We define two maps $\phi_1$ and $\phi_2$ as follows.

$\phi_1 : S^2 \setminus \{(0,0,1)\} \to \mathbb{C}$

$\phi_1(x,y,w) = \frac{x}{1-w} + i\frac{y}{1-w},$

$\phi_2 : S^2 \setminus \{(0,0,-1)\} \to \mathbb{C}$

$\phi_2(x,y,w) = \frac{x}{1+w} - i\frac{y}{1+w}.$

We want to show that $\phi_2 \circ \phi_1^{-1}(z) = 1/z.$ First, we will determine the inverse function of $\phi_1.$ Define $\psi_1 : \mathbb{C} \to S^2 \setminus \{(0,0,1)\}$

$\psi_1(z) = \left( \frac{2\Re(z)}{|z|^2+1}, \frac{2\Im(z)}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1} \right).$

Note that $\psi_1$ is well-defined because

$\left( \frac{2\Re(z)}{|z|^2+1} \right)^2 + \left( \frac{2\Im(z)}{|z|^2+1} \right)^2 + \left( \frac{|z|^2-1}{|z|^2+1} \right)^2 = \frac{4|z|^2}{|z|^4+1} + \frac{(|z|^2-1)^2}{|z|^4+1} = \frac{(|z|^2+1)^2}{|z|^4+1} = 1,$

and that $\psi_1(z)$ is never equal to $(0,0,1).$ We have

$\phi_1 \circ \psi_1(z) = \phi_1 \left( \frac{2\Re(z)}{|z|^2+1}, \frac{2\Im(z)}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1} \right) = \frac{x}{1-w} + i\frac{y}{1-w}.$
Because $1 - w = 1 - \frac{|z|^2 - 1}{|z|^2 + 1} = \frac{2}{|z|^2 + 1}$, we have

\[(*) = \frac{2 \Re(z)}{2} + i \frac{2 \Im(z)}{2} = z.
\]

Thus $\Phi_1 \circ \Phi_1 = \text{id}$. We have

\[\Psi_1 \circ \Phi_1(z) = \Psi_1 \left( \frac{x}{1 - w} + i \frac{y}{1 - w} \right) = \left( \frac{2 \Re(z)}{|z|^2 + 1}, \frac{2 \Im(z)}{|z|^2 + 1}, \frac{|z|^2 - 1}{|z|^2 + 1} \right) (**)
\]

Because $|z|^2 = \Re(z)^2 + \Im(z)^2 = \frac{x^2 + y^2}{(1 - w)^2} = \frac{1 - w^2}{1 - w} = \frac{1 + w}{1 - w}$, we get

\[|z|^2 + 1 = \frac{2}{1 - w} \quad \text{and} \quad |z|^2 - 1 = \frac{2w}{1 - w}.
\]

Thus \[(***) = \left( \frac{2x}{2}, \frac{2y}{2}, \frac{2w}{2} \right) = (x, y, w).
\]

Thus, $\Psi_1 \circ \Phi_1 = \text{id}$. Therefore, $\Phi_1 = \Phi_1^{-1}$. Next, we show that $\Phi_2 \circ \Phi_1^{-1}(z) = \text{id}$.

\[\Phi_2 \circ \Phi_1^{-1}(z) = \Phi_2 \circ \Psi_1(z) = \Phi_2 \left( \frac{2 \Re(z)}{|z|^2 + 1}, \frac{2 \Im(z)}{|z|^2 + 1}, \frac{|z|^2 - 1}{|z|^2 + 1} \right)
\]

\[= \frac{x}{1 + w} - i \frac{y}{1 + w} \quad (***)
\]

We have $1 + w = \frac{2|z|^2}{|z|^2 + 1} = \frac{2 \bar{z}^2}{|z|^2 + 1}$. Thus

\[(***) = \frac{2 \Re(z)}{2 \bar{z}^2} - i \frac{2 \Im(z)}{2 \bar{z}^2} = \frac{2 \bar{z}}{2 \bar{z}^2} = \frac{1}{\bar{z}}.
\]

Therefore, $\Phi_2 \circ \Phi_1^{-1}(z) = \frac{1}{z}$.

Define a function \( f: \mathbb{P}^1 \to S^2 \),
\[
    f([z:w]) = \frac{(2 \text{Re}(w \overline{z}), 2 \text{Im}(w \overline{z}), \sqrt{|w|^2 - |z|^2})}{|w|^2 + |z|^2}
\]

We want to show that \( f \) is a homeomorphism. However, we will show that \( f \) is an isomorphism of Riemann surfaces (a stronger statement). First, let us check that \( f \) is well-defined. For each \( \lambda \in \mathbb{C} \setminus \{0\} \), we have
\[
    f([\lambda z:w]) = \frac{(2 \text{Re}(\lambda \overline{w} \overline{z}), 2 \text{Im}(\lambda \overline{w} \overline{z}), \lambda^2 |w|^2 - |z|^2)}{|\lambda|^2 |w|^2 + |\lambda|^2 |z|^2}
    = f([z:w]) \quad \text{(after cancelling } \frac{|z|^2}{|w|^2} \text{ in the numerator and denominator).}
\]

Thus the value of \( f \) at \([z:w]\) doesn't depend on the choice of representatives of \([z:w]\). Moreover,
\[
    |f([z:w])|^2 = \frac{(2 \text{Re}(w \overline{z}))^2 + (2 \text{Im}(w \overline{z}))^2 + (|w|^2 - |z|^2)^2}{(|w|^2 + |z|^2)^2}
    = \frac{4 |w \overline{z}|^2 + (|w|^2 - |z|^2)^2}{(|w|^2 + |z|^2)^2} = \frac{(|w|^2 + |z|^2)^2}{(|w|^2 + |z|^2)^2} = 1
\]

Thus \( f([z:w]) \in S^2 \). Therefore, \( f \) is well-defined.

By definition, the projective line \( \mathbb{P}^1 = \{[z:w] | z, w \in \mathbb{C}, |z|^2 + |w|^2 \neq 0\} \) has a complex atlas consisting of two charts \((U_+, \phi_+)\) and \((U_-, \phi_-)\) defined by
\[ U_+ = \{ [z:w] \mid w \neq 0 \}, \]
\[ U_- = \{ [z:w] \mid z \neq 0 \}, \]
\[ \phi_+ : U_+ \rightarrow \mathbb{C}, \quad \phi_+([z:w]) = \frac{z}{w}, \]
\[ \phi_- : U_- \rightarrow \mathbb{C}, \quad \phi_-([z:w]) = \frac{w}{z}. \]

The inverse maps are \( \phi_+^{-1}(z) = [z:1] \) and \( \phi_-^{-1}(w) = [1:w] \).

By definition, the Riemann sphere \( S^2 = \{ (a,b,c) \mid a^2 + b^2 + c^2 = 1 \} \) has a complex atlas consisting of two charts \( (V_+, \Psi_+) \) and \( (V_-, \Psi_-) \) defined as follow.

\[ V_+ = \{ (a,b,c) \in S^2 \mid c \neq 1 \}, \]
\[ V_- = \{ (a,b,c) \in S^2 \mid c \neq 1 \}, \]
\[ \Psi_+ : V_+ \rightarrow \mathbb{C}, \quad \Psi_+(a,b,c) = \frac{a+ib}{1-c}, \quad \frac{a-ib}{1+c}, \]
\[ \Psi_- : V_- \rightarrow \mathbb{C}, \quad \Psi_-(a,b,c) = \frac{a-ib}{1+c}, \quad \frac{a+ib}{1-c}. \]

By computation, the inverse functions are obtained.

\[ \Psi_+^{-1}(z) = \frac{(2 \text{Re}(z),-2 \text{Im}(z),-1|z|^2+1)}{|z|^2+1}, \]
\[ \Psi_-^{-1}(z) = \frac{(2 \text{Re}(z),+2 \text{Im}(z),-1+|z|^2)}{1+|z|^2}. \]

Define a function \( g : S^2 \rightarrow \mathbb{R}^4 \),
\[ g(a,b,c) = \begin{cases} [1-c:(a+ib)] \text{ on } V_+, \\ [1+c:(a-ib)] \text{ on } V_. \end{cases} \]
First, we'll show that \( g \) is well-defined. On \( V_+(V_-), c \neq \pm 1 \) and so \( a^2 + b^2 \neq 0 \). We have
\[
\frac{1-c}{a-ib} = \frac{(1-c)(a+ib)}{(a-ib)(a+ib)} = \frac{(1-c)(a+ib)}{a^2+b^2} = \frac{(1-c)(a+ib)}{1+c}
\]
Thus, \([c-1] \cdot (a+ib) = [c+1] \cdot (a+ib)\). Thus the values of \( g \) by two formulae are the same in the overlap. Thus, \( g \) is well-defined.

By the definition of \( f \) and \( g \), we see that \( f(U_+) \subset V_+ \), \( f(U_-) \subset V_- \), \( g(V_+) \subset U_+ \) and \( g(V_-) \subset U_- \). These observations help us compute \( f \circ g \) and \( g \circ f \) conveniently by computing them on \( V_+, V_- \) or \( U_+, U_- \). Then we get \( f \circ g = \text{id}_{\mathbb{R}^2} \) and \( g \circ f = \text{id}_{\mathbb{R}^2} \).

Thus \( f \) is bijective and \( g \) is its inverse. Thus,
\[
V_+ = f \circ g(V_+) \subset f(U_+),
V_- = f \circ g(V_-) \subset f(U_-).
\]
Thus \( f(U_+) = V_+, f(U_-) = V_- \), \( g(V_+) = U_+ \) and \( g(V_-) = U_- \).

Put \( \mathbf{f}_\pm : C \to C \), \( \mathbf{f}_\pm(z) = \psi_\pm \circ \phi_\pm^{-1}(z) \). We have
\[ f_+(z) = \Psi_+ \circ f([z:1]) = \Psi_+ \left( \frac{2 \text{Re}(z), 2 \text{Im}(z), 1-|z|^2}{1+|z|^2} \right) \]
\[ = \Psi_+ \left( \frac{2 \text{Re}(z), -2 \text{Im}(z), 1-|z|^2}{1+|z|^2} \right) \]
\[ = \Psi_+ (\Psi_+^{-1}(z)) = z. \]

\[ f_-(w) = \Psi_- \circ f([1:w]) = \Psi_- \left( \frac{2 \text{Re}(w), 2 \text{Im}(w), |w|^2-1}{|w|^2+1} \right) \]
\[ = \Psi_- (\Psi_-^{-1}(w)) = w. \]

Thus, \( f_+ = f_- = \text{id}_C \), which are holomorphic.

Similarly, put \( g_\pm : C \to C, \ g_\pm (z) = \phi_\pm \circ g \circ \psi_\pm^{-1}(z). \) We have
\[ g_+(z) = \phi_+ \circ g \left( \frac{2 \text{Re}(z), -2 \text{Im}(z), 1-|z|^2}{1+|z|^2} \right) = \phi_+ \left( \frac{2 |z|^2}{1+|z|^2} : \frac{2 \text{Re}(z)-2i \text{Im}(z)}{1+|z|^2} \right) \]
\[ = \phi_+ (\left[ |z|^2 : \bar{z} \right]) \]
\[ = \phi_+ (\left[ z \bar{z} : \bar{z} \right]) \]
\[ = \phi_+ (\left[ : z \right]) \]
\[ = z. \]

\[ g_-(z) = \phi_- \circ g \left( \frac{2 \text{Re}(z), 2 \text{Im}(z), |z|^2-1}{|z|^2-1} \right) \]
\[ = \phi_- \left( \frac{2 \text{Re}(z)-2i \text{Im}(z)}{|z|^2-1} : \frac{2 |z|^2}{|z|^2-1} \right) \]
\[ = \phi_- (\left[ \bar{z} : \bar{z} \bar{z} \right]) = \phi_- (\left[ 1 : z \right]) = z. \]
Thus, \( g_+ = g_- = \text{id}_C \), which are holomorphic. Therefore \( f \) and \( g \) are holomorphic. Thus, \( f \) is an isomorphism.


Let \( \Omega \subset \mathbb{C} \) be an open subset and \( p \in \Omega \). Suppose \( \phi: \Omega \to \mathbb{C} \) is a holomorphic function such that \( \phi'(p) \neq 0 \). We will use the Implicit Function Theorem to show that there is an open neighborhood of \( p \) in \( \Omega \), namely \( U \), such that \( \phi|_U \) is a chart on \( \mathbb{C} \).

Write \( \phi(z) = u(x,y) + iv(x,y) \) where \( u,v: \Omega \to \mathbb{R} \) are the real and imaginary part of \( \phi(z) \). Since \( \phi \) is holomorphic, \( u \) and \( v \) satisfy the Cauchy–Riemann equations

\[
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \forall (x,y) \in \Omega.
\]

Write \( p = x_0 + iy_0 \) and \( \phi(p) = t_0 + is_0 \). Then, because \( \phi'(p) \neq 0 \),

\[
|\phi'(p)|^2 = \left( \frac{\partial u}{\partial x}(x_0, y_0) \right)^2 + \left( \frac{\partial v}{\partial x}(x_0, y_0) \right)^2 \neq 0 \quad (*)
\]

We define two functions \( f, g: \Omega \times \mathbb{R}^2 \to \mathbb{R} \) which are given by

\[
f(x,y,t,s) = u(x,y) - t,
\]

\[
g(x,y,t,s) = v(x,y) - s, \quad \forall (x,y) \in \Omega, (t,s) \in \mathbb{R}^2.
\]
Since \( u \) and \( v \) are smooth functions, so are \( f \) and \( g \). We have

\[
\begin{vmatrix}
\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\
\frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}
\end{vmatrix} = \begin{vmatrix}
\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\
\frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}
\end{vmatrix} = \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2
\]

the value of which at \((x_0, y_0)\) is non-zero by \((*)\). Thus,

\[
\begin{pmatrix}
\frac{\partial f}{\partial x} (x_0, y_0) & \frac{\partial f}{\partial y} (x_0, y_0) \\
\frac{\partial g}{\partial x} (x_0, y_0) & \frac{\partial g}{\partial y} (x_0, y_0)
\end{pmatrix}
\]

is an invertible matrix.

By the Implicit Function Theorem for complex functions, there exist an open neighborhood of \((x_0, y_0)\) in \(\mathbb{R}^2\), namely \(U\), an open neighborhood of \((b, s_0)\) in \(\mathbb{R}^2\), namely \(V\), and a holomorphic function \(\phi : V \to U\) such that \(\phi(b, s_0) = (x_0, y_0)\) and \(\phi(x) = (t, s)\) for all \((t, s) \in V\). Because of the last identity, \(\phi\) is injective. Thus \(\phi : V \to \phi(V)\) is a conformal map. Since \(\phi(V)\) is an open neighborhood of \((x_0, y_0)\), we could have taken \(U\) to be \(\phi(V)\) from the beginning. If so, \(\phi : V \to U\) is now a conformal map.

Next, we'll show that \(\phi|_U : U \to \phi(U)\) is injective. Suppose that we have \((z_1, y_1), (z_2, y_2) \in U\) such that \(\phi(z_1, y_1) = \phi(z_2, y_2)\).
Because $S : V \to U$ is surjective, there are $(u_1, v_1), (u_2, v_2) \in V$ such that $(x_1, y_1) = S(u_1, v_1)$ and $(x_2, y_2) = S(u_2, v_2)$. Thus,

$$\phi(x_1, y_1) = \phi(S(u_1, v_1)) = (u_1, v_1),$$
$$\phi(x_2, y_2) = \phi(S(u_2, v_2)) = (u_2, v_2).$$

Thus, $(u_1, v_1) = (u_2, v_2)$. Applying $S$ to both sides, we get $(x_1, y_1) = (x_2, y_2)$. Hence $\phi|_U$ is injective. Because it is also surjective and holomorphic, it is a conformal map between $U$ and $\phi(U)$. Thus, $\phi|_U$ is a chart on $C$.


On $C^3$, we define a relation $(x_1, y_1, z_1) \sim (x_2, y_2, z_2)$ if and only if there is $\lambda \in C^*$ such that $(x_1, y_1, z_1) = \lambda(x_2, y_2, z_2)$. This turns out to be an equivalent relation. The complex projective plane $P^2$ was defined to be the set $C^3/\sim$ with quotient topology.

Denote by $[x:y:z]$ the equivalence class of $(x, y, z)$. We define

$$U_0 = \{[x:y:z] \in P^2 | x \neq 0\},$$
$$U_1 = \{[x:y:z] \in P^2 | y \neq 0\},$$
$$U_2 = \{[x:y:z] \in P^2 | z \neq 0\},$$

and the maps $\phi_0 : U_0 \to C^2$, $\phi_0([x:y:z]) = \left(\frac{y}{x}, \frac{z}{x}\right)$,
\[ \phi_1: \mathbb{C} \rightarrow \mathbb{C}, \quad \phi_1([x:y:z]) = \left(\frac{x}{y}, \frac{z}{y}\right), \]
\[ \phi_2: \mathbb{C} \rightarrow \mathbb{C}, \quad \phi_2([x:y:z]) = \left(\frac{x}{z}, \frac{y}{z}\right). \]

These maps are well-defined because their values at \([x:y:z]\) do not depend on the choice of representatives. We'll show that \(\phi_0\) is a homeomorphism, and similarly for \(\phi_1\) and \(\phi_2\). First, we show that \(\phi_0\) is bijective. Put \(\Psi_0: \mathbb{C}^2 \rightarrow U_0, \quad \Psi_0(u,v) = [1:u:v]. \)

\[ \phi_0 \circ \Psi_0(u,v) = \phi_0([1:u:v]) = (u,v) \quad \forall (u,v) \in \mathbb{C}^2, \]
\[ \Psi_0 \circ \phi_0([x:y:z]) = \Psi_0\left(\frac{x}{z}, \frac{y}{z}\right) = [1: \frac{y}{z}: \frac{x}{z}] = [x:y:z], \quad \forall (x:y:z) \in U_0. \]

Thus \(\phi_0\) is bijective and its inverse map is \(\Psi_0\).

Show that \(\phi_0\) is continuous

Put \(W_0 = \{ (x_1,y_1,z_1) \in \mathbb{C}^3 \mid x_1 \neq 0 \}\). Then \(U_0 = W_0/\sim\).

Let \(q: W_0 \rightarrow W_0/\sim\) be the quotient map and put \(f = \phi_0 \circ q\). By the characterization of quotient topology, \(\phi_0\) is continuous iff \(f\) is continuous. Because \(f: W_0 \subset \mathbb{C}^3 \rightarrow \mathbb{C}^2\) and \(f(x_1,y_1,z_1) = \left(\frac{y_1}{x_1}, \frac{z_1}{x_1}\right)\), it is obviously continuous.

Show that \(\Psi_0\) is continuous

Put \(g: \mathbb{C}^2 \rightarrow W_0 \subset \mathbb{C}^3, \quad g(u,v) = (1, u, v). \)
Then $g$ is continuous and $y_0 = q \circ g$. Thus $y_0$ is also continuous.

Next, we show that $P^2$ is Hausdorff. We have

$$P^2 = \{ [x : y : z] \mid x, y, z \text{ are not zero at once} \}$$

Let $q : C^3 \rightarrow C^3/\sim$ be the quotient map and $W_0, W_1, W_2$ be the sets described above. Since $q^{-1}(U_i) = U_i$, which is open in $C^3$, $U_i$ is open in $P^2$. Because $C^2$ is Hausdorff and $U_i \cong C^2$, each $U_i$ is also Hausdorff. Let $a, b \in P^2$ such that $a \neq b$. We are looking for two neighborhoods of $a$ and $b$ in $P^2$ that are disjoint. Since $P^2 = U_0 \cup U_1 \cup U_2$, there are two cases.

Case 1: $a$ and $b$ belong to the same chart, say $U_0$ for example.

Since $U_0$ is Hausdorff, there are an open neighborhood of $a$ in $U_0$, namely $U$, and an open neighborhood of $b$ in $U_0$, namely $V$, such that $U \cap V = \emptyset$.

Since $U_0$ is open in $P^2$, $U$ and $V$ are open in $P^2$.

Case 2: $a$ and $b$ do not belong to the same chart.

WLOG, we assume $a \in U_0$ and $b \in U_1$. Then $a \in U_0 \setminus U_1$ and $b \in U_1 \setminus U_0$.

Thus, $a = [1 : 0 : z_1]$ and $b = [0 : 1 : z_2]$. It is true that for $\alpha, \beta \in C$, 

\[
\alpha \circ \beta \neq \beta \circ \alpha.
\]
\[ |x| < |y| \iff |x| \leq |y| \iff |x| \leq |y| \forall x \in \mathbb{C}^*, \]
and \[ |x| < |y| \iff |x| < |y| \forall x \in \mathbb{C}^*. \]

Thus, the ordering of \( |x|, |y|, |z| \) in \([x:y:z]\) is well-defined. Thus we can put \( U = \{ [x:y:z] : |x| > |y| \} \) and \( V = \{ [x:y:z] : |x| < |y| \} \). Then \( \phi^{-1}(U) = \{ (x,y,z) \in \mathbb{C}^3 : |x| > |y| \} \), \\
\( \phi^{-1}(V) = \{ (x,y,z) \in \mathbb{C}^3 : |x| < |y| \} \), which are open and disjoint subsets of \( \mathbb{C}^3 \). Thus, \( U \) and \( V \) are open and disjoint in \( \mathbb{P}^2 \). Since \( a = [1:0:2] \in U \) and \( b = [0:1:2] \in V \), \( U \) and \( V \) are the sets we were looking for.

Put \( D = \{ (z,w) : |z|, |w| < 1 \} \subset \mathbb{C}^2 \). We show that \( \mathbb{P}^2 = \phi_{0}^{-1}(D) \cup \phi_{1}^{-1}(D) \cup \phi_{2}^{-1}(D) \).

All we need is to show that \( \mathbb{P}^2 \) is contained in the set on the right.

\[ \phi_{0}^{-1}(D) = \{ (p,q) \in \mathbb{P}^2 : |p| < 1 \} \]

Take \([a:b:c] \in \mathbb{P}^2\) and put \( \alpha = \max\{|a|, |b|, |c|\} > 0 \) for a specific choice of representative. There are 3 cases.

\( \alpha = |a| \) Then \([a:b:c] \in U_0 \) and \( |b|, |c| \leq 1 \). Thus \\
\( \phi_0([a:b:c]) = \left( \frac{b}{a}, \frac{c}{a} \right) \in D \)

Thus, \([a:b:c] \in \phi_0^{-1}(D)\).

\( \alpha = |b| \) Then \([a:b:c] \in U_1 \) and \( |a|, |c| \leq 1 \). Thus \\
\( \phi_1([a:b:c]) = \left( \frac{a}{b}, \frac{c}{b} \right) \in D \).

Thus, \([a:b:c] \in \phi_1^{-1}(D)\).
$\alpha = |c|$ Then $[a:b:c] \in \mathbb{U}_c$ and $\frac{a}{c}, \frac{b}{c} \leq 1$. Thus,

$\phi_2([a:b:c]) = \left(\frac{a}{c}, \frac{b}{c}\right) \in D.$

Thus, $[a:b:c] \in \phi_2^{-1}(D)$. Therefore, $[a:b:c] \in \phi_0^{-1}(D) \cup \phi_1^{-1}(D) \cup \phi_2^{-1}(D)$.

Because $D$ is a compact subset of $\mathbb{C}^2$, $\phi_0^{-1}(D), \phi_1^{-1}(D), \phi_2^{-1}(D)$ are compact sets. Since $\mathbb{P}^2$ is the union of 3 compact subspaces, it is compact as well.

5) Problem I.3.C, Miranda p. 18

Let $F: \mathbb{C}^{n+1} \to \mathbb{C}$ be a homogeneous polynomial of degree $d$. Put $x = (x_0, x_1, \ldots, x_n) \in \mathbb{C}^{n+1}$. We'll show that $F(x) = \frac{1}{d} \sum_{i=0}^{n} x_i \frac{\partial F}{\partial x_i}. \ (*)$

We know that $F$ is a sum of monomials of degree $d$. Since $(*)$ is additive in $F$, it suffices to prove it for monomials. Moreover, since a scalar factor can be cancelled out from both sides of $(*)$, it suffices to consider monic monomials only.

Let $F(x) = x_0^{r_0} x_1^{r_1} \cdots x_n^{r_n}$ be a monic monomial with $r_0, \ldots, r_n > 0$ and $r_0 + r_1 + \cdots + r_n = d$. Then $\frac{\partial F}{\partial x_0} = r_0 x_0^{r_0-1} x_1^{r_1} \cdots x_n^{r_n}.$

Thus $x_0 \frac{\partial F}{\partial x_0} = r_0 x_0^{r_0} x_1^{r_1} \cdots x_n^{r_n} = r_0 F(x)$.

Since $x_0$ and any $x_i$ have the same role, $x_i \frac{\partial F}{\partial x_i} = r_i F(x)$ for all $i=1, \ldots, n.$
Thus, \[ \sum_{i=0}^{n} \frac{2F}{\partial x_i} = \sum_{i=0}^{n} r_i F(x) = \left( \sum_{i=0}^{n} r_i \right) F(x) = dF(x). \]

Therefore, \[ F(x) = \frac{1}{d} \sum_{i=0}^{n} x_i \frac{\partial F}{\partial x_i}. \]

(6) Problem I.3.E, Miranda p. 18

Let \( F(x,y,z) = ax + by + cz \) and \( G(x,y,z) = ax + by + cz \) be two homogeneous polynomials on \( \mathbb{C}^3 \). The conditions for them to be of degree one are \((a,b,c) \neq \overrightarrow{0}\) and \((a,b,c) \neq \overrightarrow{0}\). Let \( X_F \) and \( X_G \) be the projective plane curves defined by \( F \) and \( G \) respectively.

\[ X_F = \{ [x:y:z] \in \mathbb{P}^2 \mid ax + by + cz = 0^3 \}, \]

\[ X_G = \{ [x:y:z] \in \mathbb{P}^2 \mid ax + by + cz = 0^3 \}. \]

Suppose \( X_F \neq X_G \). We'll show that \( X_F \cap X_G \) has only one element.

Put \( \overrightarrow{u} = (a,b,c) \) and \( \overrightarrow{v} = (a,\beta,\gamma) \). Then \( \overrightarrow{u}, \overrightarrow{v} \neq \overrightarrow{0} \).

If \( \overrightarrow{u} \times \overrightarrow{v} = \overrightarrow{0} \) then \( \overrightarrow{u} \) and \( \overrightarrow{v} \) are parallel. Thus there is \( \lambda \in \mathbb{C}^* \) such that \( \overrightarrow{u} = \lambda \overrightarrow{v} \). Then

\[ X_F = \{ [x:y:z] \in \mathbb{P}^2 \mid (a,b,c) \cdot (x,y,z) = 0^3 \} \]

\[ = \{ [x:y:z] \in \mathbb{P}^2 \mid \lambda (a,\beta,\gamma) \cdot (x,y,z) = 0 \} \]

\[ = \{ [x:y:z] \in \mathbb{P}^2 \mid (x,\beta,\gamma) \cdot (x,y,z) = 0 \} \]

\[ = X_G, \text{ which is a contradiction.} \]
Therefore, \( \tilde{w} \times \tilde{v} \neq 0 \). Define \( T: \mathbb{C}^3 \to \mathbb{C}^2 \) to be a linear map

\[
T(x, y, z) = \begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}
\]

Then \( [x:y:z] \in X_F \cap X_G \iff \{ T(x, y, z) = 0 \}
\]

\[
\iff (x, y, z) \in (\ker T) \setminus \{0\} \quad \text{(*)}
\]

We have \( \tilde{w} \times \tilde{v} = \begin{vmatrix} b & c & | \\ \alpha & \beta & | \\ a & b & | \end{vmatrix} \neq 0 \).

Thus at least one of the determinants is nonzero. Thus the maximal size of a minor matrix of \( A \) that is invertible is 2. Thus \( \text{rank}(A) \leq 2 \).

Thus \( \dim(\text{Im} T) = 2 \). Then \( \dim(\ker T) = 3 - \dim(\text{Im} T) = 3 - 2 = 1 \).

Moreover, \( \tilde{w} \times \tilde{v} = (b \gamma - c \beta, c \alpha - a \gamma, a \beta - b \alpha) \) is in the kernel of \( T \) because

\[
a (b \gamma - c \beta) + b (c \alpha - a \gamma) + c (a \beta - b \alpha) = 0,
\]

\[
x (b \gamma - c \beta) + \beta (c \alpha - a \gamma) + x (a \beta - b \alpha) = 0.
\]

Thus, \( \ker T = \langle \tilde{w} \times \tilde{v} \rangle = \{ \lambda (\tilde{w} \times \tilde{v}) \mid \lambda \in \mathbb{C} \} \). From (*), we get

\[
[x:y:z] \in X_F \cap X_G \iff (x, y, z) = \lambda (\tilde{w} \times \tilde{v}) \text{ for some } \lambda \in \mathbb{C}^*
\]

\[
\iff [x:y:z] = [(b \gamma - c \beta); (c \alpha - a \gamma); (a \beta - b \alpha)]
\]

Therefore, \( X_F \) intersects \( X_G \) at a single point, which is

\[
[(b \gamma - c \beta); (c \alpha - a \gamma); (a \beta - b \alpha)]
\]