z_1)(z_4 - z_2)(z_4 - z_3) = (-1 - i)(-2i)(1 - i) = -2i\left(i^2 - 1^2\right) = 4i.
\]

Then we get
\[
\text{Res}_{z_1} f(z) = \lim_{z \to z_1} (z - z_1) f(z) = \frac{5z_1 + 1}{(z_1 - z_2)(z_1 - z_3)(z_1 - z_4)} = \frac{6}{4} = \frac{3}{2},
\]
\[
\text{Res}_{z_2} f(z) = \frac{5z_2 + 1}{(z_2 - z_1)(z_2 - z_3)(z_2 - z_4)} = \frac{5(-1 + i) + 1}{-4i} = \frac{1}{4} (-5 + i),
\]
\[
\text{Res}_{z_3} f(z) = \frac{5z_3 + 1}{(z_3 - z_1)(z_3 - z_2)(z_3 - z_4)} = \frac{-4}{-4} = 1,
\]
\[
\text{Res}_{z_4} f(z) = \frac{5z_4 + 1}{(z_4 - z_1)(z_4 - z_2)(z_4 - z_3)} = \frac{-8i + 1}{4i} = \frac{1}{4} (-5 - i).
\]

(2) (a) We'll compute the integral over \(C\), the unit circle traversed counterclockwise, of \(f(z) = \frac{e^{\pi z}}{4z^2 + 1}\).

The (simple) poles of \(f(z)\) are the roots of the equation \(4z^2 + 1 = 0\), and the namely \(z_1 = \frac{i}{2}\) and \(z_2 = -\frac{i}{2}\). Since \(z_1\) and \(z_2\) lie in the same connected component on the plane determined by \(C\), we
have \( n(C, z_1) = n(C, z_2) = n(C, 0) = 1 \). Let \( \Omega \) be a disk of radius a bit larger than \( 1 \). Then \( C \) is homologous with respect to \( \Omega \). Thus,

\[
\frac{2\pi i}{c} \int_C f(z) \, dz = \text{Res}_{z = z_1} f(z) + \text{Res}_{z = z_2} f(z).
\]

Since \( z_1 \) and \( z_2 \) are simple poles, we have

\[
\text{Res}_{z = z_1} f(z) = \lim_{z \to z_1} (z - z_1) f(z) = \frac{e^{\pi/2} z_1}{4(z_1 - z_2)} = \frac{e^{\pi/2} z_1}{4(2\pi i)} \quad \text{(note that } z_2 = -z_1) \]

\[
= \frac{e^{\pi/2}}{8}.
\]

Similarly,

\[
\text{Res}_{z = z_2} f(z) = \lim_{z \to z_2} (z - z_2) f(z) = \frac{e^{\pi/2} z_2}{4(z_2 - z_1)} = \frac{e^{\pi/2} z_2}{4(2\pi i)} = \frac{e^{\pi/2}}{8}.
\]

Thus,

\[
\frac{2\pi i}{c} \int_C f(z) \, dz = \frac{e^{\pi/2}}{8} + \frac{e^{\pi/2}}{8} = \frac{e^{\pi/2}}{4} \quad \text{and} \quad \int_C f(z) \, dz = \frac{e^{\pi/2} \pi i}{4} = \frac{\pi e^{\pi/2}}{2i}.
\]

(b) We'll compute \( \int_C f(z) \, dz \) where \( C \) is the same path as above and

\[
f(z) = \frac{e^z}{(z^2 + i - \frac{3}{4})^2}.
\]

The equation \( z^2 + i - \frac{3}{4} = 0 \) gives us two distinct roots \( z_1 = \frac{1}{2} \) and \( z_2 = -\frac{3}{2} \).

None of them vanishes \( e^z \). Thus \( z_1 \) and \( z_2 \) are poles of order two of \( f(z) \).

Let \( \Omega \) be the disk centered at \( 0 \) of radius \( \frac{5}{4} \). Then \( z_2 \) lies outside \( \Omega \), \( C \) is homologous to \( 0 \) with respect to \( \Omega \), and \( n(C, z_1) = 1 \).
Therefore, \( \lim_{t \to \infty} \int_c f(z) \, dz = 2\pi i \text{Res}_{z=\zeta} f(z) \).

Since \( \zeta \) is a pole of order two, we have
\[
\text{Res}_{z=\zeta} f(z) = \frac{d}{dz} \left( (z-\zeta)^2 f(z) \right) \bigg|_{z=\zeta}.
\]

We have \( f(z) = \frac{e^z}{(z-\zeta)^2} \). Thus,
\[
\frac{d}{dz} \left( (z-\zeta)^2 f(z) \right) = \frac{d}{dz} \left( \frac{e^z}{z-\zeta} \right) = \frac{e^z (z-\zeta) - e^z \cdot 2(z-\zeta)}{(z-\zeta)^3} = \frac{e^z (z-\zeta) - 2e^z}{(z-\zeta)^3} = \frac{e^z (z-\zeta)}{(z-\zeta)^3} = \frac{e^z}{z-\zeta}.
\]

Hence, \( \text{Res}_{z=\zeta} f(z) = \frac{e^\zeta \cdot (z-\zeta-2)}{(z-\zeta)^3} = 0 \) because \( z_1 - z_2 = 2 \).

Thus, \( \int_c f(z) \, dz = 0 \). A consequence of this nice accidental result is that \( f \) has an antiderivative in \( \mathbb{D} \).

We will find the following line integral
\[
I = \int_0^{2\pi} \frac{d\theta}{2 + \sin \theta}.
\]

First, we rename \( \theta \) by \( t \) because we wish to consider \( I \) as the definite form of the integral line-integral of some complex function.
\[
I = \int_0^{2\pi} \frac{dt}{2 + \sin t}.
\]

The integral \([0, 2\pi]\) hints us to choose the path \( Y \) as the unit circle,
\[
Y(t) = e^{it}.
\]

With this path, we want to find a complex function \( f(z) \) such
that \( f(\gamma(t)) \gamma'(t) = \frac{1}{2 + \text{sm} t} \), or equivalently \( f(\gamma(t)) \, d(\gamma(t)) = \frac{dt}{2 + \text{sm} t} \).

Put \( z = \gamma(t) = e^{it} \), we have \( \text{sm} t = \frac{z^2 - 1}{2iz} \), and \( dt = \frac{dz}{iz} \). Thus,

\[
\frac{dt}{2 + \text{sm} t} = \frac{dt}{2 + \frac{z^2 - 1}{2iz}} = \frac{dz}{2iz + \frac{z^2 - 1}{2}} = \frac{2}{z^2 + 4iz - 1}.
\]

Thus we choose \( f(z) = \frac{2}{z^2 + 4iz - 1} \). Then

\[
I = \int_{\gamma} f(\gamma(t)) \gamma'(t) \, dt = \int_{\gamma} f(z) \, d\zeta.
\]

The equation \( z^2 + 4iz - 1 = 0 \) has discriminant

\( \Delta = (2i)^2 + 1 = -3 \). Thus, it has two
distinct complex solutions \( z_1 = -2i + \sqrt{3}i \) and \( z_2 = -2i - \sqrt{3}i \).

Thus, we can write \( z^2 + 4iz - 1 = (z - z_1)(z - z_2) \).

Thus \( f \) has only one pole enclosed by \( \gamma \). This pole is \( z_1 \), which is a
simple pole and \( n(\gamma, z_1) = 1 \). Thus,

\[
I = \int_{\gamma} f(z) \, d\zeta = 2\pi i \text{Res}_{z=z_1} f(z) = 2\pi i \left. \text{Res}_{z=z_1} \frac{2}{z^2 + 4iz - 1} \right|_{z=z_1} = 2\pi i \left. \frac{2}{z - z_2} \right|_{z=z_1}.
\]
\[ = \frac{4\pi i}{2 - \sqrt{3}} = \frac{4\pi i}{2\sqrt{3} i} = \frac{2\sqrt{3}}{3} \pi \]

4. **Problem 1, Ahlfors p. 154**

We will find the number of roots of the equation \( z^4 - 2z^5 + 6z^3 - z + 1 = 0 \) in the disk \( |z| < 1 \).

Put \( f(z) = z^4 - 2z^5 + 6z^3 - z + 1 \). Then \( f \) is analytic everywhere in the plane.

Take \( \gamma \) to be the unit circle \( \gamma(t) = e^{i2\pi t} \), for \( 0 \leq t \leq 2\pi \). We want to find the number of zeros of \( f \) enclosed by \( \gamma \). To do so, we put \( P(z) = 6z^3 \). On \( \gamma \), we have

\[
|P(z) - f(z)| = |z^4 - 2z^5 - z + 1| \leq |z|^4 + 2|z|^5 + |z| + 1 = 5 < \frac{6}{|P(z)|}.
\]

Thus, by Rouche's theorem, \( f \) and \( P \) have the same number of zeros in the disk \( |z| < 1 \). Since the equation \( P(z) = 0 \) gives a triple solution \( z = 0 \), this number is 3. Therefore \( f \) has exactly 3 roots in \( |z| < 1 \).

5. **Problem 2, Ahlfors p. 154**

We'll find the number of roots of the equation \( z^4 - 6z + 3 = 0 \) in the annulus \( \text{Ann}(0, 1, 2) = \{ z : 1 < |z| < 2 \} \).

We'll analyze the problem first, then solve it.
Put \( f(z) = z^4 - 6z + 3 \). We want to find the number of zeros of \( f \) in \( \text{Ann}(0,1,2) \).

To apply Rouche's theorem, we need a cycle \( \Gamma \) on which we have some inequality.

The only cycle that we can choose is exactly the boundary of \( \text{Ann}(0,1,2) \), which is composed of two circles. Let \( \gamma_1 \) be the circle of radius 2 directed counterclockwise \( \gamma_1(t) = 2e^{i2\pi t} \) and \( \gamma_2 \) be the circle of radius 1 directed clockwise \( \gamma_2(t) = e^{-i2\pi t} \). Then

\[
\nu(\gamma_2, a) = \begin{cases} 
-1, & a \in \text{disk } |z| < 1 \\
0, & a \in \text{Ann}(0,1,2) \\
0, & a \in |z| > 2
\end{cases}
\]

\[
\nu(\gamma_1, a) = \begin{cases} 
1, & a \in \text{disk } |z| < 1 \\
0, & a \in \text{Ann}(0,1,2) \\
0, & a \in |z| > 2
\end{cases}
\]

Therefore, if we put \( \Gamma = \gamma_1 + \gamma_2 \) then

\[
\nu(\Gamma, a) = \begin{cases} 
0 & \text{if } a \in |z| < 1 \\
1 & \text{if } a \in \text{Ann}(0,1,2) \\
0 & \text{if } a \in |z| > 2
\end{cases}
\]
Therefore, \( R \) is eligible for Rouche's theorem (in other words, the theorem can apply to \( R \)). The domain (region) enclosed by \( R \) is \( \text{Ann}(0,1;2) \) because \( n(R,a) = 1 \) holds only if \( a \in \text{Ann}(0,1;2) \). To use Rouche's theorem, we need to find an analytic function \( g(z) \) such that \( |f(z) - g(z)| < |g(z)| \) on \( R \). Thus, we have 2 inequalities: "\( |f(z) - g(z)| < |g(z)| \) for \( |z| = 1 \)" and "\( |g(z) - g(z)| < |g(z)| \) for \( |z| = 2 \)". The problem is that it's hard to find a function \( g(z) \) satisfying both inequalities. It seems to be not a good idea to compose \( R_1 \) and \( R_2 \) into one cycle \( R \) and then try to apply Rouche's theorem for \( R \).

Instead, we'll treat \( R_1 \) and \( R_2 \) separately. By applying the theorem to \( R_2 \), we'll find the number of zeros of \( f(z) \) in \( |z| < 1 \). By applying it to \( R_1 \), we'll find the number of zeros in \( |z| < 2 \). Then by subtraction, with a caution on \( R_2 \), we get the number of zeros in \( \text{Ann}(0,1;2) \).

Put \( R_1(z) = z^4 \). We have

\[
|f(z) - R_1(z)| = |z^4 - z^2 + 3| \leq 6|z|^2 + 3 = 15 < 16 \quad \text{for} \quad |z|^2
\]

In this case,
Thus, \# of zeros of $f$ in $|z| < 2$ is equal to \# of zeros of $P_1$ in $|z| < 2$, which is 4.

Put $P_2(z) = -6z$. We have

$$|P_2(z)| = |2^6 + 3| = 12|2| + 3 = 4 < 6 = |P_2(z)|$$

Thus we have 2 conclusions:

1. $f$ has no zeros on $P_2$,
2. \# of zeros of $f$ in $|z| < 1$ is equal to \# of zeros of $P_2$ in $|z| < 1$, which is 1.

Thus, \# of zeros of $f(z)$ in $|z| < 1$ is 1. Therefore, $f(z)$ has exactly $4 - 1 = 3$ zeros in the annulus $\text{Ann}(0, 1, 2)$.

(b) Problem 1, Ahlfors p. 161.

We will find poles and residues of the following function

$$f(z) = \frac{1}{(z^2 - 1)^\nu}$$

We have $f(z) = \frac{1}{(z-1)(z+1)^\nu}$. Thus, $f$ has two poles $-1$ and $1$, each of which has order two. We have

$$\text{Res}_{z=1} f(z) = \frac{d}{dz} \left( (z-1)^\nu f(z) \right) \bigg|_{z=1} = \frac{d}{dz} \left( \frac{1}{(z+1)^\nu} \right) \bigg|_{z=1} = \frac{\nu - 2}{(z+1)^{\nu-1}} \bigg|_{z=1} = \frac{-2}{2^3} = -\frac{1}{4}.$$
\[ \text{Res}_{z=-1} f(z) = \frac{d}{dz} \left( (z+1)^2 f(z) \right) \bigg|_{z=-1} = \frac{d}{dz} \left( \frac{1}{(z-1)^2} \right) \bigg|_{z=-1} = \frac{-2}{(z-1)^3} \bigg|_{z=-1} = \frac{-2}{-8} = \frac{1}{4} \]

(c) We'll find all poles and residues of the following function
\[ f(z) = \frac{1}{\sin^2 z} \]

Write \( z = x + iy \). We'll solve the equation \( \sin(x+iy) = 0 \) for \( x \) and \( y \).

\[
0 = \sin(x+iy) = \sin(x) \cosh(y) + i \cos(x) \sinh(y)
\]

where \( \sinh(y) = \frac{e^y - e^{-y}}{2} \) and \( \cosh(y) = \frac{e^y + e^{-y}}{2} \). The equation is equivalent to

\[
\begin{cases}
\sin x \cosh y = 0 \\
\cos x \sinh y = 0
\end{cases} \iff \begin{cases}
\sin x = 0 \\
\sinh y = 0
\end{cases} \iff \begin{cases}
x = n\pi \text{ for } n \in \mathbb{Z} \\
y = 0
\end{cases}
\]

Thus, the poles of \( f(z) \) are \( z_n = n\pi \) for any \( n \in \mathbb{Z} \). Since \( \sin(z-n\pi) = (-1)^n \sin z \), we get \( \sin^2(z-n\pi) = \sin^2 z \) and thus

\[
\lim_{z \to z_n} (z-z_n)^2 f(z) = \lim_{z \to z_n} \left( \frac{1}{\sin^2 z} \right) = \lim_{z \to z_n} \frac{(z-n\pi)^2}{\sin^2(z-n\pi)} = \lim_{\omega \to 0} \frac{\omega^2}{\sin^2 \omega} = 1
\]

Hence \( n\pi \) is a pole of order 2. Then

\[
\text{Res}_{z=n\pi} f(z) = \left. \frac{d}{dz} \left( (z-n\pi)^2 f(z) \right) \right|_{z=n\pi} = \left. \frac{d}{dz} \left( \frac{\sin(z-n\pi)^2}{\sin^3 z} \right) \right|_{z=n\pi}
\]

\[
= \left. \left( (z-n\pi)^2 (z-n\pi)^2 \cosh \sin^3 z + 2 (z-n\pi) \sin \sin^2 z \right) \right|_{z=n\pi}
\]

\[
= \left. \frac{-2 \cos \sin \sin^2 z}{\sin^3 z} \right|_{z=n\pi}
\]
\[
\lim_{x \to \pi} \frac{-2 \cos x \left( x - \pi \right) + 2 \sin^2 x \left( x - \pi \right)^2 + 2 \sin x + 2 \left( x - \pi \right) \cos x}{x \cos x \sin^2 x} \quad (L'Hospital) \\
= \lim_{x \to \pi} \frac{-x \cos x \left( x - \pi \right) + 2 \sin x \left( x - \pi \right)^2}{3 \cos x \sin^2 x} \\
= \lim_{x \to \pi} \frac{-x \sin x - 2 \cos x \left( x - \pi \right) + 2 \sin x + 2 \cos x \left( x - \pi \right) \sin x}{3 \cos x \sin^2 x} \\
= \lim_{x \to \pi} \frac{-x \sin x - 2 \cos x \left( x - \pi \right) + 2 \sin x + 2 \cos x \left( x - \pi \right) \sin x}{3 \sin x + 6 \cos^2 x} \\
= \left( \lim_{x \to \pi} \frac{-x \sin x - 2 \cos x \left( x - \pi \right) + 2 \sin x + 2 \cos x \left( x - \pi \right) \sin x}{3 \sin x + 6 \cos^2 x} \right) \\
\lim_{x \to \pi} \frac{6 \sin x + 2 \left( x - \pi \right) \cos x}{-3 \sin^3 x + 6 \cos^2 x} \\
= 1.0 \\
= 0
\]

(4) We'll find the poles and residues of the following function

\[ f(x) = \frac{1}{x^m (1-x)^n} \]  

\( m, n \) are positive integers.

We see that \( f \) has two poles, \( x = 0 \) of order \( m \) and \( x = 1 \) of order \( n \). We have

\[ \text{Res}_{x=0} f(x) = \lim_{x \to 0} \frac{d^{m-1}}{dx^{m-1}} \frac{1}{(1-x)^n} \quad \text{which is equal to } 1 \text{ if } m = 1. \\
\]

We'll show by induction in \( k \geq 1 \) that

\[ \frac{d^k}{dx^k} \left( \frac{1}{(1-x)^n} \right) = n(n+1) \cdots (n+k-1) \frac{1}{(1-x)^{n+k}} \quad (4) \]

For \( k = 1 \), we have

\[ \frac{d}{dx} \left( \frac{1}{(1-x)^n} \right) = \frac{d}{dx} \left( (1-x)^{-n} \right) = (-n)(1-x)^{-n-1} = n \frac{1}{(1-x)^{n+1}}. \]
Suppose that \( (*) \) is true for \( k \). We'll show that it's true for \( k+1 \).

\[
\frac{d^{k+1}}{dz^{k+1}} \left( \frac{1}{(1-z)^n} \right) = \frac{d}{dz} \left( \frac{d^k}{dz^k} \frac{1}{(1-z)^n} \right) = \frac{d}{dz} \left( \frac{1}{(n+1)\ldots(n+k-1)(1-z)^{n+k}} \right)
\]

\[
= n(n+1)\ldots(n+k-1) \begin{pmatrix} n+k \end{pmatrix} (-1)^{n-k-1} (1-z)^{-n-k-1}
\]

\[
= n(n+1)\ldots(n+k) \begin{pmatrix} n+k \end{pmatrix} (-1)^{n+k-1} (1-z)^{-n-k-1}
\]

Thus, \( (*) \) is also true for \( k+1 \), and hence it's true for every \( k \geq 1 \).

For \( k = m-1 \), in particular, we get

\[
\frac{d^{m-1}}{dz^{m-1}} \frac{1}{(1-z)^n} = \frac{1}{(n+1)\ldots(n+m-2) \begin{pmatrix} n+m-1 \end{pmatrix} (-1)^{n+m-1}}
\]

Thus,

\[
\operatorname{Res}_{z=0} f(z) = \frac{1}{(n-1)!} \n \frac{1}{(n+1)\ldots(n+m-2) \begin{pmatrix} n+m-1 \end{pmatrix} (-1)^{n+m-1}}
\]

\[
= \frac{n(n+1)\ldots(n+m-2)}{(n-1)!} \quad \text{for } m \geq 2
\]

We have

\[
\operatorname{Res}_{z=1} f(z) = \frac{1}{(n-1)!} \left. \frac{d^{n-1}}{dz^{n-1}} \left[ (z-1)^n f(z) \right] \right|_{z=1}
\]

\[
= \frac{n^n}{(n-1)!} \left. \left( \frac{d^{n-1}}{dz^{n-1}} \frac{1}{1-z^n} \right) \right|_{z=1}
\]

Put \( w = 1-ze \), \( (*) \) gives us

\[
(-1)^k \frac{d^k}{dw^k} \frac{1}{w^n} = n(n+1)\ldots(n+k-1) \begin{pmatrix} n+k \end{pmatrix} \frac{1}{w^{n+k}}
\]

Replacing \( n \) by \( m \), and \( k \) by \( m-1 \), we get

\[
(-1)^{m-1} \frac{d^{m-1}}{dw^{m-1}} \frac{1}{w^m} = m(m+1)\ldots(m+n-2) \begin{pmatrix} n \end{pmatrix} \frac{1}{w^{m+n-1}}
\]
Thus,

\[ \text{Res}_{z=1} f(t) = \frac{(-1)^n}{(n-1)!} \left. \frac{(-1)^{n-1} \frac{m(m+1) \ldots (m+n-2)}{t^{m+n-1}}} \right|_{t=1} \]

\[ = \frac{m(m+1) \ldots (m+n-2)}{(n-1)!} \quad \text{for } n \geq 2 \]

For \( n = 1 \), \( \text{Res}_{z=1} f(t) = \lim_{t \to 1} (t-1) f(t) = \lim_{t \to 1} \frac{-1}{2t} = -\frac{1}{2} \).

(1) Problem 3, Ahlfors p. 161

5/5 We'll evaluate the following integrals by the method of residues.

(c) \( I = \int_{-\infty}^{\infty} f(t) \, dt \) where \( f(t) = \frac{t^2 - t + 2}{t^4 + 10t^2 + 9} \).

Since \( f(t) = \frac{t^2 - t + 2}{t^4 + 10t^2 + 9} \) is a rational function on the plane and has no poles on the real line, \( I \) can be computed through the residues of \( f(t) \) but first we should find the poles of \( f(t) \).

\[ t^4 + 10t^2 + 9 = 0 \]
\[ \Leftrightarrow (t^2 + 5)^2 - 4^2 = 0 \]
\[ \Leftrightarrow (t^2 + 1)(t^2 + 9) = 0 \]
\[ \Leftrightarrow t = \pm i \text{ or } t = \pm 3i \]

Therefore \( f(t) \) has four poles \( \pm i, \pm 3i \), each of which are simple. We can write

\[ f(t) = \frac{t^2 - t + 2}{(t-i)(t+i)(t-3i)(t+3i)} \]
Therefore,
\[
\text{Res}_{z=i} f(z) = \frac{z^2 - z + 2}{(z + 3i)(z - 3i)(z + 3i)} \bigg|_{z=i} = \frac{i^2 - i + 2}{2i(-2i)(2i)} = \frac{1 - i}{16i} = \frac{-i + 1}{16} = \frac{1}{16} - \frac{1}{16}i
\]
\[
\text{Res}_{z=-i} f(z) = \frac{z^2 - z + 2}{(z - 3i)(z + 3i)(z + 3i)} \bigg|_{z=-i} = \frac{(-i + 2)}{(-2i)(-4i)(2i)} = \frac{1 + i}{16i} = \frac{1}{16} + \frac{1}{16}i
\]
\[
\text{Res}_{z=3i} f(z) = \frac{z^2 - z + 2}{(z - i)(z + 3i)(z + 3i)} \bigg|_{z=3i} = \frac{9i^2 - 3i + 2}{2i(4i)(6i)} = \frac{-7 - 3i}{-48i} = \frac{-7i + 3}{48} = \frac{3 - 7i}{48}
\]
\[
\text{Res}_{z=-3i} f(z) = \frac{z^2 - z + 2}{(z + 3i)(z + 3i)(z + 3i)} \bigg|_{z=-3i} = \frac{-4i + 3i + 2}{(2i + 3i)(2i + 3i)} = \frac{-7 + 3i}{-48i} = \frac{-7i - 3}{48} = \frac{3 + 7i}{48}
\]

(d) \( I = \int_{0}^{\infty} f(x) \, dx \), where \( f(x) = \frac{1}{(x^2 + a^2)^{\frac{3}{2}}} \). \( a \) is real.

For \( a = 0 \), we have \( f(x) = \frac{1}{x^6} = \frac{1}{x^4} \). Then \( I = \int_{0}^{\infty} \frac{1}{x^4} \, dx = \infty \).

For \( a \neq 0 \), we can assume \( a > 0 \) without loss of generality. Since \( f(x) = \frac{2x}{(x^2 + a^2)^{\frac{3}{2}}} \) is a rational function such that the numerator's degree is two less than the denominator's degree, and \( f(x) \) has no zeros on the real line, \( I \) can be computed via the residue of \( f(x) \). We have
\[
f(x) = \frac{2x}{(x - ia)^{\frac{3}{2}}(x + ia)^{\frac{3}{2}}}
\]

Thus the pole in the upper half plane is \( ia \), which is of order three. Thus
\[
\int_{-\infty}^{\infty} 2\pi i \text{ Res}_{z=i\alpha} f(z) = 2\pi i \left. \frac{1}{\alpha} \frac{d^2}{dz^2} \frac{z^2}{(z+i\alpha)^3} \right|_{z=i\alpha}
\]
\[
= \pi i \left. \frac{d^2}{dt^2} \left( 2t (z+i\alpha)^{-3} \right) \right|_{t=i\alpha}
\]
\[
= \pi i \left. \frac{d}{dt} \left( 2 (z+i\alpha)^{-3} - 3 z^2 (z+i\alpha)^{-4} + 12 z^4 (z+i\alpha)^{-5} \right) \right|_{t=i\alpha}
\]
\[
= \pi i \left( 2 (2i\alpha)^{-3} - 6 (i\alpha)^{-4} - 6 (i\alpha)^{-4} + 12 (i\alpha)^{-5} \right)
\]
\[
= \pi i \left( \frac{2}{(2\alpha)^3} - \frac{6}{(2\alpha)^4} - \frac{6}{(2\alpha)^4} + \frac{12}{(2\alpha)^5} \right)
\]
\[
= \pi i \left( \frac{4}{8\alpha^3} - 3 \frac{3}{8\alpha^4} + \frac{3}{8\alpha^5} \right)
\]
\[
= -\frac{\pi}{\alpha^3} \left( -\frac{1}{8} \right)
\]
\[
= \frac{\pi}{8\alpha^3}
\]

Since \( f \) is an even function, \[ I = \frac{1}{2} \int_{-\infty}^{\infty} f(x) \, dx = \frac{\pi}{16\alpha^3} \]

(e) \[ I = \int_{0}^{\infty} \frac{\cos x}{x^2 + a^2} \, dx \]

If \( a = 0 \) then \[ I = \int_{0}^{\infty} \frac{\cos x}{x^2} \, dx \]. On the interval \( x \in (0, \frac{\pi}{3}) \), we have \( \cos x \geq \frac{1}{2} \). Thus \( \frac{\cos x}{x^2} \geq \frac{1}{2x^2} \) and hence \( \int_{0}^{\infty} \frac{\cos x}{x^2} \, dx = \int_{0}^{\pi/3} \frac{1}{2x^2} \, dx = \infty \).

Therefore, \( I \) diverges.
For $a \neq 0$, we can assume $a > 0$ without loss of generality. We see that

$$J = \int_{-\infty}^{\infty} \frac{e^{inx}}{x^2 + a^2} \, dx = \int_{-\infty}^{\infty} \frac{\cos x}{x^2 + a^2} \, dx + i \int_{-\infty}^{\infty} \frac{\sin x}{x^2 + a^2} \, dx$$

Thus, $\text{Re}(J) = \int_{-\infty}^{\infty} \frac{\cos x}{x^2 + a^2} \, dx = 2 \int_{0}^{\infty} \frac{\cos x}{x^2 + a^2} \, dx$ (since the integrand is even)

$$= 2I$$

Hence, we'll evaluate $J$ instead of $I$, since $J$ is of the form we know how to deal with. The function $f(t) = \frac{1}{t^2 + a^2}$ has only one pole in the upper half plane $t = ia$. Thus,

$$J = 2\pi i \text{ Res}_{t=a} f(t) = 2\pi i \frac{e^{ia}}{2ia} = \frac{\pi}{a} e^{-a}$$

Therefore,

$$I = \frac{1}{2} \text{ Re}(J) = \frac{\pi}{2a} e^{-a} \checkmark$$

(8) $I = \int_{0}^{\infty} \frac{x^{1/3}}{1 + x^2} \, dx$

Put $x = t^3$, then $dx = 3t^2 \, dt$. We get

$$I = \int_{0}^{\infty} \frac{3t}{1 + t^6} \, 3t^2 \, dt = \int_{0}^{\infty} \frac{3t^3}{1 + t^6} \, dt$$

Put $u = t^2$, then $du = 2t \, dt$. We get

$$I = \frac{3}{2} \int_{0}^{\infty} \frac{t^2}{1 + t^6} 2t \, dt = \frac{3}{2} \int_{0}^{\infty} \frac{8u}{1 + u^3} \, du$$

Here we're facing difficulty because what we want is an integral over the whole real line, not the positive half like this. Even the change of variable $v = -u$ doesn't help. We should follow a different approach.
Put $f(z) = \frac{1}{1+z^2}$. Then $f(z)$ is analytic everywhere on the plane except at two poles $z = i$ and $z = -i$. We have $I = \int_0^\infty x^{\frac{1}{3}} f(x) \, dx$.

To relate $I$ to an integral over $(-\infty, \infty)$, we should find an analytic extension of $x^{\frac{1}{3}}$ on the upper half plane and the negative real line. By definition,

$$x^{\frac{1}{3}} := \exp \left( \frac{1}{3} \log x \right)$$

Thus, we need an extension of $\log x$ to the upper half plane and the negative real line. $\log x$ is not a good choice because it's not analytic on the negative real line. We choose $g(z) = \log(-x) + i\pi$.

Then $g(z)$ coincides $\log x$ on the positive real ray, and $g(z)$ is analytic on the domain we want.

Then $h(z) = \exp \left( \frac{1}{3} g(z) \right)$ is the extension of $x^{\frac{1}{3}}$ that we want. For $y > 0$, we have $g(-y) = \log(iy) + i\frac{\pi}{2} = \log y + i\sigma = \log(-iy) + i\frac{\pi}{2} + i\pi = g(y) + i\pi$.

Thus, $h(-y) = \exp \left( \frac{1}{3} g(-y) \right) = \exp \left( \frac{1}{3} g(y) + \frac{i\pi}{3} \right) = \exp \left( \frac{i\pi}{3} \right) \exp \left( \frac{1}{3} g(y) \right)$

$$= \exp \left( \frac{i\pi}{3} \right) h(y).$$

Using the fact that $f(x)$ is even, we get

$$\int_0^\infty h(x) f(x) \, dx \xrightarrow{\text{even}} \int_0^\infty h(-y) f(y) \, dy = \exp \left( \frac{i\pi}{3} \right) \int_0^\infty h(y) f(y) \, dy = \exp \left( \frac{i\pi}{3} \right) I.$$
Thus, \[ \text{p.v.} \int_{-\infty}^{0} h(z) f(z) \, dz = \int_{0}^{\infty} h(x) f(x) \, dx + \int_{-\infty}^{0} h(z) f(z) \, dz \]

\[ = I + \exp \left( \frac{i\pi}{3} \right) I \]

\[ = I \left( 1 + \exp \left( \frac{i\pi}{3} \right) \right) \quad (1) \]

Put \( C_L = [-L, 0] + [0, L] - C_0 + C \) as in the figure. We have

\[ |h(z)| = \left| \exp \left( \frac{1}{2} \log (-iz) \right) \right| \]

\[ = \left| \exp \left( \frac{1}{2} \log (-i\ell) \right) \right| \]

\[ = (\text{Im} \, \ell)^{1/2} \leq |L|^{1/3} \]

Thus, \[ \left| \int_{C_L} h(z) f(z) \, dz \right| \leq \int_{C_L} |h(z)||f(z)| \, dz \leq \int_{C_L} \left| \frac{1}{1 + z^2} \right| \, |dz| \]

\[ \leq \int_{C_L} \frac{|z|^{1/3}}{|z|^{2} - 1} \, |dz| = \frac{L^{1/3}}{L^2 - 1} \pi L \rightarrow 0 \]

as \( L \rightarrow \infty \).

On the other hand, \[ \left| \int_{-C_0} h(z) f(z) \, dz \right| \leq \int_{-C_0} \left| \frac{1}{1 + z^2} \right| \, |dz| \]

\[ \leq \int_{-C_0} \frac{1}{1 - |z|^2} \, |dz| = \frac{\delta^{1/3}}{1 - \delta^{2/3}} \pi \delta \rightarrow 0 \text{ as } \delta \rightarrow 0 \]

Thus, \[ \lim_{L \to 0, \delta \to 0} \int_{C_L} h(z) f(z) \, dz = \text{p.v.} \int_{-\infty}^{\infty} h(z) f(z) \, dz \xrightarrow{(1)} I \left( 1 + \exp \left( \frac{i\pi}{3} \right) \right) \quad (2) \]

Since \( h(z) \) has no pole in the the upperhalf plane and \( f(z) \) has only one (simple) pole \( z = i \) in the upperhalf plane, we get
\[
\lim_{\beta \to 0} \int_{-\infty}^{\infty} h(x) f(x) \, dx = 2\pi i \int_{x=i}^{x=i} h(x) \, f(x) \frac{1}{2\pi i} \bigg|_{x = i} = 2\pi i \cdot h(i) \frac{1}{2i} = \pi \exp \left( \frac{1}{2} g(i) \right) \\
= \pi \exp \left( \frac{1}{3} \left( \log 1 + i \frac{\pi}{2} \right) \right) = \pi \exp \left( i \frac{\pi}{6} \right)
\]

Then, together with (2), we get
\[
\pi \exp \left( i \frac{\pi}{6} \right) = \frac{1}{1 + \exp \left( i \frac{\pi}{3} \right)}
\]

Thus
\[
I = \pi \frac{\exp \left( i \frac{\pi}{6} \right)}{1 + \exp \left( i \frac{\pi}{3} \right)} = \pi \frac{\frac{1}{2} + i\frac{\sqrt{3}}{2}}{1 + \frac{1}{2} + i\frac{\sqrt{3}}{2}} = \pi \frac{1}{3 + i\sqrt{3}} = \frac{\pi}{\sqrt{3}}.
\]

Therefore,
\[
\int_{0}^{\infty} \frac{x^{1/3}}{1 + x^2} \, dx = \frac{\pi}{3 \sqrt{3}}.
\]

(h) \[
I = \int_{0}^{\infty} \frac{\log x}{1 + x^2} \, dx
\]

The idea is similar to the above problem. We want to extend \( \log x \) to an analytic function on the upper half plane and the negative real ray. Thus, we choose \( g(x) = \log(-ix) + i\pi/2 \) as above.

For \( y > 0 \), we have \( g(-y) = g(y) + i\pi \).

Put \( f(x) = \frac{1}{1 + x^2} \). Using the fact that \( f \) is even, we get
\[
\int_{-\infty}^{\infty} g(x) f(x) \, dx = \int_{0}^{\infty} g(-y) f(y) \, dy \\
= \int_{0}^{\infty} (g(y) + ia) f(y) \, dy \\
= \int_{0}^{\infty} g(y) f(y) \, dy + i\pi \int_{0}^{\infty} \frac{dy}{1 + y^2} \\
= I + i\pi \left. \arctan(y) \right|_{y=0}^{y=\infty} \\
= I + i \frac{\pi^2}{2}
\]

Thus, \( \text{pr. v.} \int_{-\infty}^{\infty} g(x) f(x) \, dx = \int_{0}^{\infty} g(x) f(x) \, dx + \int_{-\infty}^{0} g(x) f(x) \, dx \)

\[
= I + I + i \frac{\pi^2}{2} = 2I + i \frac{\pi^2}{2} \quad (\dagger)
\]

Put \( \gamma_8 = [-L, -\delta] - C_8 + [S, L] + C_L \) as in the figure. We get have

\[
|g(z)| = \left| \log(-iz) + i\frac{\pi}{2} \right| \\
= \left| \log(-ix + y) + i\frac{\pi}{2} \right| \\
= \left| \log|x| + (\text{Im}(z) + \frac{\pi}{2}) \right| \\
\leq \sqrt{(|\log|x|^2 + (\arg(-iz) + \frac{\pi}{2})^2} \\
\leq \sqrt{(\log|x|)^2 + \frac{\pi^2}{4}}
\]
We have
\[
\left| \int_{\mathbb{C}_L} g(z) f(z) \, dz \right| \leq \int_{\mathbb{C}_L} |g(z)||f(z)| \, |dz| \leq \int_{\mathbb{C}_L} \sqrt{\log|z|^2 + \frac{g^2}{4}} \cdot \frac{1}{|z|^2 + 1} \, |dz|
\]
\[
\leq \int_{\mathbb{C}_L} \sqrt{\log L + \frac{g^2}{4}} \cdot \frac{1}{L^2 - 1} \, |dz|
\]
\[
= 2\pi \int_{\mathbb{C}_L} \sqrt{\log L + \frac{g^2}{4}} \cdot \frac{1}{L^2 - 1} \, |dz|
\]
\[
\to 0 \quad \text{as} \quad L \to \infty \quad \text{because the logarithmic function creeps to infinity slower than the identity function.}\]

On the other hand,
\[
\left| \int_{-\mathbb{C}_L} g(z) f(z) \, dz \right| \leq \int_{-\mathbb{C}_L} |g(z)||f(z)| \, |dz| \leq \int_{-\mathbb{C}_L} \sqrt{\log|z|^2 + \frac{g^2}{4}} \cdot \frac{1}{|z|^2 + \varepsilon} \, |dz|
\]
\[
\leq \int_{-\mathbb{C}_L} \sqrt{\log \delta + \frac{g^2}{4}} \cdot \frac{1}{\delta^2 - \varepsilon} \, |dz|
\]
\[
= \frac{\pi}{\delta} \sqrt{\log \delta + \frac{g^2}{4}} \cdot \frac{1}{\delta^2 - \varepsilon}
\]
\[
= \frac{\pi}{1 - \varepsilon} \sqrt{\log \delta + \frac{g^2}{4}}
\]
\[
\to 0 \quad \text{as} \quad \delta \to 0 \quad \text{because} \quad \log \delta \to 0 \quad \text{as} \quad \delta \to 0.
\]

Thus,
\[
\lim_{L \to \infty} \int_{\mathbb{C}_L} g(z) f(z) \, dz = \text{p.v.} \int_{-\infty}^{\infty} g(z) f(z) \, dz = \frac{1}{2} \pi + i \frac{\pi^2}{2} \quad (2)
\]

Since \( g(z) \) has no zero in the upper half plane and \( f(z) \) has only one pole in this domain, namely \( z = i \), which is simple, we have
\lim_{L \to \infty} \int_{L+i\delta}^{L-i\delta} g(z) f(z) \, dz = 2\pi i \text{Res}_{z=c} g(z) f(z) = 2\pi i \frac{g(c)}{2i} = \pi g(c) = \pi \left( \log 1 + i \frac{\pi}{2} \right) = i \frac{\pi^2}{2}

Together with (2), we get

\[ \frac{\pi^2}{2} \mp 2I + i \frac{\pi^2}{2}. \text{ Then } I = 0 \]

Therefore

\[ \int_{0}^{\infty} \frac{\log x}{1+x^2} \, dx = 0. \]