Problem 3, Ahlfors, p. 238

First, we look for a conformal map from the unit disk to a familiar strip, namely \( \Omega = \{ z \in \mathbb{C} : -\pi < \text{Im}(z) < \pi \} \). We have the following chain of conformal maps:

\[
\begin{align*}
D & \xrightarrow{w \mapsto i \frac{1-w}{1+w}} \Omega \xrightarrow{z \mapsto z^2} \Omega \xrightarrow{z \mapsto -z = \eta} \Omega \xrightarrow{\eta \mapsto \log \eta} \Omega \xrightarrow{\zeta \mapsto \log \zeta} \mathbb{C} \setminus \{ \zeta \in \mathbb{C} : \text{Re}(\zeta) > 0, \text{Im}(\zeta) = 0 \} \xrightarrow{\zeta \mapsto \frac{1-w}{1+w}} D
\end{align*}
\]

Here we are taking the principal branch of logarithm, namely
\[
\log z := \log |z| + i \arg(z), \text{ where } -\pi < \arg z < \pi.
\]

Thus a conformal map from \( D \) to \( \Omega \) is
\[
\eta = \log \zeta = \log (-z) = \log (-e^{i \theta}) = \log \left(-i \frac{1-w}{1+w} \right) = \log \left(\frac{1-w}{1+w} \right).
\]

We put \( F : D \to \Omega \),
\[
F(w) = \log \left(\frac{1-w}{1+w} \right).
\]

Now we try to present \( F(w) \) in form of Schwarz-Christoffel formula.
the chain rule, we have \[ F'(w) = \frac{4}{w^2-1} = \frac{4}{(w-1)(w+1)} \]

Since \( F \) is analytic in the unit disk and \( F(0) = 0 \), we have

\[
F(w) = \int_0^w F'(z) \, dz = 4 \int_0^w \frac{dz}{(z-1)(z+1)} \tag{\*}
\]

The formula (\*) is now in form of Schwarz–Christoffel formula. The generic conformal map from \( D \) to a polygon whose outer angles are \( \beta_1, \ldots, \beta_n \) is

\[
G(w) = C \int_0^w \frac{dz}{(z-w_1)^{\beta_1} \cdots (z-w_n)^{\beta_n}} + C'
\]

Therefore (\*) is a degenerate form with \( n = 2 \), \( w_1 = -1, w_2 = 1 \), \( \beta_1 = \beta_2 = 1 \).

The image of \( D \) under \( F \) is a "polygon" with only two vertices at \( \infty \) (therefore the inner angles are \( \alpha_1 = 0 \), \( \alpha_2 = 0 \)). Two "line segments" connecting these vertices should be parallel because they have two intercepts, both at infinity. Thus the polygon looks like an infinite strip.

Next, we'll look for a conformal map from the unit disk to a familiar infinite half strip \( \Omega = \{ z \in \mathbb{C} : \text{Im} \, z > 0, -\frac{\pi}{2} < \text{Re} \, z < \frac{\pi}{2} \} \).
We put $H: \mathbb{D} \to \mathbb{H}$, $H(\omega) = i \frac{1-\omega}{1+\omega}$.

$G: \mathbb{H} \to \partial \mathbb{D}$, $G(\xi) = \int_0^\xi \frac{dt}{(t^2-1)^{1/2}}$ (principal branch of logarithm).

Then we know (from the lecture note) that $H$ and $G$ are conformal maps. Thus $F = G \circ H: \mathbb{D} \to \partial \mathbb{D}$ is a conformal map. For any $\omega \in \mathbb{H}$, we have

$-\pi < \text{arg}(1-\omega) < 0$, and $0 < \text{arg}(1+\omega) < \pi$. Thus $-\pi < \text{arg}(1-\omega) + \text{arg}(1+\omega) < \pi$.

Thus $\log(1-\omega) = \log(1-\omega) + \log(1+\omega)$. Thus $(1-\omega)^{1/2} = (1-\omega)^{1/2} (1+\omega)^{1/2}$.

Because $-\pi < \text{arg}(1-\omega) < 0$, we have

$(1-\omega)^{1/2} = -i (\omega-1)^{1/2}$

Then $(1-\omega)^{1/2} = -i (\omega-1)^{1/2} (\omega+1)^{1/2}$. Thus,

\[ G(\xi) = i \int_0^\xi \frac{dt}{(t-1)^{1/2} (t+1)^{1/2}} \]

we have

\[ H(\omega) = \frac{-2i}{(1+\omega)^2} \]

$H(\omega) - 1 = (i+1)(i-\omega)/(1+\omega),$

$H(\omega) + 1 = (i-1)(-i-\omega)/(1+\omega).$

Then

\[ F'(\omega) = G'(H(\omega)) H'(\omega) = \frac{i}{(H(\omega)-1)^{1/2} (H(\omega)+1)^{1/2}} \frac{-2i}{(1+\omega)^2} \]
\[
\frac{2}{[w^2 + (1-w)^2]^2 \left[\frac{1}{2}((1+i)(1-w))\right]^{1/2} (1+w)} = \frac{2}{-i\sqrt{2} (w-i)^{1/2} (w+i)^{1/2} (w+1)}
\]

Thus,

\[
F(w) = \int_0^\infty F(\xi) d\xi + F(0) = i\sqrt{2} \int_0^\infty \frac{d\xi}{(\xi-i)^{1/2} (\xi+i)^{1/2} (\xi+1)} + F(0)
\]

This expression is now in the form of Schwartz Christoffel's formula with \(n=3\), \(w_1 = i\), \(w_2 = -i\), \(w_3 = -1\); \(\beta_1 = \frac{1}{2}\), \(\beta_2 = \frac{1}{2}\), \(\beta_3 = 1\). We can think of the image as a triangle with three vertices at \(\frac{\pi}{2}\), \(-\frac{\pi}{2}\) and \(\infty\). Because \(\alpha_1 = \alpha_2 = \frac{1}{2}\), this triangle has two right angles.

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**Problem 5, Allgor's p. 238**

We'll show that the map \(F(w) = \int_0^\infty (1-\xi)^{-2/\alpha} d\xi\) maps \(|w|<1\) onto the interior of a regular polygon of \(n\) sides.

First, we notice that for two different branches of the logarithm, the results for \((1-\xi)^{-2/\alpha}\) differ by a constant factor \(C\) with \(|C|<1\) for all \(\xi \in \mathbb{D}\). Thus \(F\) should be defined up to a constant factor \(C\) with...
\[ |c| = 1. \] Then the images of \( F \) corresponding to different choices of branches of logarithm differ by rotations about the origin. Such rotations, however, do not change the shape of \( F(1D) \). If \( F(1D) \) is the interior of a regular polygon of \( n \) sides, then \( F(1D) \) will still be the interior of another regular polygon of \( n \) sides when a different branch of logarithm is chosen. Thus we can assume that what we want. It cannot be assumed!

\[ (1-w)^{-2/n} = \exp \left( -\frac{2}{n} \log (1-w) \right) \quad \forall w \in 1D \]

with the choice of \( \log \) such that \[ -\frac{\pi}{2} \leq \arg z < \frac{3\pi}{2} \].

For each \( w \in 1D \), we have

\[ \log (1-w) = i\pi + \log (w-1). \]

Thus

\[ (1-w)^{-2/n} = \exp \left( -\frac{2\pi i}{n} + \frac{2}{n} \log (w-1) \right) \]

\[ = C_1 (w-1)^{-2/n} \]

where \( C_1 = \exp \left( -\frac{2\pi i}{n} \right) \).

Thus,

\[ F(w) = \int_0^w (1-x^n)^{-2/n} \, dx = C_1 \int_0^w (x^n-1)^{-2/n} \, dx. \]

Since the constant factor \( C_1 \) just plays a role as a rotation, it doesn't matter. In fact, we have seen this. Thus we can ignore \( C_1 \) by considering \( C_1 \) as \( 1 \). Now let \( w_k = \exp \left( \frac{2\pi i k}{n} \right) \), \( k=1,2,...,n \) be...
the $n$th roots of unity. Then $z^n - 1 = (z - w_1) \cdots (z - w_n)$ and

$$F(w) = \int_{w} \frac{dz}{[(z-w_1) \cdots (z-w_n)]^{2/n}}$$

For each $k = 1, \ldots, n$, we define $\log_k$ to be a single valued function $\log_k z = \log |z| + i \arg z$ with

$$\frac{2\pi (k-1)}{n} < \arg \frac{\partial}{\partial z} < \frac{2\pi k}{n}$$

On this branch of logarithm, we can speak of $w[(z-w_k)^{2/n}]$. Consider the following map $\eta: \mathbb{D} \to \mathbb{C}$, $\eta(z) = \log [(z-w_1) \cdots (z-w_n)] - \sum_{k=1}^{n} \log_k (z-w_k)$.

For each value $z \in \mathbb{D}$, $\eta(z)$ is an integer multiple of $2\pi i$. Since $\eta$ is analytic, it must be a constant function. Thus there is $c_2$ such that $\eta(z) = c_2$ for all $z \in \mathbb{D}$. Then

$$-\frac{2}{n} \eta(z) = -\frac{2c_2}{n} = -\frac{2}{n} \log [(z-w_1) \cdots (z-w_n)] + \frac{2}{n} \sum_{k=1}^{n} \log_k (z-w_k)$$

Now take the exponential both sides:

$$[(z-w_1) \cdots (z-w_n)]^{-\frac{2}{n}} = (z-w_1)^{-\frac{2}{n}} \cdots (z-w_n)^{-\frac{2}{n}} \exp \left(-\frac{2c_2}{n}\right)$$

Thus

$$F(w) = \exp \left(-\frac{2c_2}{n}\right) \int_{w} \frac{dz}{[z-w_1]^{2/n} \cdots [z-w_n]^{2/n}}$$

Since $c_2$ is an integer multiple of $2\pi i$, $c_2$ has modulus $1$ and accounts for a rotation about the origin. As we mentioned, this factor doesn't matter. Thus, we should ignore it (by considering $c_2$ as if $1$). Then we
have \[ F(w) = \int_0^w \frac{ds}{(s-w_1)^{2/n} \cdots (s-w_n)^{2/n}}. \]

Let \( \Omega \) be the interior of the regular polygon whose vertices are \( w_1, \ldots, w_n \). By Riemann mapping theorem, there is a conformal map \( G \) from \( \mathbb{D} \) to \( \Omega \). Because \( \Omega \) has inner angles equal to \( \frac{(n-2)\pi}{n} \), then \( \beta_1 = \beta_2 = \cdots = \beta_n = \frac{2}{n} \).

By Schwarz-Christoffel's theorem, \( G \) has the form

\[ G(w) = C \int_0^w \frac{ds}{(s-w_1)^{2/n} \cdots (s-w_n)^{2/n}} + C' = CF(w) + C'. \]

Thus the image of \( G \) is obtained by a rotation, a dilation followed by a translation of the image of \( F \). Since \( G(\mathbb{D}) = \Omega \), \( F(\mathbb{D}) \) is also a \( p \)-regular polygon of \( n \)-sides.

(3) Problem 6, Ahlfors, p. 238.

Put \( \mathcal{S} = \{ z = x + iy : x, y > 0, \text{Im}(z) < 1 \} \). We'll look for a conformal map \( f : \mathbb{H} \to \mathcal{S} \). On the real line, we consider four consecutive segments \( A = (-\infty, -1) \), \( B = (-1, 0) \), \( C = (0, 1) \) and \( D = (1, \infty) \). On the boundary of \( \mathcal{S} \), we also name four segments.
\[ A' = \{ z = 0 + iy : y > 0 \}, \]
\[ B' = \{ z = 1 + iy : y > 0 \}, \]
\[ C' = \{ z = x + i : x > 1 \}, \]
\[ D' = \{ z = x + 0i : x > 0 \}. \]

We want to find a conformal map \( f : \mathbb{H} \to \mathbb{D} \) that maps \( A \) to \( A' \), \( B \) to \( B' \), \( C \) to \( C' \) and \( D \) to \( D' \) with the direction specified in the picture. The argument function \( \arg : \mathbb{C} \setminus \{0\} \to \mathbb{R}/2\pi\mathbb{Z} \) is additive. We have
\[
\arg f'(z) = \arg \frac{df}{dz} = \arg df - \arg dz.
\]

Thus we have

\[
\arg f'(z) = \begin{cases} 
\frac{\pi}{2} \pmod{2\pi} & \text{on } A \\
-\frac{\pi}{2} \pmod{2\pi} & \text{on } B \\
0 \pmod{2\pi} & \text{on } C \\
\pi \pmod{2\pi} & \text{on } D 
\end{cases}
\]

This hints us to put \( f(z) = \alpha e^{\pi i \theta} (z+1)^{\eta_1} z^{\eta_2} (z-1)^{\eta_3} \) where \( \alpha > 0 \), \( \theta \in \mathbb{R} \), \( -1 < \eta_1, \eta_2, \eta_3 \leq 1 \) and the power functions are defined on the principal branch of the logarithm, i.e. \( -\pi < \arg(z) < \pi \).
We see that $\forall z \in \mathbb{H}$, $0 < \text{arg} z$, \text{arg}(z-1), \text{arg}(z+1) < \pi$. Moreover,
\[
\text{arg}(z+1) \equiv \eta_1 \text{arg}(z+1) \pmod{2\pi},
\]
\[
\text{arg} z \equiv \eta_2 \text{arg} z \pmod{2\pi},
\]
\[
\text{arg}(z-1) \equiv \eta_3 \text{arg}(z-1) \pmod{2\pi}.
\]
Thus we get $\text{arg} f(z) = \pi \theta + \eta_1 \text{arg}(z+1) + \eta_2 \text{arg} z + \eta_3 \text{arg}(z-1) \pmod{2\pi}$.

On $A$, we have
\[
\frac{\pi}{2} = \pi \theta + \eta_1 \pi + \eta_2 \pi + \eta_3 \pi \pmod{2\pi},
\]
on $B$, we have
\[
-\frac{\pi}{2} = \pi \theta + \pi + \eta_2 \pi + \eta_3 \pi \pmod{2\pi},
\]
on $C$, we have
\[
0 = \pi \theta + \pi \eta_3 \pmod{2\pi},
\]
on $D$, we have
\[
\pi = \pi \theta \pmod{2\pi}.
\]
Thus
\[
\begin{align*}
\frac{\pi}{2} & = \theta + \eta_1 + \eta_2 + \eta_3 \pmod{2} \\
-\frac{\pi}{2} & = \theta + \eta_2 + \eta_3 \pmod{2} \\
0 & = \theta + \eta_3 \pmod{2} \\
1 & = \theta \pmod{2}
\end{align*}
\]
Since $f(z)$ blows up to infinity when $z$ approaches $\pm 1$ on the real line, we want $f'(z)$ does to. Thus we want $\eta_1, \eta_3 < 0$. Then we can choose
$\theta = 1$, $\eta_1 = \eta_3 = -\frac{1}{2}$, $\eta_2 = -\frac{1}{2}$. Thus
\[
f'(z) = \frac{-2}{12(z+1)(z-1)} \text{ for } z \in \mathbb{H}.
\]
The function $z \mapsto \sqrt{z}$ can extend analytically to a domain covering $\mathbb{R} \setminus \{0\}$. 


by defining \( \sqrt{z} := \exp \left( \frac{1}{2} \log |z| + \frac{1}{2} i \arg z \right) \), where \( -\frac{\pi}{4} < \arg z < \frac{5\pi}{4} \).

This definition agrees with the function \( \sqrt{z} \) from the beginning on \( \mathbb{H} \).

The A function \( f \) we are looking for should look like

\[
f(z) = -\alpha \int_0^z \frac{d\xi}{\sqrt{\xi (\xi-1)(\xi+1)}} + \beta \quad (*)
\]

Note that with this definition, \( f(z) \) is continuous at 0. We'll adjust \( \alpha, \beta \) such that \( \mathbb{H} \) is mapped exactly to \( \mathbb{R} \). We have

\[
\int_0^z \frac{d\xi}{\sqrt{\xi (\xi-1)(\xi+1)}} = \frac{1}{2} \int_0^z \frac{d\xi}{\sqrt{\xi (\xi-1)}} - \frac{1}{2} \int_0^z \frac{d\xi}{\sqrt{\xi (\xi+1)}}
\]

Put \( u = \sqrt{\xi} \) for each \( \xi \in \mathbb{H} \). Then \( u \) belongs to the first quadrant \( \mathbb{H} \).

Thus \( 0 < \arg(u-1), \arg(u+1) < \pi \). Then

\[
\int_0^z \frac{d\xi}{\sqrt{\xi (\xi-1)(\xi+1)}} = \frac{1}{2} \int_0^{\sqrt{z}} \frac{2u \, du}{u(u^2-1)} - \frac{1}{2} \int_0^{\sqrt{z}} \frac{2u \, du}{u(u^2+1)}
\]

\[
= \frac{1}{2} \int_0^{\sqrt{z}} \frac{du}{u-1} - \frac{1}{2} \int_0^{\sqrt{z}} \frac{du}{u+1} - \int_0^{\sqrt{z}} \frac{du}{u^2+1}
\]

\[
= \frac{1}{2} \left( \log(\sqrt{z}-1) - \pi i \right) - \frac{1}{2} \left( \log(\sqrt{z}+1) \right) - \int_0^{\sqrt{z}} \frac{du}{u^2+1}
\]

Therefore,

\[
f(z) = -\frac{\alpha}{2} \left[ \log(\sqrt{z}-1) - \log(\sqrt{z}+1) - \frac{\pi i}{2} - \int_0^{\sqrt{z}} \frac{du}{u^2+1} \right] + \beta
\]

\((***)\)
On $B$  

Put \( f_1(x) = f(-x) = -\alpha \int_0^x \frac{dt}{\sqrt{t} (t^2 - 1)} + \beta \) for \( 0 < x < 1 \).

By the substitution \( t = -u \), we get

\[
    f_1(x) = -\alpha \int_0^x \frac{-dt}{\sqrt{-u} (u^2 - 1)} + \beta = i \alpha \int_0^x \frac{dt}{\sqrt{-t} (1-t^2)} + \beta.
\]

Thus \( f_1(0) = \beta \). Because we want \( f(0) = 0 \), we choose \( \beta = 0 \). Because the map \( x \in (0,1) \mapsto \int_0^x \frac{dt}{\sqrt{t} (t^2 - 1)} \) is an increasing function which goes to infinity as \( x \to 1^- \), the image of \( f_1 \) is exactly \( B' \). Thus \( f \) maps \( B \) to \( B' \) and respects the desired direction on \( B' \).

On \( C \)  

\[
    f(x) = -\alpha \int_0^x \frac{dt}{\sqrt{t} (t^2 - 1)} \quad \forall x \in (0,1)
\]

\[
    \left\{ \begin{array}{l}
    \emptyset \text{ at } x = 0 \\
    \text{increasing as } x \to 1^- \\
    \to \infty \text{ as } x \to 1^-
    \end{array} \right.
\]

Thus \( f \) maps \( C \) to \( C' \) with the desired direction on \( C' \).

On \( A \)  

Put \( f_2(x) = f(-x) \) for all \( x > 0 \). By (**) we have

\[
    f(x) = -\frac{\alpha}{2} \left[ \log \left( \sqrt{x} - 1 \right) - \log \left( \sqrt{-x} + 1 \right) - \pi i - 2 \int_0^{\sqrt{x}} \frac{du}{u^2 + 1} \right]
\]

\[
    = -\frac{\alpha}{2} \left[ \log \left( \sqrt{i} - 1 \right) - \log \left( \sqrt{i} + 1 \right) - \pi i - 2 \int_0^{\sqrt{i}} \frac{du}{u^2 + 1} \right] \tag{1}
\]

Put \( \theta = \arg (\sqrt{i} - 1) \). Then \( \frac{\pi}{2} < \theta < \pi \) as shown in the picture.
Then \( \arg(i^n+1) = -\theta + \pi \in (0, \pi) \). Thus,
\[
\arg \frac{i^n-1}{i^n+1} = \arg(i^n-1) - \arg(i^n+1) = 2\theta - \pi
\]

Since \( \left| \frac{i^n-1}{i^n+1} \right| = 1 \), we have
\[
\log \frac{i^n-1}{i^n+1} = i \arg \frac{i^n-1}{i^n+1} = c(2\theta - \pi) \quad (2)
\]

We have
\[
\int_0^{i^n} \frac{du}{u^2+1} = \int_0^{i^n} \frac{du}{(1-iu)(1+iu)}
\]
\[
= -\frac{1}{2} \left[ \int_0^{i^n} \frac{du}{iu-1} + \frac{1}{2} \int_0^{i^n} \frac{du}{iu+1} \right]
\]
\[
= \frac{i}{2} \left[ \log (u+c) \right]_0^{i^n} - \frac{i}{2} \left[ \log (u-c) \right]_0^{i^n}
\]
\[
= \frac{i}{2} \left[ \log (i^n+c) - \log c \right] - \frac{i}{2} \left[ \log (i^n-c) - \log c \right]
\]
\[
= \frac{i}{2} \log (i^n+c) - \frac{i}{2} \left[ \log (i^n-1) + i\pi \right]
\]
\[
= \frac{i}{2} \log \frac{\sqrt{x+1}}{\sqrt{x-1}} + \frac{\pi}{2} \quad (3)
\]

Substituting (2) and (3) into (1), we get
\[
f_x(a) = -\frac{a}{2} \left( i(2\theta - \pi) - \pi - i \log \frac{\sqrt{x+1}}{\sqrt{x-1}} - \pi \right)
\]
Thus
\[ f_2(x) = \frac{x \pi}{2} + \frac{ix}{2} \left( \log \frac{\sqrt{x} + 1}{\sqrt{x} - 1} + 2 \pi - 2 \theta \right) \]

\[ \text{Strictly increasing in } x \]
\[ \rightarrow \infty \text{ as } x \rightarrow 1^+ \]
\[ \rightarrow 2 \pi - 2 \cdot \frac{\pi}{2} = \pi \text{ as } x \rightarrow \infty \]

Because we want the image of \( f \) to be exactly \( D' \), we must have
\[ \frac{x \pi}{2} = 1 \implies x = \frac{2}{\pi} . \]

Thus we obtain two forms of \( f(x) \), one from (*) and one from (**):

\[ f(z) = -\frac{2}{\pi} \int_0^z \frac{d \xi}{\sqrt{\xi} (\sqrt{\xi} - 1)(\sqrt{\xi} + 1)} \quad (4) \]

\[ f(z) = -\frac{4}{\pi} \int \log (\sqrt{\xi} - 1) - \log (\sqrt{\xi} + 1) - \pi i - 2 \int_0^{\sqrt{\xi}} \frac{du}{u^2 + 1} \] \quad (5)

On \( D \) we have \( \forall x > 1, \)
\[ f(x) = -\frac{1}{\pi} \left( \log \frac{\sqrt{x} - 1}{\sqrt{x} + 1} - \pi i - 2 \int_0^{\sqrt{x}} \frac{du}{u^2 + 1} \right) \]
\[ = \frac{1}{\pi} \left( 2 \arctan(\sqrt{x}) - \log \frac{\sqrt{x} - 1}{\sqrt{x} + 1} \right) + i \]
\[ \text{derivative } = \frac{-2}{\sqrt{x}(x-1)} < 0 , \]
\[ \rightarrow \infty \text{ as } x \rightarrow 1^+, \]
\[ \rightarrow \pi \text{ as } x \rightarrow \infty \]

Thus \( f \) maps \( D \) to \( D' \) and respects the direction on \( D' \), as desired.
With the computation
\[2 \int_0^{\sqrt{2}} \frac{du}{u^2 + 1} = i \int_0^{\sqrt{2}} \frac{du}{u + i} - i \int_0^{\sqrt{2}} \frac{du}{u - i}
\]
\[= i \left[ \log(u + i) \right]_0^{\sqrt{2}} - i \left[ \log(u - i) \right]_0^{\sqrt{2}}
\]
\[= i (\log(\sqrt{2} + i) - \log(i)) - i (\log(\sqrt{2} - i) - \log(-i))
\]
\[= + i \log(\sqrt{2} + i) - i \log(\sqrt{2} - i) + \pi
\]
\[= i \log \left( \frac{\sqrt{2} + i}{\sqrt{2} - i} \right) + \pi,
\]

The form (5) can be reduced as
\[f(z) = -\frac{1}{\pi} \left( \log \frac{\sqrt{2} - i}{\sqrt{2} + 1} - i \log \frac{\sqrt{2} + i}{\sqrt{2} - i} \right) + (1 + i).
\]

4. Let \(A_1, A_2, \ldots, A_n\) be real numbers and \(\Omega = \mathbb{C} \setminus \bigcup_{k=1}^{n} \{A_k + iy : y \leq 0\} \). We'll show that \(\Omega\) is simply connected. Let \(\Gamma = \{z \in \mathbb{C} : \text{Im} z = y = 1\}\). Then \(\Gamma\) is just a line and thus simply connected. To show that \(\Omega\) is simply connected, we will show that \(\Omega\) is a retract of \(\Omega\). We define a function \(H\) as follows:

\[H : \Omega \times [0, 1] \to \Omega
\]

\[H(z, t) = (z, (1-t)y + t)
\]

\[= z + it(1-y)
\]
To check if $H$ is well-defined, we take $z = A_t + i(y)$ ($y > 0$) and check if $H(z, t) \in \mathbb{R}$ for all $t \in [0, 1]$. We have $H(z, t) = A_t + i[(1-t)y + t]$ and $(1-t)y + t \geq 0$ for all $t \in [0, 1]$. Thus $H$ is well-defined. We see that $H$ is also continuous and $H(z, 0) = z \quad \forall z \in \mathbb{R}$, $H(z, 1) = x + i \forall x \in \mathbb{R}$. Thus $H$ is a deformation retraction of $\Omega$ onto $\Gamma$. Thus $\Pi_1(\Omega, z) \simeq \Pi_1(\Gamma, z) \simeq \mathbb{Z}$. Therefore $\Omega$ is simply connected. While the argument works, it is easier for everyone to stay in the $C$-analytic realm.

5. Define a function $u : \mathbb{R}^2 \setminus \{(0, 0)\} \to \mathbb{R}$, $u(x, y) = \text{Re} \left( \frac{(x + y + i) \cdot y}{x^2 + (1-y)^2} \right)$.

Because $u$ is the real part of an analytic function on $\mathbb{C} \setminus \{0\}$, $u$ is harmonic in $\mathbb{R}^2 \setminus \{(0, 0)\}$. In particular, $u$ is harmonic in the unit disk. We have

$$u(x, y) = \text{Re} \left( \frac{x + (y+1)i}{x + (1-y)i} \right) = \text{Re} \left[ \frac{(x+(y+1)i)(-x-(1-y)i)}{x^2 + (1-y)^2} \right]$$

$$= \text{Re} \left[ \frac{-x^2 + (1-y)^2}{x^2 + (1-y)^2} + i \frac{-2x}{x^2 + (1-y)^2} \right]$$

$$= \frac{1 - (x^2 + y^2)}{x^2 + (1-y)^2}$$

For every $(x, y)$ on the unit disk except $(0, 1)$, we have $u(x, y) = 0$. At the point $(0, 1)$, $u$ was defined to be 0 by the problem.