Problem 1, Ahlfors, p. 274

Let \( f(z) \) be an even elliptic function whose the set of periods is \( \Lambda = \{m \omega_1 + n \omega_2 : m, n \in \mathbb{Z} \} \).

Put \( \wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda^*} \left( \frac{1}{(z+\omega)^2} - \frac{1}{\omega^2} \right), \quad \forall z \in \mathbb{C} \setminus \Lambda. \)

Then \( \wp(z) \) is the Weierstrass \( \wp \)-function corresponding to the lattice \( \Lambda \).

(a) Suppose that 0 is not a zero nor a pole of \( f(z) \). We'll show that there are \( a_1, \ldots, a_n, b_1, \ldots, b_n, C \in \mathbb{C} \) such that

\[
    f(z) = C \frac{(\wp(z) - \wp(a_1)) \cdots (\wp(z) - \wp(a_n))}{(\wp(z) - \wp(b_1)) \cdots (\wp(z) - \wp(b_n))} \quad \forall z \in \mathbb{C} \setminus \Lambda.
\]

We will consider a special fundamental parallelogram. Put

\[
    \wp_1(t) = \frac{\omega_1}{2} + t \frac{\omega_1}{2}, \quad 0 < t < 1
\]

\[
    \wp_2(t) = \frac{\omega_1}{2} + t \frac{\omega_2}{2}, \quad 0 < t < 1
\]

\[
    \wp_3(t) = -\frac{\omega_2}{2} - t \frac{\omega_1}{2}, \quad 0 < t < 1
\]

\[
    \wp_4(t) = -\frac{\omega_1}{2} - t \frac{\omega_2}{2}, \quad 0 < t < 1
\]

\[
    P_{int} = \left\{ z = \frac{\omega_1}{2} t + \frac{\omega_2}{2} s : -1 < t, s < 1 \right\}
\]
Then we get a fundamental parallelogram of the lattice
\[
P = P \cap \{ \frac{w_1}{2}, \frac{w_2}{2}, \frac{w_1 + w_2}{2} \}
\]
Put
\[
p^+ = P \cap \{ \frac{w_1}{2}, \frac{w_2}{2}, \frac{w_1 + w_2}{2}, s(-1 \leq t \leq 1, s > 0) \}
\]
\[
p^- = P \setminus (p^+ \cup \{0\})
\]
Thus \(P\) is a disjoint union \(P = p^+ \cup p^- \cup \{0\} \).

**Editors Note:**

These are the half periods in \(P\). We have an important property:
\[
a \in p^+ \setminus A \implies -a \in p^-
\]

For each \(c \in P \setminus \{0\}\), \(P(t) - P(c)\) is an elliptic function with periods \(w_1\) and \(w_2\). Because \(t = 0\) is the only pole, which is a double pole, of \(P(t) - P(c)\) in \(P\), it is of order 2. If \(c \in A\) then \(\pm t\) are zeros of \(P(t) - P(c)\) in \(P\). These are the only zeros of \(P(t) - P(c)\) in \(P\) because \#(zeros) = #(poles). If \(c \in A\), we know that \(P(c) = 0\) and \(c\) is a single zero of \(\bar{g}\). Thus \(c\) is a double zero of \(P(t) - P(c)\). It is the only zero of \(P(t) - P(c)\) in \(P\) because \#(zeros) = #(poles).

Because 0 is not a zero nor a pole of \(f(t)\), all zeros and poles of \(f(t)\) lie in \(p^+ \cup p^-\). We see that \(a \in p^+ \setminus A\) is a zero of \(f(t)\) iff \(-a \in p^-\).
is also a zero. By the previous paragraph, ±a are the only zeros of \( P(z) - P(a) \). Thus \( \frac{f(z)}{P(z) - P(a)} \) is an elliptic function whose orders of zeros at ±a decrease by 1 compared to that of \( f(z) \). This function gives rise to a double zero at \( z = 0 \). Other than 0 and ±a, \( \frac{f(z)}{P(z) - P(a)} \) and \( f(z) \) have the same zeros with the same orders.

Let \( a_1, \ldots, a_r \) be all zeros of \( f(z) \) in \( P^+ \backslash A \) (enumerated with multiplicity). Then \( -a_1, \ldots, -a_r \) are all zeros of \( f(z) \) in \( P^- \) (enumerated with multiplicity). The function

\[
f_1(z) = \frac{f(z)}{(P(z) - P(a_1)) \cdots (P(z) - P(a_r))}
\]

thus has no zero in \( P \backslash (A \cup \{0\}) \). Note that \( f_1(z) \) is also elliptic.

We do similarly to poles: \( b \in P^+ \backslash A \) is a pole of \( f(z) \) iff \( -b \in P^- \) is also a pole. Also, ±b are the only zeros of \( P(z) - P(b) \). Thus \( f(z)(P(z) - P(b)) \) is elliptic function whose orders of poles at ±b decrease by 1 compared to that of \( f(z) \). This function gives rise to a double pole at \( z = 0 \). Other than 0 and ±b, \( f(z)(P(z) - P(b)) \) and \( f_2(b) \) have the same poles with the same orders.

Let \( b_1, \ldots, b_s \) be all poles of \( f(z) \) in \( P^+ \backslash A \) (counted with multiplicity).
Then \(-b_1, \ldots, -b_5\) are all poles of \(f(z)\) in \(P^\times\) (counted with multiplicity).

Then the function
\[
 f_2(z) = f(z) \frac{(P(z) - P(b_1)) \ldots (P(z) - P(b_5))}{(P(z) - P(a_1)) \ldots (P(z) - P(a_5))}
\]
is elliptic and has no pole nor zero in \(P \setminus (A \cup \Theta)\).

Now we will show that the order of poles and zeros at the half period must be even (which can be 0). We write the Laurent series of \(f(z)\) about \(\omega/2, \omega_1/2, (\omega_1 + \omega)/2\) as follow.

\[
 f(z) = c_1 \left(z - \frac{\omega_1}{2}\right)^{b_1} + \ldots
\]
\[
 f(z) = c_2 \left(z - \frac{\omega_1}{2}\right)^{b_2} + \ldots
\]
\[
 f(z) = c_3 \left(z - \frac{\omega_1 + \omega}{2}\right)^{b_3} + \ldots
\]

If \(b_i > 0\) then we have a zero; \(b_i < 0\) a pole; \(b_i = 0\) neither a zero nor a pole. Since \(f(z)\) is elliptic, \(\#(\text{zeros}) = \#(\text{poles})\) in \(P\). Thus,

\[
 0 = \#(\text{zeros}) - \#(\text{poles}) = \#(\text{zeros in } P \setminus (A \cup \Theta)) - \#(\text{poles in } P \setminus (A \cup \Theta)) + \#(\text{zeros in } A) - \#(\text{poles in } A)
\]

\[
 = 2r - 2s + (b_1 + b_2 + b_3)
\]

Thus \(b_1 + b_2 + b_3\) is even. (1)

Also, we know that \(\sum \text{zeros} - \sum \text{poles} \in \mathbb{N}\). Thus we must have
\[ \sum \text{zeros} - \sum \text{poles} = \sum (\text{zeros in } P \setminus (A \cup \{0\})) - \sum (\text{poles in } P \setminus (A \cup \{0\})) \\
+ \sum (\text{zeros in } A) - \sum (\text{poles in } A) \]

\[ = h_1 \frac{w_1}{2} + h_2 \frac{w_2}{2} + h_3 \frac{w_1 + w_2}{2} \]

Here we use the fact that \(\pm a_1, \ldots, \pm a_r\) are all zeros of \(f(x)\) in \(P \setminus (A \cup \{0\})\). That is why the second is zero. The same is for the poles \(\pm b_1, \ldots, \pm b_s\). Thus, \[\frac{w_1}{2} (h_1 + h_3) + \frac{w_2}{2} (h_2 + h_3) \leq \Lambda.\]

Thus \(h_1 + h_3\) and \(h_2 + h_3\) are even. \(\text{(2)}\)

From (1) and (2), we conclude that \(h_1, h_2, h_3\) are even. Thus the poles and zeros of \(f(x)\) in \(A\) are of even orders. Let \(a_{r+1}, \ldots, a_{r+k}\) be all zeros of \(f(x)\) in \(A\) which are enumerated with half of the multiplicity. E.g., if \(a\) is a zero of order 4 then \(a\) will be listed only twice. Then the since \(P(z) - P(a_{r+1})\) has a double zero at \(a_{r+1}\),

the function \[\frac{f(x)}{(P(z) - P(a_{r+1})) \cdots (P(z) - P(a_{r+k}))}\]

has no zeros in \(A\).

Let \(b_{s+1}, \ldots, b_{s+t}\) be all poles of \(f(x)\) in \(A\) which are enumerated with half of the multiplicity. Then

\[\frac{f(x)}{(P(z) - P(b_{s+1})) \cdots (P(z) - P(b_{s+t}))}\]

\[\frac{(P(z) - P(a_{r+1})) \cdots (P(z) - P(a_{r+k}))}{(P(z) - P(b_{s+1})) \cdots (P(z) - P(b_{s+t}))}\]
has no zero nor pole in $A$. Then the function
\[
g(z) = f(z) \frac{(P(z) - P(b_1)) \cdots (P(z) - P(b_k))}{(P(z) - P(a_1)) \cdots (P(z) - P(a_l))}
\]
is elliptic and has no pole nor zero in $P \setminus \{\rho\}$. We have
\[
h_1 + h_2 + h_3 = \#(\text{zeros in } A) - \#(\text{poles in } A) = 2k - 2l
\]
By (1), we get $0 = 2r - 2s + (2k - 2l)$. Thus $r + k = s + l (= \text{some } n)$.
Thus,
\[
g(z) = f(z) \frac{(P(z) - P(b_1)) \cdots (P(z) - P(b_k))}{(P(z) - P(a_1)) \cdots (P(z) - P(a_l))}
\]
Because the numerator and denominator have a pole at $0$ of order $2n$, the
function has no pole nor zero at $0$. Thus $g(z)$ has no pole nor
zero at every point in $P$. Thus $g(z)$ is holomorphic and doubly periodic
in $C$. Thus $g(z) \equiv C$ - which is a nonzero constant. Thus,
\[
f(z) = C \frac{(P(z) - P(a_1)) \cdots (P(z) - P(a_l))}{(P(z) - P(b_1)) \cdots (P(z) - P(b_k))}
\]
(b) Now suppose that $f(z)$ can have zeros or poles at $z = 0$. We note
the $f(z)$ is even. Thus the Laurent series of $f(z)$ about $z = 0$ contains
only the even powers of $z$. We are interested in the first term only.
\[
f(z) = az^{2k} + \cdots
\]
$P(z)$ also has Laurent series about $z = 0$, namely $P(z) = z^2 + \cdots$
Thus
\[
f(z) P(z)^k = (az^{2k} + \cdots)(z^2 + \cdots)^k = (az^{2k} + \cdots)(z^{-2k} + \cdots) = az^{2k} + \cdots
\]
This is an elliptic function with no poles nor zeros at \( z = 0 \).
Moreover, \( f(z) \Phi(z)^4 \) is even because \( f(z) \) and \( \Phi(z)^4 \) are even.
By part (a), there are complex numbers \( a_1, \ldots, a_n, b_1, \ldots, b_n, C \in \mathbb{C} \) such that
\[
f(z) \Phi(z)^4 = C \frac{(\Phi(z) - \Phi(a_1)) \cdots (\Phi(z) - \Phi(a_n))}{(\Phi(z) - \Phi(b_1)) \cdots (\Phi(z) - \Phi(b_n))}.
\]
Therefore,
\[
f(z) = C \Phi(z)^{-4} \frac{(\Phi(z) - \Phi(a_1)) \cdots (\Phi(z) - \Phi(a_n))}{(\Phi(z) - \Phi(b_1)) \cdots (\Phi(z) - \Phi(b_n))}.
\]

(2) Problem 2, Ahlfors, p. 275.
Let \( f(z) \) be an elliptic function whose Weierstrass\( \Lambda \) is
\[
\Lambda = \{ m \omega_1 + n \omega_2 \mid m, n \in \mathbb{Z} \}.
\]
For any \( z_1, z_2 \in \mathbb{C} \), we shall write \( z_1 \equiv z_2 \pmod{\Lambda} \) if \( z_1 - z_2 \in \Lambda \).
The function \( \sigma : \mathbb{C} \rightarrow \mathbb{C} \) defined in Ahlfors, p. 274 is
\[
\sigma(z) = \prod_{w \in \Lambda^*} \left( 1 - \frac{z}{w} \right) \exp \left( \frac{z}{w} + \frac{1}{2} \left( \frac{z}{w} \right)^2 \right).
\]
Accordingly, \( \sigma \) is an entire function whose zeros are exactly at the lattice points. These zeros are of order 1. Moreover, there are complex constants \( \eta_1 \) and \( \eta_2 \) such that
\[
\sigma(z + \omega_1) = - \sigma(z) \exp \left( \eta_1 z + \frac{\eta_1 \omega_1}{2} \right) \quad \forall z \in \mathbb{C}, \quad (1)
\]
\[
\sigma(z + \omega_2) = - \sigma(z) \exp \left( \eta_2 z + \frac{\eta_2 \omega_2}{2} \right) \quad \forall z \in \mathbb{C}. \quad (2)
\]
Let $P$ be a fundamental parallelogram of the lattice $\Lambda$, because $P$ is elliptic, 

$$\#(\text{zeros in } P) = \#(\text{poles in } P).$$

Let $x_1, \ldots, x_n$ be the zeros of $f$ in $P$ and $\beta_1, \ldots, \beta_n$ be the poles of $f$ in $P$. These were enumerated with multiplicity. Also, we know that $x_1 + \cdots + x_n \equiv \beta_1 + \cdots + \beta_n \pmod{\Lambda}$. Thus there are integers $k$ and $l$ such that 

$$\beta_1 + \cdots + \beta_n - x_1 - \cdots - x_n = k \omega_1 + l \omega_2.$$ 

Put 

$$a_j = x_j, \quad b_j = \beta_j \quad \forall 1 \leq j \leq n,$$

$$a_n = x_n + k \omega_1, \quad b_n = \beta_n - l \omega_2.$$ 

We have 

$$\sum_{j=1}^{n} b_j - \sum_{j=1}^{n} a_j = \sum_{j=1}^{n-1} b_j + b_n - \sum_{j=1}^{n-1} a_j - a_n$$

$$= \sum_{j=1}^{n} b_j - k \omega_1 - \sum_{j=1}^{n} a_j - l \omega_2$$

$$= 0.$$ 

By the definition of $a_j$ and $b_j$, we have $a_j \equiv x_j \pmod{\Lambda}$ and $b_j \equiv \beta_j \pmod{\Lambda}$. Then, the function 

$$g(x) = \frac{\delta(x - a_1) \cdots \delta(x - a_n)}{\delta(x - b_1) \cdots \delta(x - b_n)} \quad \forall \mathbf{x} \in \{\beta_1, \ldots, \beta_n\} \pmod{\Lambda}$$

has zeros $x_1, \ldots, x_n$ and poles $\beta_1, \ldots, \beta_n$ in $P$. Thus the quotient $\frac{f(x)}{g(x)}$ has no pole nor zero in $P$. We'll show that $g$ is doubly periodic.
For \( j = 1, 2, \ldots, n \) and \( i = 1, 2 \), we have

\[
\sigma(z - a_j + w_i) = -\sigma(z - a_j) \exp \left( \eta_i(z - a_j) + \frac{\eta_i w_i}{2} \right),
\]

\[
\sigma(z - b_i + w_i) = -\sigma(z - b_i) \exp \left( \eta_i(z - b_i) + \frac{\eta_i w_i}{2} \right).
\]

Thus,

\[
\frac{\sigma(z - a_j + w_i)}{\sigma(z - b_i + w_i)} = \frac{\sigma(z - a_j)}{\sigma(z - b_i)} \exp \left( \eta_i(b_j - a_j) \right)
\]

Thus,

\[
g(z + w_i) = \prod_{j=1}^{n} \frac{\sigma(z - a_j + w_i)}{\sigma(z - b_j + w_i)} = g(z) \exp \left( \eta_i \left( \sum_{j=1}^{n} b_j - \sum_{j=1}^{n} a_j \right) \right)
\]

\[
= g(z).
\]

Therefore \( g \) is doubly periodic. Thus \( f(z)/g(z) \) is doubly periodic and has no pole nor zero in \( \mathbb{P} \). Thus \( f(z)/g(z) \) is holomorphic. Thus it must be a constant \( C \). Therefore,

\[
f(z) = Cg(z) = C \prod_{j=1}^{n} \frac{\sigma(z - a_j)}{\sigma(z - b_j)}.
\]

(3) Problem 1, Ahlfors p. 276

Let \( \Lambda \) be the lattice generated by \( w_1 \) and \( w_2 \):

\[
\Lambda = \{ mw_1 + nw_2 : m, n \in \mathbb{Z} \}.
\]

With \( \delta \) and \( \sigma \) defined as in the previous problems, we'll show that

\[
\delta(z) - \delta(u) = -\frac{\delta(z - u) \delta(z + u)}{\delta(z)^2 \delta(u)^2} \quad \forall z, u \neq 0 \ (\text{mod} \ \Lambda).
\]

Regarding \( u \) as a constant, we put \( f(z) = \delta(z) - \delta(u) \quad \forall z \neq 0 \ (\text{mod} \ \Lambda) \).
and \[ g(z) = \frac{\delta(z-u) \delta(z+u)}{\delta(z)^2 \delta(u)^2} \quad \forall z \neq 0 \pmod{n}. \]

We'll use the identities (1) and (2) in Problem 2 to show that \( g \) is doubly periodic. For \( i = 1, 2 \), we have

\[
\delta(z-u+i\omega) = \delta(z-u) \exp \left( \frac{i \omega}{2} \right),
\]

\[
\delta(z+u+i\omega) = -\delta(z+u) \exp \left( \frac{i \omega}{2} \right),
\]

\[
\delta(z+i\omega) = -\delta(z) \exp \left( \frac{i \omega}{2} \right),
\]

\[
\delta(z+i\omega)^2 = \delta(z)^2 \exp \left( 2 \frac{i \omega}{2} \right).
\]

Thus

\[
\frac{\delta(z-u+i\omega) \delta(z+u+i\omega)}{\delta(z+i\omega)^2} = \frac{\delta(z-u) \delta(z+u)}{\delta(z)^2} \exp(0) = \frac{1}{\delta(z)^2}.
\]

Thus \( g(z+i\omega) = g(z) \). This means \( g \) is doubly periodic with periods \( i \omega \).

We know that \( \delta \) has simple zeros at exactly the lattice points. Thus \( g(z) \) has zeros at \( \pm u \pmod{n} \), each of which is of order one. When \( u = -u \pmod{n} \), i.e. \( u \) is a half period, then two simple zeros at \( u \) and \( -u \) become a double zero at \( u \). By saying \( g(z) \) has simple zeros at \( \pm u \pmod{n} \), we have already included this special case. Also, \( g(z) \) has double poles at the lattice points. As we discussed in Problem 1, \( f(z) = \Phi(z) - \Phi(u) \) has double
poles at the lattice points. Also, \( f(z) \) has simple zeros at \( \pm u \) (\( \text{mod } \Lambda \)). Therefore, the quotient \( f(z)/g(z) \) is an entire and doubly periodic function. Thus, it is a constant \( C_u \).

\[
P(z) - P(u) = C_u \frac{\delta(z-u) \delta(z+u)}{\delta(z)^2 \delta(u)^2} \quad \forall z \neq 0 \pmod{\Lambda} \quad (*)
\]

We'll show that \( C_u \) is independent of \( u \). Recall the definition of \( \delta \) as follow

\[
\delta(z) = \frac{\Gamma(\frac{z}{2 \pi})}{\Gamma(\frac{z}{2 \pi} + \frac{1}{2})} \exp \left( \frac{z}{\pi} + \frac{1}{2} \left( \frac{z}{\pi} \right)^2 \right).
\]

Thus

\[
\delta(-z) = -\frac{\Gamma(\frac{z}{2 \pi})}{\Gamma(\frac{z}{2 \pi} + \frac{1}{2})} \exp \left( \frac{-z}{\pi} + \frac{1}{2} \left( \frac{z}{\pi} \right)^2 \right).
\]

By replacing \( w \) by \(-w\), we get \( \delta(-z) = -\delta(z) \). Thus \( \delta \) is an odd function. Now we swap \( z \) and \( u \) in (*) to get

\[
P(u) - P(z) = C_u \frac{\delta(u-z) \delta(u+z)}{\delta(u)^2 \delta(z)^2} \quad \forall u \neq 0 \pmod{\Lambda}
\]

With this form, we proved earlier that the factor \( C_u \) is independent of \( u \) (because it depends only in \( z \)). Thus \( C_u = C \), which is independent of both \( z \) and \( u \).

\[
P(z) - P(u) = C \frac{\delta(z-u) \delta(z+u)}{\delta(z)^2 \delta(u)^2} \quad \forall u, z \neq 0 \pmod{\Lambda} \quad (**)
\]

Consider only \( u \) such that \( 2u \neq 0 \pmod{\Lambda} \), By the definition of \( \delta \), we have \( \delta(z) = \sqrt[4]{\delta(z)} \) where \( \delta \) is entire and \( \delta(0) = 1 \). Consequently

\[
\delta'(0) = \lim_{z \to 0} \frac{\delta(z)}{z} = \delta(0) = 1.
\]
Dividing both sides of (**) by \( z-u \), we get

\[
\frac{P(z) - P(u)}{z-u} = C \frac{\delta(z-u)}{z-u} \frac{\delta(z+u)}{\delta(z)^2 \delta(u)^2}
\]

Let \( z \) approach \( u \), we get

\[
P'(u) = C \delta'(0) \frac{\delta(2u)}{\delta(u)^4} = C \frac{\delta(2u)}{\delta(u)^4}
\]

We know that

\[
P'(u) = -\frac{2}{u^3} + O(u),
\]

\[
\frac{\delta(2u)}{\delta(u)^4} = \frac{2u \delta(2u)}{u^4 \delta(u)^4} = \frac{2}{u^3} \frac{\delta(2u)}{\delta(u)^4} = \frac{2}{u^3} \left( \frac{\delta'(0) + O(u)}{\delta(u)^4} \right)
\]

Thus, (***) gives us

\[
-\frac{2}{u^3} + O(u) = C \frac{2}{u^3} \left( 1 + O(u) \right) = \frac{2C}{u^3} + C \cdot O(u^2).
\]

Thus \( C = -1 \), therefore we have the identity

\[
P'(z) - P'(u) = \frac{\delta(z-u) \delta(z+u)}{\delta(z)^2 \delta(u)^2} \quad \forall u \neq 0 \pmod{n}.
\]

(4) With the function \( \xi(z) \) defined by \( \xi(x) = \frac{\delta(4)}{\delta(x)} \) for all \( x \neq 0 \pmod{n} \),
we'll show that

\[
\frac{P'(z)}{P(z)' - P(u)'} = \xi(z-u) + \xi(z+u) - 2 \xi(z) \quad \forall z \neq 0, \quad u \neq u \pmod{n}.
\]

Put \( f(z) = \delta(z) - \delta(u) \) and \( g(z) = -\frac{\delta(z-u) \delta(z+u)}{\delta(z)^2 \delta(u)^2} \).

For each \( z_0 \) such that \( z_0 \neq 0, u \pmod{n} \), there is a neighborhood of \( z_0 \) and a constant \( C \) such that for all \( z \) in that neighborhood,

\[
\log g(z) = \log \delta(z-u) + \log \delta(z+u) - 2 \log \delta(z) - \log \delta(u) + C.
\]
Here the logarithms do not necessarily refer to the same branch cut. Also, we can take a logarithm of \( f(z) \) for \( z \) in the neighborhood of \( z_0 \):

\[
\log f(z) = \log (F(z) - F(w)). \tag{2}
\]

Also, the logarithms in (1) and (2) may not be the same. However, since they differ by a constant, we can take the derivatives to get an identity. Namely,

\[
\frac{d}{dz} \log (F(z) - F(w)) = \frac{d}{dz} \log F(z-u) + \frac{d}{dz} \log F(z+u) - 2 \frac{d}{dz} \log F(z).
\]

This is equivalent to

\[
\frac{f'(z)}{F(z) - F(w)} = \frac{5'(z-u)}{5(z-u)} + \frac{5'(z+u)}{5(z+u)} - 2 \frac{5'(z)}{5(z)}
\]

\[
= 5(z-u) + 5(z+u) - 25(z).
\]


By the previous problem, when \( z, w \neq 0 \) and \( z \neq \pm u \ (\mod \Lambda) \) we have

\[
\frac{f'(z)}{F(z) - F(w)} = 5(z-u) + 5(z+u) - 25(z). \tag{3}
\]

Because the conditions are symmetric, we can swap \( z \) and \( w \) to get

\[
\frac{f'(w)}{F(w) - F(z)} = 5(w-z) + 5(w+z) - 25(w). \tag{4}
\]

As mentioned in Problem (3), \( 5 \) is an odd function. Thus \( 5' \) is even.

Then \( 5 = \frac{5'}{5} \) is odd. Thus \( 5(z-u) + 5(u-z) = 0 \).
Summing up (3) and (4) term by term, we get
\[
\frac{\varphi'(z) - \varphi'(u)}{\varphi(z) - \varphi(u)} = 2\varphi(z+u) - 2\varphi(z) - 2\varphi(u).
\]
Therefore,
\[
\varphi(z+u) = \varphi(z) + \varphi(u) + \frac{1}{2} \frac{\varphi'(z) - \varphi'(u)}{\varphi(z) - \varphi(u)}. \tag{5}
\]

(6) Problem 4, Ahlfors p. 277.

With the relation \( \varphi = -\varphi' \), we differentiate (5) with respect to \( t \) and get
\[
-\varphi(z+u) = -\varphi(z) + \frac{1}{2} \frac{\varphi''(z) - \varphi''(u)}{(\varphi(z) - \varphi(u))^2}
\]
we swap \( t \) and \( u \) in (6) to get
\[
-\varphi(z+u) = -\varphi(u) + \frac{1}{2} \frac{\varphi''(u) - \varphi''(z)}{(\varphi(z) - \varphi(u))^2} \tag{7}
\]
Adding (6) to (7) term by term, we get
\[
-2\varphi(z+u) = -\varphi(z) - \varphi(u) + \frac{1}{2} \frac{\varphi''(z) - \varphi''(u)}{\varphi(z) - \varphi(u)} - \frac{1}{2} \left( \frac{\varphi'(z) - \varphi'(u)}{\varphi(z) - \varphi(u)} \right)^2 \tag{8}
\]
We know that \( \varphi \) satisfies the following differential equation
\[
\varphi'(z)^2 = 4\varphi(z)^3 - g_2\varphi(z) - g_3,
\]
where \( g_1 \) and \( g_3 \) are constants related to the Eisenstein series. Taking derivatives both sides, we get
\[
2\varphi'(z)\varphi''(z) = 12\varphi(z)\varphi'(z)^2 - g_2\varphi'(z).
\]
For \( z \) not equal to half periods, \( \varphi'(z) \neq 0 \). Then
\[ 2 \Phi''(z) = 12 \Phi(z)^2 - g_2 \]

Thus, \[ \Phi''(z) = 6 \Phi(z)^2 - \frac{1}{2} g_2. \]
Replacing \( z \) by \( u \), we get
\[ \Phi''(u) = 6 \Phi(u)^2 - \frac{1}{2} g_2. \]

Thus, \[ \Phi''(z) - \Phi''(u) = 6 \Phi(z)^2 - 6 \Phi(u)^2 = 6(\Phi(z) - \Phi(u))(\Phi(z) + \Phi(u)). \]

Thus,
\[ \frac{\Phi''(z) - \Phi''(u)}{\Phi(z) - \Phi(u)} = 6(\Phi(z) + \Phi(u)). \]

Replacing this identity into (8), we get
\[ -2 \gamma(z+u) = - \Phi(z) - \Phi(u) + 3(\Phi(z) + \Phi(u)) - \frac{1}{2} \left( \frac{\Phi'(z) - \Phi'(u)}{\Phi(z) - \Phi(u)} \right)^2 \]
\[ 2(\Phi(z) + \Phi(u)) \]

Therefore,
\[ \gamma(z+u) = - \Phi(z) - \Phi(u) + 4 \left( \frac{\Phi'(z) - \Phi'(u)}{\Phi(z) - \Phi(u)} \right)^2. \]  

(9)

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By l'Hopital's rule, we have \( \lim_{z \to u} \frac{\Phi(z) - \Phi(u)}{\Phi'(z)} = \Phi''(z) \).

As \( z \) approaches \( u \), (9) becomes
\[ \gamma(z+u) = -2 \Phi(z) + \frac{1}{4} \left( \Phi'(z) \right)^2. \]

(8) In problem (3), we derived Equation (3) and then proved that the constant \( C \) therein is \(-1\). Thus, \( \gamma(u) = - \frac{5(2u)}{6(u)^4} \) for all \( u \neq 0 \) (mod \( a \)). Renaming \( u \) with \( z \), we write \( \gamma'(z) = - \frac{5(2z)}{6(z)^4} \).
9. The additional problem.

Suppose \( \text{Im} \tau > 0 \). We'll show that the series \( \sum_{(m,n) \neq (0_0)} \frac{1}{|n+\tau m|^2} \) diverges.

Suppose by contradiction that it converges. \( S = \text{max}\{1, |c|\} > 0 \). We have \( |n+\tau m| \leq |n|+|\tau| |m| \leq S(|n|+|m|) \). Thus,

\[
\infty > \sum_{(m,n) \neq (0_0)} \frac{1}{|n+\tau m|^2} \geq \frac{1}{S^2} \sum_{(m,n) \neq (0_0)} \frac{1}{(|n|+|m|)^2} \geq \frac{1}{S^2} \sum_{m,n \geq 1} \frac{1}{(n+m)^2}.
\]

Thus, \( A = \sum_{m,n \geq 1} \frac{1}{(n+m)^2} < \infty \).

For each \( k \geq 2 \), the term \( \frac{1}{k^2} \) occurs exactly \( (k-1) \) times in \( A \) as

\[
\frac{1}{(1+(k-1))^2}, \frac{1}{(2+(k-2))^2}, \ldots, \frac{1}{(k-1)+1}.
\]

Thus, \( A = \sum_{k=2}^{\infty} \frac{k-1}{k^2} = \sum_{k=2}^{\infty} \left( \frac{1}{k} - \frac{1}{k^2} \right) \).

Since \( A < \infty \),

\[
\sum_{k=2}^{\infty} \frac{1}{k} = A + \sum_{k=2}^{\infty} \frac{1}{k^2} < \infty.
\]

This is a contradiction.