Problem 1, Ahlfors, p. 232

Let \( D \) be a symmetric domain with respect to the real line, and \( \alpha \in \mathbb{R} \). Let \( f : D \rightarrow \mathbb{D} \) be the conformal map such that \( f(\alpha) = 0 \) and \( f'(\alpha) > 0 \).

Put \( g : D \rightarrow \mathbb{D}, \quad g(z) = \frac{1}{f(z)}. \) Then \( g \) is well-defined because both \( D \) and \( \mathbb{D} \) are symmetric with respect to the real axis. For any \( \alpha \in \mathbb{R} \), we have

\[
\lim_{z \to \alpha} \frac{g(z) - g(\alpha)}{z - \alpha} = \lim_{z \to \alpha} \frac{\frac{1}{f(z)} - \frac{1}{f(\alpha)}}{z - \alpha} = \lim_{z \to \alpha} \left( \frac{f(z) - f(\alpha)}{z - \alpha} \right) = f'(\alpha).
\]

Thus \( g \) is analytic and \( g'(z) = \frac{1}{f'(z)} \) for all \( z \in \mathbb{R} \).

Moreover, \( g \) is bijective because \( g(z) = w \)

\[
(\Rightarrow) \quad f(w) = z
\]

\[
(\Rightarrow) \quad f(\bar{w}) = \bar{z}
\]

\[
(\Rightarrow) \quad \bar{z} = f^{-1}(\bar{w})
\]

\[
(\Rightarrow) \quad z = f^{-1}(w)
\]

Thus \( g \) is a conformal map. Moreover,

\[
g(\alpha) = \frac{1}{f(\alpha)} = \frac{1}{f(\bar{\alpha})} = \bar{0} = 0,
\]

\[
g'(\alpha) = \frac{1}{f'(\alpha)} = \frac{1}{f'(\bar{\alpha})} = f'(\alpha) > 0.
\]
By the uniqueness, \( g = f \). Therefore, \( f(\overline{z}) = f(z) \) for all \( z \in \mathbb{C} \).

(2) Suppose that \( \Omega \) is a domain symmetric with respect to the point \( z_0 \).

Let \( f: \Omega \to D \) be the conformal map such that \( f(z_0) = 0 \), \( f'(z_0) > 0 \). We define \( g: \Omega \to D \), \( g(z) = -f(2z_0 - z) \). Then \( g \) is well-defined because \( \Omega \) is symmetric with respect to \( z_0 \) and \( D \) is symmetric with respect to 0. We have \( g'(z) = f'(2z_0 - z) \) by the chain rule. Thus \( g \) is analytic.

In addition, \( g \) is bijective because \( f \) is bijective. Thus \( g \) is a conformal map. Moreover, \( g(z_0) = -f(2z_0 - z_0) = -f(z_0) = 0 \), \( g'(z_0) = f'(2z_0 - z_0) = f'(z_0) > 0 \).

By the uniqueness, \( g = f \). Therefore, \( -f(2z_0 - z) = f(z) \) for all \( z \in \Omega \).

As a comment, if \( \Omega \) is symmetric with respect to the real line then \( f \) maps symmetric points to symmetric points (wrt to the real line); if \( \Omega \) is symmetric with respect to \( z_0 \) then \( f \) maps symmetric points wrt \( z_0 \) to symmetric points wrt the origin.

(3) Problem 1, additional problem.

Put \( U = \{ z \in \mathbb{C} : \Re(z) > 0 \} \) and consider a holomorphic map \( f: U \to D \) with \( f(1) = 0 \). We'll look for the maximum possible the value of \( |f(z)| \).
We know that the map \( \phi_1 : \Delta \to \mathbb{D} \),
\[
\phi_1(z) = i \frac{1-z}{1+z}
\]
is a conformal map. By rotating \( 90^\circ \) about the origin, we can map \( \mathbb{D} \) onto \( U \). Thus \( \phi_2 : \mathbb{D} \to U \), is a conformal map. Thus \( \phi = \phi_2 \circ \phi_1 : \Delta \to U \) is a conformal map. We have
\[
\phi(z) = \phi_2(\phi_1(z)) = -i\phi_1(z) = \frac{1-z}{1+z}.
\]
Put \( g = \phi \circ \phi_1 : \Delta \to \Delta \). We have \( g(0) = \phi_1(\phi_0) = \phi(1) = 0 \). Therefore by Schwarz's lemma, \( |g(z)| \leq |z| \) for all \( z \in \Delta \). Thus, \( |f(\phi(z))| \leq |z| \) for all \( z \in \Delta \). If \( z \) is replaced by \( \phi^{-1}(z) \) then \( |f(z)| \leq |\phi^{-1}(z)| \) for all \( z \in U \). We have \( \phi^{-1}(z) = \phi(z) = \frac{1-z}{1+z} \). Thus,
\[
|f(z)| \leq \left| \frac{1-z}{1+z} \right|, \text{ for all } z \in U.
\]
Thus \( |f(z)| \leq \frac{1}{3} \). The equality happens when \( g(z) = z \), for which \( f = \phi^{-1} \). Thus in that case,
\[
f(z) = \frac{1-z}{1+z} \quad \forall z \in U.
\]

4. Put \( F = \{ f : \Delta \to \mathbb{C} \text{ holomorphic, } f(0)=0, \text{ diam } (f(\Delta)) \leq 2 \frac{3}{4} \} \), where
\[
\text{diam } (f(\Delta)) := \sup_{z, w \in \Delta} |f(z)-f(w)|.
\]
We see that \( F \) is a family of holomorphic functions on \( \Delta \). To show that \( F \) is a normal family, by Montel's theorem,
it suffices to show that $F$ is locally bounded. We have

$$|f(z)| = |f(z) - f(0)| \leq \text{diam}(f(D)) \leq 2 \quad \forall z \in D, \forall f \in F.$$  

Thus $F$ is uniformly bounded in $D$, and hence it's a normal family.

To show that $F$ is compact, we take a sequence $(f_n)$ in $F$ and show that it has a convergent subsequence to some element in $F$. Since $F$ is normal, there exists a subsequence $(f_{n_k})$ of $(f_n)$ that converges to some $f : D \to \mathbb{C}$ uniformly on every compact subset of $D$. By Weierstrass's theorem, $f$ is holomorphic.

We have

$$f(0) = \lim_{k \to \infty} f_{n_k}(0) = \lim_{k \to \infty} 0 = 0.$$  

Moreover,

$$|f(z_1) - f(z_2)| = \lim_{k \to \infty} \left| f_{n_k}(z_1) - f_{n_k}(z_2) \right| \leq \text{diam}(f_{n_k}(D)) \leq 2,$$

Thus $\text{diam}(f(D)) \leq 2$. We have showed that $f \in F$. Therefore, $F$ is compact.

5. We consider the family $F = \{f : D \to \mathbb{C} \text{ holomorphic, } f(0) = 0 \}$. Then $F$ is a normal family because $|f(z)| < 1$ for all $z \in D, f \in F$. Consider $f \in F$ and put $f_n(z) = f \circ f \circ \cdots \circ f(z)$. Suppose that $\lim_{n \to \infty} f_n(z) = h(z)$ for all $z \in D$.

We'll show that either $h(z) = z$ for all $z$, or $h(z) = 0$ for all $z$.

Because $f_n$ is holomorphic and $f_n(0) = f \circ f \circ \cdots \circ f(0) = 0$, $f_n \in EF$. Since $F$ is normal, there exists a subsequence $(f_{n_k})$ which converges uniformly
to $h$ on every compact set. Thus $h: D \to \mathbb{C}$ is a holomorphic map by Weierstrass's theorem. Moreover, $h(0) = \lim_{n \to \infty} f_n(0) = 0$ and $|h(2)| = \lim_{n \to \infty} |f_n(2)| \leq 1$. By the Maximum Principle, $|h(z)| < 1$ for all $z \in D$. Thus $h: D \to D$ and $h(0) = 0$. We will consider two cases $|f(0)| = 1$ and $|f'(0)| < 1$.

$|f'(0)| = 1$.

Applying the Schwartz's Lemma for $f(t)$, there is a constant $c$ with $|c| = 1$ and $f(z) = cz$ for all $z \in D$. Then $f_n(z) = c^n z$.

For $z \neq 0$, the sequence $c^n z^3_n$ converges if and only if the sequence $\{c^n\}$ converges. In that case, $\lim_{n \to \infty} (c^{n+1} - c^n) = 0$. Consequently, $|c^{n+1} - c^n| \to 0$.

we have $|c^{n+1} - c^n| = |c^n| |c-1| = |c-1|$. Thus $c = 1$. Then $f(z) = z$, and $f_n(z) = z$, and hence $h(z) = z$ for all $z \in D$.

$|f'(0)| < 1$.

Because $|f'(0)|$ is continuous around zero, there exists $\varepsilon > 0$ and $\delta < 1$ such that $|f'(z)| \leq \delta$ for all $z \in \Phi = B(0, \varepsilon)$.

Applying the Schwartz's Lemma to $f$, we have $|f(z)| \leq |z|$ for all $z \in D$. Thus $f(z) \in \Phi$ for all $z \in D$. Thus, we can view $f$ as a
map from $\mathbb{C}$ to $\mathbb{C}$. Then $f_n$ also map $\mathbb{C}$ to $\mathbb{C}$ for any $n \in \mathbb{N}$.

For $z \in \mathbb{C}$, we have $f_{n+1}(z) = f(f_n(z))$. Then by the chain rule,

$$f_{n+1}'(z) = f'(f_n(z)) f_n'(z).$$

Thus

$$|f_{n+1}'(z)| \leq |f'(f_n(z))| |f_n'(z)| \leq M |f_n'(z)|.$$

Repeatedly using this result, we get

$$|f_n'(z)| \leq M |f_n'(z)| \leq M^2 |f_n'(z)| \leq \cdots \leq M^n |f'(z)| \leq p^n |f'(z)|.$$ 

Thus $|f_n'(z)| \leq p^n$ for all $z \in \mathbb{C}$. By Weierstrass's theorem, we also know that $h'(z) = \lim_{n \to \infty} f_n'(z)$. Thus $h'(z) = 0$ for all $z \in \mathbb{C}$. Since $h'$ is holomorphic, and $\mathbb{D}$ is connected, $h'(z) = 0$ for all $z \in \mathbb{D}$. Thus $h$ is a constant function. Since $h(0) = 0$, $h$ must be the zero function.