(2) Put \( \Sigma = D \setminus \{ -\frac{1}{2}, \frac{1}{2} \} \).

We want to find all analytic functions \( f : D \to \Sigma \) such that for any cycle \( \gamma \) in \( D \) that is not homologous to 0 (mod \( \Sigma \)),
\[ f \circ \gamma \] is not homologous to 0 (mod \( \Sigma \)) either.

Suppose that \( f \) is such a function. Then \( f \) is not a constant function. Indeed, if \( f \) were a constant function, it would map every cycle to a cycle whose image is a point in \( \Sigma \), such a cycle of course has winding number zero with respect to both \(-1/2\) and \(1/2\); and so it would be homologous to 0 (mod \( \Sigma \)).

Next, we will show that \( f \) can extend to an analytic function from \( D \) to \( D \). Note that currently \( f \) has two isolated singularities at \( \pm 1/2 \). Since \( f \) maps \( D \) onto \( \Sigma \), it is bounded. Thus,
\[
\lim_{t \to -\frac{1}{2}} (t + \frac{1}{2}) f(t) = 0 \quad \text{and} \quad \lim_{t \to \frac{1}{2}} (t - \frac{1}{2}) f(t) = 0.
\]
This means \( f \) can extend to an analytic function on \( D \) by defining
\[ f\left(\frac{1}{2}\right) := \lim_{t \to \frac{1}{2}^-} f(t), \quad f\left(\frac{1}{2}\right) := \lim_{t \to \frac{1}{2}^+} f(t). \]

Then \( f\left(\frac{1}{2}\right), f\left(-\frac{1}{2}\right) \in \overline{D} \). Thus \( f \) is an analytic function from \( D \) to \( \overline{D} \). Since \( f \) is nonconstant, it is an open mapping. Thus \( f(D) \) is an open subset of \( \mathbb{C} \). Thus, \( f(D) \) must be contained in the interior of \( \overline{D} \), which is \( \mathbb{D} \). This means \( f \) is a map from \( D \) to \( \mathbb{D} \).

Next, we will show that \( f\left(-\frac{1}{2}\right), f\left(\frac{1}{2}\right) \in \left\{-\frac{1}{2}, \frac{1}{2}\right\} \). Suppose that this is not true. Then we can assume WLOG that \( f\left(\frac{1}{2}\right) = \alpha \in \mathbb{D} \).

Then we can choose an open disk \( D(a,r) \) that lies entirely in \( \mathbb{D} \).

Since \( f \) is continuous, there exists \( \varepsilon > 0 \) such that \( f\left(D\left(\frac{1}{2}, \frac{\varepsilon}{2}\right)\right) \subset D(a,r) \). Let \( \gamma \) be the simple closed path \( \partial D\left(\frac{1}{2}, \frac{\varepsilon}{2}\right) \). Then \( \gamma \) is not homologous to \( 0 \) (mod \( \mathbb{Z} \)) because \( n\left(\gamma, \frac{1}{2}\right) = 1 \neq 0 \). We know, however, that \( f_*\gamma \) is a closed path lying in \( D(a,r) \). Thus its winding numbers with respect to \( -\frac{1}{2} \) and \( \frac{1}{2} \) are both zero. Then \( f_*\gamma \) is homologous to \( 0 \) (mod \( \mathbb{Z} \)). This is a contradiction.

We have proved that \( f\left(\frac{1}{2}\right) = \pm \frac{1}{2} \) and \( f\left(-\frac{1}{2}\right) = \pm \frac{1}{2} \). Next, we will show that \( f\left(\frac{1}{2}\right) \neq f\left(-\frac{1}{2}\right) \). Suppose by contradiction that
\[ f\left(\frac{1}{2}\right) = f\left(-\frac{1}{2}\right) = \frac{1}{2} \] (the case they are equal to \(-\frac{1}{2}\) will be treated similarly). Then \(\frac{1}{2}\) and \(-\frac{1}{2}\) are zeros of the function \(f(z) - \frac{1}{2}\). Since \(f\) is nonconstant, they are isolated zeros. Denote by \(\mathcal{M}_1\) the order of zero at \(\frac{1}{2}\) of \(f(z) - \frac{1}{2}\), and by \(\mathcal{M}_2\) the order of zero at \(-\frac{1}{2}\) of \(f(z) - \frac{1}{2}\).

Since \(-\frac{1}{2}\) is an isolated zero of \(f(z) - \frac{1}{2}\), there is \(\varepsilon_1 > 0\) such that \(-\frac{1}{2}\) is the only zero of \(f(z) - \frac{1}{2}\) in \(\overline{\mathcal{D}}\left(-\frac{1}{2}, \varepsilon_1\right)\). Moreover, by the continuity of \(f\), we could have chosen \(\varepsilon_1 > 0\) such that
\[ f\left(\overline{\mathcal{D}}\left(-\frac{1}{2}, \varepsilon_1\right)\right) \subset \mathcal{D}\left(\frac{1}{2}, \frac{1}{4}\right). \] Put \(\mathcal{V}_1 = \mathcal{D}\left(-\frac{1}{2}, \varepsilon_1\right)\) which is positively oriented. Then \(\mathcal{V}_1\) has the following properties:

- \(\mathcal{V}_1\) does not pass through any zero of \(f(z) - \frac{1}{2}\).
- \(-\frac{1}{2}\) is the only zero of \(f(z) - \frac{1}{2}\) that is enclosed in \(\mathcal{V}_1\),
- \(\partial \mathcal{V}_1\) doesn't enclose \(-\frac{1}{2}\) i.e \(\nu(\partial \mathcal{V}_1, -\frac{1}{2}) = 0\) because it is a loop in \(\mathcal{D}\left(\frac{1}{2}, \frac{1}{4}\right)\).

Similarly, since \(\frac{1}{2}\) is an isolated zero of \(f(z) - \frac{1}{2}\), there is \(\varepsilon_2 > 0\) such that \(\frac{1}{2}\) is the only zero of \(f(z) - \frac{1}{2}\) in \(\overline{\mathcal{D}}\left(\frac{1}{2}, \varepsilon_2\right)\). Moreover,
by the continuity of $f$, we could have chosen $\varepsilon_2 > 0$ such that
\[ \int \mathbb{D}(\frac{1}{2}, \varepsilon_2) < \mathbb{D}(\frac{1}{2}, \frac{1}{4}). \]
Put $\delta_2 = \mathbb{D}(\frac{1}{2}, \varepsilon_2)$ which is positively oriented. Then $\delta_2$ has 3 following properties.

1. $\delta_2$ does not pass through any zero of $f(x) - \frac{1}{2}$.
2. $\delta_2$ is the only zero of $f(x) - \frac{1}{2}$ that is enclosed in $\delta_2$.
3. $f(\delta_2)$ does not enclose $-1/2$, i.e. $n(f(\delta_2), -\frac{1}{2}) = 0$, because it is a loop in $\mathbb{D}(\frac{1}{2}, \frac{1}{4})$.

Consider a cycle $Y = \ell \delta_1 - m \delta_2$. It is not homologous to 0 (mod $\mathbb{R}$) because $n(Y, -1/2) = \ell n(\delta_1, -1/2) - m n(\delta_2, -1/2) = \ell \neq 0$.

Thus $f(\delta)$ must be not homologous to 0 (mod $\mathbb{R}$). We have

\[ \ell f(\delta_1) = \ell f(\delta_1) - m f_0(\delta_2) = f(\delta_0 \delta_1) - m f(\delta_0 \delta_2). \]

Then
\[ n(f(\delta), -\frac{1}{2}) = \ell n(f(\delta_0 \delta_1), -\frac{1}{2}) - m n(f(\delta_0 \delta_2), -\frac{1}{2}) = 0. \]

Also,
\[ n(f(\delta), \frac{1}{2}) = \ell n(f(\delta_0 \delta_1), \frac{1}{2}) - m n(f(\delta_0 \delta_2), \frac{1}{2}). \quad (\ast) \]

\[
\begin{align*}
n(f(\delta_0 \delta_1), \frac{1}{2}) &= \frac{1}{2\pi i} \int_{f(\delta_0 \delta_1)} \frac{dw}{w - \frac{1}{2}} = \frac{1}{2\pi i} \int_0^1 \frac{(f(\delta_0 \delta_1)(t) \, dt}{f(\delta_0 \delta_1)(t) - \frac{1}{2}} \\
&= \frac{1}{2\pi i} \int_0^1 \frac{f'(\delta_0 \delta_1(t)) \, dx(t)}{f(\delta_0 \delta_1(t) - \frac{1}{2}} \, dt \\
&= \frac{1}{2\pi i} \int_{\delta_1} \frac{f'(x)}{f(x) - \frac{1}{2}} \, dx.
\end{align*}
\]
By the Argument Principle, this is the number of zeros of \( f(z) - \frac{1}{2} \), counted with multiplicity, that are enclosed in \( \gamma_1 \). By the choice of \( \gamma_1 \), we get \( n(f; \gamma_1, \frac{1}{2}) = m \).

Similarly, \( n(f; \gamma_2, \frac{1}{2}) = \frac{1}{2\pi i} \oint_{\gamma_2} \frac{f'(z)}{f(z) - \frac{1}{2}} \, dz \),

which is equal to the number of zeros of \( f(z) - \frac{1}{2} \), counted with multiplicity, that are enclosed in \( \gamma_2 \). By the choice of \( \gamma_2 \), we get \( n(f; \gamma_2, \frac{1}{2}) = l \). By \((*)\), \( n(f; \gamma_1, \frac{1}{2}) + n(f; \gamma_2, \frac{1}{2}) = m - m - l = 0 \). Thus,

\[
 n(f; \gamma_1, \frac{1}{2}) = n(f; \gamma_2, \frac{1}{2}) = 0.
\]

This means \( f(x) \) is homologous to 0 (mod \( 2\pi i \)), which is a contradiction.

We have proved that \( f(\frac{1}{2}) \neq f(-\frac{1}{2}) \). Therefore, there are only two possibilities, namely \((f(\frac{1}{2}) = \frac{1}{2}, f(-\frac{1}{2}) = -\frac{1}{2})\) and \((f(\frac{1}{2}) = -\frac{1}{2}, f(-\frac{1}{2}) = \frac{1}{2})\).

**Case 1** \( f(\frac{1}{2}) = \frac{1}{2} \) and \( f(-\frac{1}{2}) = -\frac{1}{2} \).

Let \( g \) be the linear fractional transformation under which

\[
1 \mapsto 1, \\
\frac{1}{2} \mapsto 0, \\
2 \mapsto \infty.
\]

This results in \( g(z) = (z, 1, 0, 2) = \frac{-2z + 1}{z - 2} \).

The inverse map is \( g^{-1}(w) = \frac{2w + 1}{w + 2} \).
Since $1$ is on the unit circle $(C)$, $2$ is the reflection point of $1$, and $w$ is the reflection point of $0$ with respect to $(C)$, $g$ maps $(C)$ to $(C)$ and maps the unit disk $D$ conformally to itself.

Put $h = g \circ f \circ g^{-1} : D \to D$. Then $g$

\[ h(1) = g \circ f \circ g^{-1}(1) = g \circ f \left(\frac{1}{2}\right) = g\left(\frac{1}{2}\right) = 0, \]

\[ h(-\frac{4}{5}) = g \circ f \circ g^{-1}\left(-\frac{4}{5}\right) = g \circ f\left(-\frac{1}{2}\right) = g\left(-\frac{1}{2}\right) = -\frac{4}{5}. \]

Therefore, by Schwartz's Lemma, $h$ is a rotation about the origin. Since $h$ fixes $-\frac{4}{5}$, $h$ is the identity map. Thus $f = g^{-1} \circ h \circ g$ is also the identity map.

**Case 2** $f\left(\frac{1}{2}\right) = -\frac{1}{2}$ and $f\left(-\frac{1}{2}\right) = \frac{1}{2}$.

Put $\tilde{f}(z) = f(-z)$. Then $\tilde{f}\left(\frac{1}{2}\right) = \frac{1}{2}$ and $\tilde{f}\left(-\frac{1}{2}\right) = -\frac{1}{2}$. We return to case 1 and conclude $\tilde{f}(z) = z \ \forall z \in D$. Thus $f(z) = -z \ \forall z \in D$.

We have proved that there are at most two functions $f$ satisfying the condition in the problem, namely $f_1(z) = z \ \forall z \in \Omega$ and $f_2(z) = -z \ \forall z \in \Omega$. Conversely, any cycle $\gamma$ in $\Omega$ is mapped to itself under $f_1$. Thus if $\gamma$ is not homologous to $0 \ (\text{mod} \ \Omega)$, $(f_1)_*(\gamma)$ is not either. Now suppose that $\gamma$ is a cycle that is not homologous to $0 \ (\text{mod} \ \Omega)$. Then either $n(\gamma, \frac{1}{2}) \neq 0$ or $n(\gamma, -\frac{1}{2}) \neq 0$. 
Since \( f_2 \) is the reflection about \( O \), \( (f_2)^e(x) \) is the reflection of \( x \) about \( 0 \). Thus \( n((f_2)^e(x), \frac{1}{2}) = n(x, -\frac{1}{2}) \) and \( n((f_2)^e(x), -\frac{1}{2}) = n(x, \frac{1}{2}) \). This means \( (f_2)^e(x) \) is not homologous to \( 0 \) (mod \( R \)).

In conclusion, all functions \( f \) satisfying the problem are \( f_1(x) \equiv x \) and \( f_2(x) \equiv -x \).

1. Put \( t = it \) where \( t \) is a positive real number. Consider the lattice in the complex plane generated by \( 1 \) and \( t \):

\[
\Lambda = \{ m + nt : m, n \in \mathbb{Z} \}
\]

Let \( \wp(t) \) be the Weierstrass function associated with the lattice \( \Lambda \).

\[
\wp(t) = \frac{1}{t^2} + \sum_{(m,n) \neq (0,0)} \left[ \frac{1}{(z + m + nt)^2} - \frac{1}{(z + m)^2} \right]
\]

(a) We'll show that \( \wp(t) \) is real-valued on the horizontal lines \( y = \frac{mt}{2} \) and on the vertical lines \( x = \frac{n}{2} \). Since \( \wp(t) \) is doubly periodic with periods \( 1 \) and \( t \), it suffices to show that \( \wp(t) \) is real-valued on 4 lines, namely the imaginary axis (except at the lattice points), the real axis (except at the lattice points), the line \( x = \frac{1}{2} \) and \( y = \frac{t}{2} \).

Consider the imaginary axis.
Let \( z \) be a point on the imaginary axis which is not a lattice point. We have \( \overline{z} = -z \) and \( \overline{z} = -z \). Then

\[
\overline{\Phi}(z) = \frac{1}{z^2} + \sum_{(m,n) \neq (0,0)} \left[ \frac{1}{(z+n+mc)^{n}} - \frac{1}{(n+mc)^{n}} \right]
\]

\[
= \frac{1}{z^2} + \sum_{(m,n) \neq (0,0)} \left[ \frac{1}{(-z+n+mc)^{n}} - \frac{1}{(n+mc)^{n}} \right] \quad (1)
\]

Since the lattice is invariant under the transformation \((m,n) \rightarrow (m,-n)\), we can replace \((m,n)\) by \((m,-n)\) in the summands. Then (1) becomes

\[
\overline{\Phi}(z) = \frac{1}{z^2} + \sum_{(m,n) \neq (0,0)} \left[ \frac{1}{(-z+n+mc)^{n}} - \frac{1}{(n+mc)^{n}} \right] = \Phi(z).
\]

Thus \( \Phi(z) \in \mathbb{R} \).

Consider the real axis

Let \( z \) be a point on the real axis which is not a lattice point. Then \( \overline{z} = z \).

\[
\overline{\Phi}(z) = \frac{1}{z^2} + \sum_{(m,n) \neq (0,0)} \left[ \frac{1}{(z+n+mc)^{n}} - \frac{1}{(n+mc)^{n}} \right]
\]

\[
= \frac{1}{z^2} + \sum_{(m,n) \neq (0,0)} \left[ \frac{1}{(z+n+mc)^{n}} - \frac{1}{(n+mc)^{n}} \right] \quad (2)
\]

Since the lattice is invariant under the transformation \((m,n) \rightarrow (-m,n)\), we can replace \((m,n)\) by \((-m,n)\) in the summands. Then (2) becomes

\[
\overline{\Phi}(z) = \frac{1}{z^2} + \sum_{(m,n) \neq (0,0)} \left[ \frac{1}{(z+n+mc)^{n}} - \frac{1}{(n+mc)^{n}} \right] = \Phi(z).
\]
Thus \( B(x) \in \mathbb{R} \).

Consider the line \( x = \frac{1}{2} \).

Let \( z \) be a point on the vertical line \( x = \frac{1}{2} \). Then \( z = \frac{1}{2} + iy \) and \( \overline{z} = \frac{1}{2} - iy \). Thus \( z + \overline{z} = 1 \), or equivalently \( \overline{z} = 1 - z \).

\[
\overline{B(z)} = \frac{1}{z^2} + \sum_{(m,n) \neq (0,0)} \left[ \frac{1}{(z + n + m \overline{z})^2} - \frac{1}{(n + m \overline{z})^2} \right]
\]

\[
= \frac{1}{(z - 1)^2} + \sum_{(m,n) \neq (0,0)} \left[ \frac{1}{(z - (n+1) + m \overline{z})^2} - \frac{1}{(z - n + m \overline{z})^2} \right]
\]

\[
= \frac{1}{(z - 1)^2} + \sum_{(m,n) \neq (0,0)} \left[ \frac{1}{(z - (n+1) + m \overline{z})^2} - \frac{1}{(z - n + m \overline{z})^2} \right]
\]

\[
+ \sum_{(m,n) \neq (0,0)} \left[ \frac{1}{(z - n + m \overline{z})^2} - \frac{1}{(z - n + m \overline{z})^2} \right] \quad (3)
\]

By replacing \((m,n)\) with \((m,-n)\), the second sum of RHS(3) is equal to \( B(z) - \frac{1}{2z} \). The first sum of RHS(3) can be rewritten as follows

\[
\sum_{m \in \mathbb{Z}^*} \left\{ \sum_{n \in \mathbb{Z}} \left[ \frac{1}{(z - (n+1) + m \overline{z})^2} - \frac{1}{(z - n + m \overline{z})^2} \right] \right\} + \sum_{n \in \mathbb{Z}^*} \left[ \frac{1}{(z - (n+1))^2} - \frac{1}{(z - n)^2} \right] \quad (4)
\]

We have

\[
A = \lim_{N \to \infty} \sum_{n=0}^{N} \left[ \frac{1}{(z - (n+1) + m \overline{z})^2} - \frac{1}{(z - n + m \overline{z})^2} \right]
\]
\[
B = \lim_{N \to \infty} \sum_{n=-N}^{N} \left[ \frac{1}{(2-(n+1))^{\nu}} - \frac{1}{(2-n)^{\nu}} \right] + \lim_{N \to \infty} \sum_{n=1}^{N} \left[ \frac{1}{(2-(n+1))^{\nu}} - \frac{1}{(2-n)^{\nu}} \right] \\
= \lim_{N \to \infty} \left( \frac{1}{2^{\nu}} - \frac{1}{(2+N)^{\nu}} \right) + \lim_{N \to \infty} \left( \frac{1}{(2-(N+1))^{\nu}} - \frac{1}{(2-1)^{\nu}} \right) \\
= \frac{1}{2^{\nu}} - \frac{1}{(2-1)^{\nu}}.
\]

Now, \(B = \frac{1}{2^{\nu}} - \frac{1}{(2-1)^{\nu}} \). Then (3) becomes

\[
\overline{f}(z) = \frac{1}{(z-1)^{\nu}} + \left( \frac{1}{z^{\nu}} - \frac{1}{(z-1)^{\nu}} \right) + (f(\bar{z}) - \frac{1}{\bar{z}^{\nu}}) = \overline{f}(\bar{z}).
\]

Thus, \(\overline{f}(\bar{z}) \in \mathbb{C}\).

Consider the line \(y = \frac{x}{2}\)

Let \(z\) be a point on the horizontal line \(y = \frac{\nu}{2}\). Then \(z = x + \frac{\nu}{2}\).

Thus \(\overline{z} = x - \frac{\nu}{2}\) and \(z - \overline{z} = ti = \tau\). Equivalently, \(\overline{z} = z - \tau\).

\[
\overline{f}(\tau) = \frac{1}{\tau^{\nu}} + \sum_{(m,n) \neq (0,0)} \left[ \frac{1}{(\overline{z}+n+mc)^{\nu}} - \frac{1}{(n+mc)^{\nu}} \right] \\
= \frac{1}{(2-\tau)^{\nu}} + \sum_{(m,n) \neq (0,0)} \left[ \frac{1}{(z+n-(m+1)c)^{\nu}} - \frac{1}{(n-mc)^{\nu}} \right] \\
= \frac{1}{(2-\tau)^{\nu}} + \sum_{(m,n) \neq (0,0)} \left[ \frac{1}{(z+n-(m+1)c)^{\nu}} - \frac{1}{(2+n-mc)^{\nu}} \right] \\
+ \sum_{(m,n) \neq (0,0)} \left[ \frac{1}{(z+n-mc)^{\nu}} - \frac{1}{(n-mc)^{\nu}} \right] (5)
\]
By replacing \((\min)\) with \((-\min)\), the second sum of \(\text{RHS}(5)\) is equal to \(\Phi(x) - \frac{1}{x^2}\). The first sum of \(\text{RHS}(5)\) can be rewritten as follows.

\[
\sum_{n \in \mathbb{Z}^*} \left\{ \sum_{m \in \mathbb{Z}} \left[ \frac{1}{(z+n-(m+1)c)^2} - \frac{1}{(z+n-mc)^2} \right] \right\} + \sum_{m \in \mathbb{Z}} \left[ \frac{1}{(z-(m+1)c)^2} - \frac{1}{(z-mc)^2} \right]
\]

We have

\[
C = \lim_{N \to \infty} \sum_{m=-N}^{N} \left[ \frac{1}{(z+n-(m+1)c)^2} - \frac{1}{(z+n-mc)^2} \right]
\]

\[
= \lim_{N \to \infty} \left[ \frac{1}{(z+n-(N+1)c)^2} - \frac{1}{(z+n+mc)^2} \right]
\]

\[
= 0,
\]

\[
D = \lim_{N \to \infty} \sum_{m=-N}^{N-1} \left[ \frac{1}{(z-(m+1)c)^2} - \frac{1}{(z-mc)^2} \right] + \lim_{N \to \infty} \sum_{m=1}^{N} \left[ \frac{1}{(z-(m+1)c)^2} - \frac{1}{(z-mc)^2} \right]
\]

\[
= \lim_{N \to \infty} \left( \frac{1}{z^2} - \frac{1}{(z+Nc)^2} \right) + \lim_{N \to \infty} \left( \frac{1}{(z-(N+1)c)^2} - \frac{1}{(z-c)^2} \right)
\]

\[
= \frac{1}{z^2} - \frac{1}{(z-c)^2}.
\]

Now \((6) = D = \frac{1}{z^2} - \frac{1}{(z-c)^2}\). Then \((5)\) becomes

\[
\Phi(x) = \frac{1}{(z-c)^2} + \left( \frac{1}{z^2} - \frac{1}{(z-c)^2} \right) + \left( \Phi(x) - \frac{1}{x^2} \right) = \Omega(x).
\]

Thus \(\Phi(x) \in \mathbb{R}\).

\((6)\) The grid lines \(x = \frac{n}{2}\) and \(y = \frac{mt}{2}\) divide the complex plane
into disjoint open rectangles. Put
\[ H^+ = \{ z \in \mathbb{C} : \text{Im} z > 0 \}, \]
\[ H^- = \{ z \in \mathbb{C} : \text{Im} z < 0 \}. \]
We'll show that \( \varphi \) maps each of these rectangles conformally to \( H^+ \) or \( H^- \). We'll prove this for the open rectangle \( \Omega \) whose vertices are at \( 0, \frac{1}{2}, \frac{1+i}{2}, \frac{i}{2} \).
The method shown in the following also applies for other rectangles.

First, we will show that \( \varphi \) is injective on \( \Lambda \). One of the problems in Homework 5 was Problem 1, Ahlfors page 276. It said that for any \( z, u \in \mathbb{C} \setminus \Lambda \),
\[ \varphi(z) - \varphi(u) = -\frac{\delta (z-u) \delta (z+u)}{\delta (z)^2 \delta (u)^2}, \]
where
\[ \delta(z) = z \prod_{w \in \Lambda^*} \left( 1 - \frac{z}{w} \right) e^{\frac{z^2}{2} + \frac{i}{2} \left( \frac{z}{w} \right)^2}. \]
We see that the zeros of \( \delta \) are exactly at the lattice points. Thus,
\[ \varphi(z) = \varphi(u) \iff \delta(z-u) = 0 \text{ or } \delta(z+u) = 0 \]
\[ \iff z-u \in \Lambda \text{ or } z+u \in \Lambda. \]
In case \( z, u \in \Omega \), we can write \( z = \alpha + \beta \omega, \ u = \gamma + \delta \omega \),
where \( 0 < \alpha, \beta, \gamma, \delta \leq \frac{1}{2} \). Then \( \frac{-1}{2} < \alpha - \gamma, \beta - \delta < \frac{1}{2} \) and \( \varphi(z) \in \Omega \).
Since \( z + u = (x+y) + (\beta + \delta)z \) and \( 0 < x+y, \beta + \delta < 1 \), it never lies on the lattice \( \Lambda \). Thus

\[
\begin{align*}
\{ \Phi(z) = \Phi(u) \} & \Rightarrow \{ z + u \in \Lambda \} \Rightarrow \{ (x-y) + (\beta - \delta)z \in \Lambda \}, \\
0 < x, \beta, \delta < 1/2.
\end{align*}
\]

This happens only if \( x - y = 0 \) and \( \beta - \delta = 0 \), i.e. \( z = u \). Therefore, \( \Phi \) is injective on \( \Lambda \).

Consequently, \( \Phi \) maps \( \Lambda \) conformally to a simply connected domain \( \Omega \) in \( \mathbb{C} \). We'll show that \( \Omega = H^+ \) or \( \Omega = H^- \).

In the following, we'll use the following theorem (Theorem 2, Ahlfors page 233):

Let \( f \) be a topological mapping of a region \( U \) to \( V \). If \( (z_n) \) tends to the boundary of \( U \) then \( (f(z_n)) \) tends to the boundary of \( V \).

Let \( (z_n) \) be any sequence in \( \Lambda \) which tends to a point \( a \in \partial \Lambda \). Since \( a \) lies on the boundary of \( \Lambda \), \( \Phi(a) = \lim_{n \to o} \Phi(z_n) \) lies on the boundary of \( \Omega \) by the above Theorem. As shown in part
(a), if \( x \in \mathcal{A} \) and \( a \neq 0 \) then \( \mathcal{P}(a) \in \mathbb{R} \). Thus \( \mathcal{P}(a) \neq \emptyset \).

Next, we'll show that \( \mathcal{D} \subset \mathbb{R} \). Suppose by contradiction that there exists a point \( w_0 \in \mathcal{D} \setminus \mathbb{R} \). Then \( w_0 \) is the limit of a sequence \( (w_n) \) in \( \mathcal{D} \). We apply the above Theorem for the map \( (\mathcal{P}(a))^{-1} \) and the sequence \( (w_n) \). Since \( \lim_{n \to \infty} w_n = w_0 \in \mathcal{D} \), \( \{ \mathcal{P}^{-1}(w_n) \} \) tends to the boundary of \( \mathcal{A} \). Since \( \overline{A} \) is compact, there exists a subsequence \( \{ \mathcal{P}^{-1}(w_{n_k}) \} \) that converges to some \( z_0 \in \mathcal{D} \).

Thus, from the beginning we could have taken \( (w_n) \) to be the subsequence \( (w_{n_k}) \) to get \( \lim_{n \to \infty} \mathcal{P}^{-1}(w_n) = z_0 \in \mathcal{D} \). We consider two cases of \( z_0 \), namely \( z_0 = 0 \) and \( z_0 \in \mathcal{D} \setminus \{0\} \).

- \( z_0 = 0 \) Then \( \lim_{n \to \infty} \mathcal{P}^{-1}(w_n) = 0 \), since \( 0 \) is a pole of \( \mathcal{P} \),

  \[
  \lim_{n \to \infty} \mathcal{P}(\mathcal{P}^{-1}(w_n)) = \infty, \text{ i.e. } \lim_{n \to \infty} w_n = \infty.
  \]

  This is a contradiction because \( (w_n) \) converges in \( \mathbb{C} \).

- \( z_0 \in \mathcal{D} \setminus \{0\} \)

  Then \( \mathcal{P} \) is continuous at \( z_0 \). Then \( \lim_{n \to \infty} \mathcal{P}(\mathcal{P}^{-1}(w_n)) = \mathcal{P}(z_0) \).

  Thus, \( \lim_{n \to \infty} w_n = \mathcal{P}(z_0) \in \mathbb{R} \). This contradicts the fact that
\[ \lim w_n = w_0 \in \mathbb{R}. \]

We have shown that \( \mathcal{D} \subset \mathbb{R} \). Next, we’ll show that \( \mathcal{D} = \mathbb{H}^+ \) or \( \mathcal{D} = \mathbb{H}^- \). Since \( \mathcal{D} \) is a nonempty open subset of \( \mathbb{C} \), it must intersect either \( \mathbb{H}^+ \) or \( \mathbb{H}^- \).

Suppose that \( \mathcal{D} \cap \mathbb{H}^+ \neq \emptyset \). Then \( \mathcal{D} \cap \mathbb{H}^+ \) is a nonempty open subset of \( \mathbb{H}^+ \). We’ll show that it is also closed in \( \mathbb{H}^+ \). Let \( (w_n) \) be a sequence in \( \mathcal{D} \cap \mathbb{H}^+ \) that converges to some \( w_0 \in \mathbb{H}^+ \). We need to show that \( w_0 \in \mathcal{D} \cap \mathbb{H}^+ \). Suppose this is not true. Then \( w_0 \notin \mathcal{D} \).

Since \( w_0 \) is the limit of a sequence in \( \mathcal{D} \), \( w_0 \in \mathbb{D} \mathcal{D} \). Thus \( w_0 \in \mathbb{R} \). This contradicts the fact that \( w_0 \in \mathbb{H}^+ \). Therefore, \( \mathcal{D} \cap \mathbb{H}^+ \) is nonempty open and closed in \( \mathbb{H}^+ \). Since \( \mathbb{H}^+ \) is connected, \( \mathcal{D} \cap \mathbb{H}^+ = \mathbb{H}^+ \). Thus, \( \mathbb{H}^+ \subset \mathcal{D} \).

We have proved that if \( \mathcal{D} \cap \mathbb{H}^+ \neq \emptyset \) then \( \mathbb{H}^+ \subset \mathcal{D} \). Similarly, we can show that if \( \mathcal{D} \cap \mathbb{H}^- \neq \emptyset \) then \( \mathbb{H}^- \subset \mathcal{D} \). Therefore, to show that \( \mathcal{D} = \mathbb{H}^+ \) or \( \mathcal{D} = \mathbb{H}^- \), we only need to show that \( \mathcal{D} \) cannot intersect both \( \mathbb{H}^+ \) and \( \mathbb{H}^- \). Suppose by contradiction that \( \mathcal{D} \cap \mathbb{H}^+ \neq \emptyset \) and \( \mathcal{D} \cap \mathbb{H}^- \neq \emptyset \). Then \( \mathbb{H}^+, \mathbb{H}^- \subset \mathcal{D} \). We showed earlier that \( \mathcal{D} \neq \emptyset \) and \( \mathcal{D} \subset \mathbb{R} \). Now we’ll show that \( \mathbb{R} \setminus \mathcal{D} \) is bounded.
Let \( x_1(t) = \frac{1}{2} - \frac{1}{2} t, \quad 0 < t < 1, \)
and \( x_2(s) = \frac{s}{2} - \frac{3}{2} s, \quad 0 < s < 1, \)
be the parametrization of the edges of \( G \) that contain \( O \). We know by the Theorem mentioned earlier that \( P \circ x_1(t) \) and \( P \circ x_2(s) \) lie on \( \mathcal{W} \) for all \( 0 < s, t < 1 \).

Since \( O \) is a pole of \( P \), \( \lim_{t \to 0} P(t) = \infty \). Thus,
\[
\lim_{t \to 1^+} P \circ x_1(t) = \pm \infty \quad \text{and} \quad \lim_{s \to 1^-} P \circ x_2(s) = \pm \infty. \quad (*)
\]

For \( s, t \in (0,1) \), we have
\[
x_1(t) + x_2(s) = \frac{1-t}{2} + \frac{1-s}{2},
\]
and
\[
x_1(t) - x_2(s) = \frac{1-t}{2} - \frac{1-s}{2}.
\]

Thus, \( x_1(t) + x_2(s) \) and \( x_1(t) - x_2(s) \) do not belong to the lattice \( \mathcal{W} \).

Thus, \( P(x_1(t)) \neq P(x_2(s)) \) for all \( t, s \in (0,1) \). Since \( P \circ x_1 \) and \( P \circ x_2 \) are continuous, 4 possibilities in (*) collapse into 2, namely
\[
\lim_{t \to 1^-} P(x_1(t)) = \infty, \quad \lim_{s \to 1^-} P(x_2(s)) = -\infty,
\]
and
\[
\lim_{t \to 1^-} P(x_1(t)) = -\infty, \quad \lim_{s \to 1^-} P(x_2(s)) = \infty.
\]

Due to the continuity of \( P \circ x_1 \) and \( P \circ x_2 \), the set \( P \circ x_1((0,1)) \cup P \circ x_2((0,1)) \) contains a subset \((-\infty, a) \cup (b, \infty)\) for some \( a, b \in \mathbb{R} \). Thus, \((-\infty, a) \cup (b, \infty) \subset \mathcal{W}\). Consequently, \( \mathbb{R} \setminus \mathcal{W} \subset [a, b] \). We have
proved that $IR \setminus \Omega$ is bounded in some interval $[a,b] \subset IR$. Since $(\partial \Omega) \cap IR \neq \emptyset$, there is $c \in (\partial \Omega) \cap [a,b]$. Let $\gamma$ be any loop enclosing $c$ but does not intersect the segment $[a,b]$. Then $\gamma \subset \Omega$ and $n(\gamma,c) \neq 0$. This is a contradiction because $\Omega$ is simply connected.

$(\Rightarrow)$ Suppose that $f$ is harmonic. Consider a closed disk $\overline{D}(z_0,r) \subset \Omega$. We want to show that $f(z_0) = \frac{1}{2\pi} \int_{\partial \overline{D}(z_0,r)} f(re^{it}) \, dt$.

By the compactness of $\overline{D}(z_0,r)$, there is $r' > r$ such that $\overline{D}(z_0,r) \subset D(z_0,r') \subset \Omega$. Since $f$ is harmonic in $D(z_0,r')$, which is a simply connected domain, there exists an analytic function $F: D(z_0,r') \to \mathbb{C}$ such that $f(z) = \text{Re} F(z)$, $\forall z \in D(z_0,r')$. Since $F$ is analytic, it satisfies the Mean-Value property (which is a consequence of Cauchy Integral Theorem):
\[ F(z_0) = \frac{1}{2\pi} \int_0^{2\pi} F(z_0 + re^{i\theta}) \, d\theta \]

By taking the real parts of both sides, we get
\[ f(t) = \frac{1}{2\pi} \int_0^{2\pi} f(t + re^{i\theta}) \, d\theta. \]

(\(\Leftarrow\)) Suppose that \( f \) satisfies the Mean Value property. We'll show that \( f \) is harmonic. First, we'll show prove the following lemma:

If \( u: \overline{D(z_0, r)} \to \mathbb{R} \) is a continuous function satisfying the Mean Value property, then \( u \) attains maximum and minimum on \( \partial D(z_0, r) \).

**Proof of the Lemma**

Since \( \overline{D}(z_0, r) \) is compact, \( u \) attains maximum and minimum in \( \overline{D}(z_0, r) \). Suppose that \( u \) attains maximum at some point in \( D(z_0, r) \), and the minimum value is \( a \). Then the set
\[ S = \{ z \in D(z_0, r) : u(z) = a \} \]

is nonempty. Since \( u \) is continuous, \( S \) is closed in \( D(z_0, r) \). Take any \( z_1 \in D(z_0, r) \) and any \( r_1 > 0 \) such that \( \overline{D}(z_1, r_1) \subset D(z_0, r) \).

Since \( u \) satisfies the Mean Value property, we have
\[ a = u(z_1) = \frac{1}{2\pi} \int_0^{2\pi} u(z_1 + r_1 e^{i\theta}) \, d\theta \]

Since \( a = \max_{\overline{D}(z_0, r)} u \), \( a \geq u(z_1 + r_1 e^{i\theta}) \) for all \( \theta \in [0, 2\pi] \). Thus
\[ a > \frac{i}{2\pi} \int_{0}^{2\pi} u(z + r e^{i\theta}) d\theta \]

Since the equality actually happens and \( u \) is continuous, we have \( u(z + r e^{i\theta}) = a \) for all \( \theta \in [0, 2\pi] \). Since \( r \) was chosen arbitrarily at as long as \( \overline{D}(z, r) \subset D(z_0, r) \), \( u(z) = a \) for all \( z \) in a neighborhood of \( z_0 \) in \( D(z_0, r) \). Thus \( S \) is open in \( D(z_0, r) \). Now that \( S \) is nonempty, open and closed in \( D(z_0, r) \), and that \( D(z_0, r) \) is connected, \( S = D(z_0, r) \). Thus \( u \) is a constant function. Thus, it attains maximum on \( \partial D(z_0, r) \).

For the case \( u \) attains minimum in \( D(z_0, r) \), \(-u\) attains maximum in \( D(z_0, r) \). By the above arguments, \(-u\) attains a constant map. Thus \( u \) attains minimum on \( \partial D(z_0, r) \).

To show that \( f \) is harmonic, we take any closed disk \( \overline{D}(z_0, r) \subset \mathbb{C} \) and show that \( f \) is harmonic in \( D(z_0, r) \). Since \( f \) is continuous on \( \partial D(z_0, r) \), the Schwartz's theorem says that the function \( g : D(z_0, r) \to \mathbb{C} \) defined by

\[ g(z) = \frac{1}{2\pi} \int_{0}^{2\pi} f(z_0 + re^{i\theta}) \text{Re} \left( \frac{re^{i\theta} - z + z_0}{re^{i\theta} - z + z_0} \right) d\theta \]

is harmonic, and that \( g \) extends continuously to \( \overline{D}(z_0, r) \), and that \( g = f \) on \( \partial D(z_0, r) \).
Since \( g \) is harmonic in \( \overline{D}(0, r) \), it satisfies Mean Value property as shown in the first part of this problem. Thus \( u = g - f \) also satisfies the Mean Value property. Since \( u \) is continuous in \( \overline{D}(0, r) \), by the above lemma, \( u \) attains minimum and maximum on \( \partial \overline{D}(0, r) \). On the other hand, \( u = 0 \) on \( \partial D(0, r) \). Thus \( u \) is identically zero in \( \overline{D}(0, r) \). Thus \( g = f \) in \( \overline{D}(0, r) \). Thus \( f \) is harmonic in \( D(0, r) \).

\( \textbf{4)} \) Let \( w \) be a meromorphic 1-form on the Riemann sphere \( \mathbb{C} \). We'll show that \( \sum_{p \in \partial \mathbb{C}} \text{Res}(w) = 0 \) \((\ast)\).

We know that \( \mathbb{C} \cup \{\infty\} \) has two charts: the finite chart, which is commonly identified with the complex plane, and the infinite chart, which contains \( \infty \). The transition map is \( z \mapsto \frac{1}{z} \). Also, we know that every meromorphic function on \( \mathbb{C} \cup \{\infty\} \) has the representation in \( \mathbb{C} \) as a rational function, namely \( r(z) \). Thus \( w = r(z) \, dz \) in \( \mathbb{C} \). The representation of \( w \) in the infinite chart is then obtained by replacing \( z \) with \( \frac{1}{z} \) and then applying the chain rule. Specifically, 

\[ w = r\left(\frac{1}{z}\right) \, d\left(\frac{1}{z}\right) = -\frac{1}{z^2} \, r\left(\frac{1}{z}\right) \, dz. \]
This means any meromorphic 1-form on \( \mathbb{C}^* \) is determined by its representation in the finite chart. Thus, we can just write \( w = r(z)dz \) to indicate a 1-form in \( \mathbb{C}^* \).

Every rational function \( r(z) \) on \( \mathbb{C} \) admits a partial fraction decomposition

\[
r(z) = Q(z) + \sum_{j=1}^{n} \sum_{k=1}^{e_j} \frac{p_{jk}(z)}{(z-z_j)^k},
\]

where \( Q(z), p_{jk}(z) \) are polynomials and \( \deg p_{jk} < k \). Thus \( w \) is a sum of simple meromorphic 1-forms.

\[
w = Q(z)dz + \sum_{j=1}^{n} \sum_{k=1}^{e_j} \frac{p_{jk}(z)dz}{(z-z_j)^k}.
\]

We know that residues satisfy the linear laws

\[
\text{Res}_p(w_1 + w_2) = \text{Res}_p(w_1) + \text{Res}_p(w_2),
\]

\[
\text{Res}_p(cw) = c\text{Res}_p(w), \quad \forall c \in \mathbb{C}.
\]

Thus it suffices to prove (*) for \( w \) of the following forms:

- \( t^k dt \), where \( k \geq 0 \),

- \( \frac{t^k}{(t-a)^j} \), where \( a \in \mathbb{C} \), \( 0 \leq k < j \).

Consider the case \( w = t^k dt \), \( k \geq 0 \)

Then \( w \) has no pole in the finite chart. In the infinite chart,

\[
w = \left( \frac{1}{t} \right)^k d\left( \frac{1}{t} \right) = -t^{k-2} dt.
\]
Then \( \text{Res}_\infty(w) = \text{Res}_{z=0}(-z^{-k-2}) = 0 \) because \(-k-2 \leq -2\). Thus \( \text{Res}_p(w) = 0 \) for all \( p \in \mathbb{C} \). Therefore (*) is satisfied.

Consider the case \( w = \frac{z^k}{(z-a)^j}, \quad a \in \mathbb{C}, \quad 0 \leq k < j \).

In the finite chart, \( w \) has a pole at \( z = a \). In the \( \wp \)-finite chart,

\[
 w = \frac{(\frac{1}{z})^k}{(\frac{1}{z}-a)^j} \quad d\left(\frac{1}{z}\right) = -\frac{1}{z^{k+2}}(1-a^2)^j \quad dt = -\frac{z^{-j-k-2}}{(1-at)^j} \quad dt.
\]

Then

\[
\sum_{p \in \mathbb{C}} \text{Res}_p(w) = \text{Res}_a(w) + \text{Res}_0(w)
\]

\[
= \underbrace{\text{Res}_{z=a} \left( \frac{z^k}{(z-a)^j} \right)}_{A} + \underbrace{\text{Res}_{z=0} \left( \frac{-z^{-j-k-2}}{(1-at)^j} \right)}_{B}
\]

Now we consider two cases of \( a \), namely \( a = 0 \) and \( a \neq 0 \).

- \( a = 0 \)

\[
 A = \text{Res}_{z=0} \left( \frac{z^k}{(z-a)^j} \right) = \text{Res}_{z=0} (z^{j-k}) = \begin{cases} 1 & \text{if} \quad k = j-1, \\ 0 & \text{if} \quad k < j-1. \end{cases}
\]

\[
 B = \text{Res}_{z=0} (-z^{-j-k-2}) = \begin{cases} -1 & \text{if} \quad k = j-1, \\ 0 & \text{if} \quad k < j-1. \end{cases}
\]

By (1) and (2), \( A + B \) is always zero.

- \( a \neq 0 \) We have \( z^k = [(z-a) + a]^k = \sum_{l=0}^{k} C_k^l (z-a)^l a^{k-l} \).

Thus, \( A = \text{Res}_{z=a} \left[ \sum_{l=0}^{k} C_k^l (z-a)^{l-j} a^{k-l} \right] \).
Since \( l \leq k < j \), we have \( l-j \leq -1 \). The equality occurs only if \( l=k=j-1 \). Thus,
\[
A = \begin{cases} 
\sum_{l} (a_l) \ell & \text{if } k=j-1, \\
0 & \text{if } k<j-1 
\end{cases}
\]
\[
= \begin{cases} 
1 & \text{if } k=j-1, \\
0 & \text{if } k<j-1 
\end{cases} 
\tag{3}
\]
We have \( \frac{1}{(1-az)^l} = \sum_{\ell=0}^\infty (az)\ell \) for all \( z \) in some neighborhood of 0.

Thus, \( B = \text{Res}_{z=0} \left[ -z^{j-k-2} \sum_{\ell=0}^\infty (az)\ell \right] = \text{Res}_{z=0} \left[ \sum_{\ell=0}^\infty -a \ell \ell^{j-\ell-2} \right] \)

Since \( j-1 \geq k \) and \( l \geq 0 \), \( j+l-k-2 \geq -1 \). The equality occurs only if \( l=0 \) and \( k=j-1 \). Thus,
\[
B = \begin{cases} 
-a^0 & \text{if } k=j-1, \\
0 & \text{if } k<j-1 
\end{cases} = \begin{cases} 
-1 & \text{if } k=j-1, \\
0 & \text{if } k<j-1 
\end{cases} 
\tag{4}
\]

By (3) and (4), \( A+B \) is always zero.

Let \( a \) and \( b \) be two nonnegative integers. Consider a polynomial over \( \mathbb{C} \) given by \( f(z,w) = z^{2a} - 2w^b z^a + 1 \). Let \( Y \) be the locus of roots of \( f(z,w) \). To have the connectedness of \( Y \), \( f \) is necessarily irreducible. If \( a = 0 \) then \( f(z,w) = -2w^b + 1 \), which is reducible because \( w = \sqrt[2]{\frac{1}{2}} \) is a root. If \( b = 0 \) then \( f(z,w) = z^{2a} - 2z^a + 1 = (z^{a} - 1)^2 \),
which is reducible. Therefore, a and b must be positive integers.

Next, we'll show that Y is actually a Riemann surface. To do so, we only need to show that \( f \) is irreducible and nonsingular on \( Y \).

**Show that \( f(z,w) \) is irreducible**

We can view \( f(z,w) \) as a polynomial \( \tilde{f}(w) = (-2z^a)w^b + (z^{2a} + 1) \) in \( \mathbb{C}[[z]][w] \). Let \( \xi \in \mathbb{C} \) be a root of \( z^{2a} + 1 \). Then \( z - \xi \) is a prime in \( \mathbb{C}[[z]] \) and \( (z - \xi) \) \( | \) \( z^{2a} + 1 \). Since \( \xi \neq 0 \), \( (z - \xi) \) \( | \) \( (2z^a) \). For the same reason, \( \xi \) is not a root of the derivative of \( z^{2a} + 1 \). Thus, \( (z - \xi)^2 \nmid (z^{2a} + 1) \). By the Eisenstein's Criterion of irreducibility, \( \tilde{f}(w) \) is irreducible over the field of fractions of \( \mathbb{C}[[z]] \). On the other hand, \( \text{cont}(\tilde{f}) = \gcd((-2z^a), z^{2a} + 1) = 1 \). Thus, \( \tilde{f}(w) \) is a primitive polynomial. Thus \( \tilde{f}(w) \) is also irreducible over \( \mathbb{C}[[z]] \). This means \( f(z,w) \) is irreducible in \( \mathbb{C}[[z]][w] \).

**Show that \( f(z,w) \) is nonsingular on \( Y \)**

Suppose by contradiction that \( f \) is singular at \( (z_0, w_0) \in Y \). Then

\[
f(z_0, w_0) = \frac{\partial f}{\partial z}(z_0, w_0) = \frac{\partial f}{\partial w}(z_0, w_0) = 0
\]

Equivalently,

\[
\begin{align*}
z_0^{2a} - 2w_0^b z_0^a + 1 &= 0 \quad (1) \\
2a z_0^{2a-1} - 2aw_0^b &= 0 \quad (2) \\
2bw_0^{b+1} z_0^a &= 0 \quad (3)
\end{align*}
\]
(3) implies \( u_0 = 0 \) or \( v_0 = 0 \). Then (2) implies \( u_0 = v_0 = 0 \). However, \( (u_0, v_0) = (0, 0) \) does not satisfy (1). This is a contradiction.

We have proved that \( Y \) is a Riemann surface. However, \( Y \) is not compact because it is an unbounded subset of \( \mathbb{C}^2 \). We'll compactify \( Y \) by adding two points \( (z, w) = (0, \infty) \) and \( (z, w) = (\infty, 0) \).

This process must be done carefully because we want the resulting space \( X \) to be a Riemann surface. Thus we need to resolve the singularities \( (0, \infty) \) and \( (\infty, 0) \) on \( Y \cup \{(0, \infty), (\infty, 0)\} \).

Resolving the singularity \( (0, \infty) \)

For each \( (z, w) \in Y \), \[ w^b = \frac{1}{2} z^{-a} (1 + z^{2a}) \quad (4). \]

For \( |z| < \frac{1}{2} \), \( 1 + z^{2a} \) lies on the right half plane. \[ \begin{array}{c}
| \end{array} \]

Thus \( 1 + z^{2a} \) has an analytic \( b \)th root, i.e. there exists an analytic function \( g : \text{ID}(0, \frac{1}{2}) \to \mathbb{C} \) such that \( g(z)^b = 1 + z^{2a} \). Then (4) \[ w^b = \frac{1}{2} z^{-a} g(z)^b \quad (5) \]

We consider 3 following cases:

- \( \gcd(a, b) = 1 \) Then there exist \( m \in \mathbb{Z} \) such that \( -an + bm = 1 \).

Define \( r(t) = (z, w) = \left( t^b, \frac{1}{\sqrt{2}} t^{-a} g(t^b) \right) \in Y \) for all \( t \in \text{ID}(0, \frac{1}{12}) \).

Then \( r \) is holomorphic and has an inverse map:
\[ s(z,w) = t = z^n \left( \frac{w^{\frac{1}{n}}}{g(z)} \right)^n. \]

\( s \) is also holomorphic and thus gives us a hole chart on \( Y \) around the point \((0,0)\). By plugging the hole, we have resolved the singularity. Then the coordinate representation of the projection map \( \pi : X \to \mathbb{C}_w \)
\[ \pi(z,w) = z \] around the new point \( p \) is \( t \mapsto z = t^b \). Thus \( \text{mult}_p \pi = b \).

\[ a = b \]

Then (5') \[ w^b = \frac{j}{2} \hat{z}^{-b} g(z) \Rightarrow \frac{j}{2} \hat{z}^{-b} (w - \sum_{j=0}^{b-1} z^j \frac{1}{j!} x^j y(z)) = 0, \]

where \( \hat{z} \) is a primitive \( b \)'th root of unity. Each factor defines a smooth curve which passes through the point \((0,0)\). Thus, removing \((0,0)\) then gives a space which decomposes into \( b \) smooth curves, each with a hole in it. For the \( j \)'th curve, we define

\[ r(t) = (z,w) = (t, s^j \frac{1}{\sqrt{2}} \frac{1}{t} \hat{z} x(t)) \quad \forall t \in \Omega(4 \frac{1}{2}) \setminus \{0\}. \]

Then \( r \) is holomorphic and has an inverse \( s(z,w) = t = z \). Since \( s \) is also holomorphic, it gives us a hole chart on the \( j \)'th curve. Plugging the hole resolves the singularity in this case. (This means we plugged \( n \) holes total, one for each smooth curve). The coordinate representation of the map \( \pi : X \to \mathbb{C}_w \) around each new point \( p \) is \( t \mapsto z = t \). Thus \( \text{mult}_p \pi = 1 \).
a \neq b \text{ and } \gcd(a_1, b_1) = k > 1

Then \( a = ka_1 \) and \( b = kb_1 \) with \( \gcd(a_1, b_1) = 1 \). Then

\[
(5) \Rightarrow (w^{b_1})^k = \frac{1}{2} \left( z^{-a} \right)^k \left( g(z^{b_1}) \right)^k
\]

\[
= \prod_{j=0}^{k-1} \left( w^{b_1} - \frac{1}{k^{b_1}} z^{-a_1} g(z^{b_1}) \right) = 0.
\]

Then we have \( k \) factors, each of which we know how to resolve (this was our first case). We have also seen that there is one hole to plug for each factor. There are \( k \) plugged holes and the multiplicity of \( \pi \) at each new point is multiple \( a_1 \).

We see that the last statement is true for all 3 cases.

Resolving the singularity \((0, 0)\)

We will use almost the same argument as we dealt with the singularity \((0, \infty)\). For each \((x, w) \in Y\),

\[
w^b = \frac{1}{2} z^a (1 + z^{-2a}) \quad (6)
\]

For \(|z| > 2\), \( 1 + z^{-2a} \) lies on the right half plane \( \uparrow \)

Thus, \( 1 + z^{-2a} \) has an analytic \( b' \)th root, namely \( h(z) : C \setminus D(0, 2) \rightarrow C \).

Then \( (6) \Rightarrow w^b = \frac{1}{2} z^a h(z)^b \).

From now, the procedure is just repeated as in the case we resolved \((0, \infty)\). The only difference is that \( a \) is replaced by \(-a\), and \( g \) is
replaced by \( \mathfrak{b} \). We get the similar conclusion: with \( \gcd(a,b) = k \) and \( b = b_1 + k \), there are \( k \) plugged holes and the multiplicity of \( \pi \) at each new point \( p \) is \( \text{mult}_p \pi = b_1 \).

Now we can compute \( \deg(\pi) \). Note that \( \pi^{-1}(0) \) has \( k \) elements, which are \( k \) plugged holes of type \( (0, \infty) \). Then

\[
\deg(\pi) = d_0(\pi) = \sum_{p \in \pi^{-1}(0)} \text{mult}_p(\pi) = kn = b.
\]

So far, we have determined 2 branch points of \( \pi \), which are not in \( \pi(Y) \), namely \( z = 0 \) and \( z = \infty \). Next, we will determine all branch points in \( \pi(Y) \). For each \( (z,w) \in Y \), we have \( w = \frac{1}{2} (z^a + z^{-a}) \).

Thus \( z \in \mathbb{C} \setminus \{0\} \) is a branch point of \( \pi \) if and only if \( z^a = -1 \).

Thus if \( b > 2 \), there are exactly \( 2a \) branch points in \( \mathbb{C} \setminus \{0\} \). They are the \( (2a) \) roots of \( -1 \), namely \( z_1, z_2, \ldots, z_{2a} \). Since the equation \( w^b = 0 \) has a zero at 0 with multiplicity \( b \), we have \( \text{mult}_{(z_j, 0)} \pi = b \) for all \( j = 1, 2, \ldots, 2a \). Therefore, we have a conclusion as follows.

Put \( b_1 = \frac{b}{\gcd(a,b)} \). If \( b = 1 \) then \( \pi \) has no branch point.

If \( b > 2 \):

\[ b_1 = \frac{b}{\gcd(a,b)} \]
If \( b_1 = 1 \): there are 2a branch points \( \xi_1, \xi_2, \ldots, \xi_a \) - the 2a \( \mu \)th roots of \(-1\).

If \( b_1 > 2 \): there are 2a + 2 branch points \( \xi_1, \xi_2, \ldots, \xi_a, 0, \infty \).

Now applying Hurwitz formula for the map \( \pi \), we have

\[
2g(X) - 2 = \deg(\pi)(2g(\Theta) - 2) + \sum_{j=1}^{2a} \left( \text{mult}_{\xi_j, 0} - 1 \right) + \sum_{b} \left( \text{mult}_{\xi_j, b} - 1 \right) + \sum_{\rho \in \pi^{-1}(\infty)} \left( \text{mult}_{\xi_j, b} - 1 \right)
\]

\( \Rightarrow 2g(X) - 2 = b(-2) + 2a(b-1) + k(b_1-1) + k(b_1-1) \)

\( \Rightarrow g(X) = ab - a - b + k(b_1 - 1) + 1 \)

\( \Rightarrow g(X) = ab - a - k + 1 \)

\( \Rightarrow a \chi_{\pi_1} - ab + a = a \chi_{\pi_1}(a, b) + 1 \).

6. Let \( X \) be the hyperelliptic surface defined by \( y^2 = x^5 - x \), put \( \pi_1, \pi_2 : X \to \mathbb{C}^\times \), \( \pi_1(x, y) = x \) and \( \pi_2(x, y) = y \). We want to find \( \text{div}(\pi_1) \) and \( \text{div}(\pi_2) \). By definition, \( \text{div}(\pi_1) = \sum_{p \in X} \text{ord}_p(\pi_1) \cdot p \) and \( \text{div}(\pi_2) = \sum_{p \in X} \text{ord}_p(\pi_2) \cdot p \).

Then the problem becomes finding the order of \( \pi_1 \) and \( \pi_2 \) at each point on \( X \).

Let \( X_1 = \{ (x, y) \in \mathbb{C}^2 : y^2 = x^5 - x \} \),

\( X_2 = \{ (x, w) \in \mathbb{C}^2 : w^2 = -z^5 + z \} \).
Then there is an isomorphism \( \phi: X_1 \backslash \{(0,0)\} \rightarrow X_2 \backslash \{(0,0)\} \)

\[
\phi(x,y) = (z,w) = \left( \frac{y}{x^2}, \frac{y^2}{x^3} \right).
\]

By the definition of hyperelliptic surfaces, \( X = (X_1 \cup X_2) / \phi \). We will determine the coordinate of \( X \) around \((x,y) = (0,0)\) and around \((x,y) = (\infty, \infty)\). Let

\[
\begin{align*}
    f(x,y) &= y^2 - x^5 + x, \\
    g(z,w) &= w^2 + z^5 - 2.
\end{align*}
\]

Let \( p_1 \in X \) be the point corresponding to \((x,y) = (0,0)\), and \( p_2 \in X \) be the point corresponding to \((x,y) = (\infty, \infty)\). Note that \( p_2 \) corresponds to \((z,w) = (0,0)\).

Since \( \frac{\partial f}{\partial x}(p_1) = \frac{\partial f}{\partial y}(0,0) = 1 \neq 0 \), \( x \) is a coordinate of \( X \) around \( p_1 \).

Since \( \frac{\partial g}{\partial z}(p_2) = \frac{\partial g}{\partial w}(0,0) = -1 \neq 0 \), \( w \) is a coordinate of \( X \) around \( p_2 \).

Consider \( \pi_1 \)

We concern only the zeros and the poles of \( \pi_1 \), because for any other point \( p \in X \), \( \text{ord}_p(\pi_1) = 0 \). We have

\[
\begin{align*}
    \pi_1(x,y) &= 0 \iff x = 0 \iff \begin{cases} x = 0, \\
    y = 0.\end{cases} \\
    \pi_1(x,y) &= \infty \iff x = \infty \iff \begin{cases} x = \infty, \\
    y = \infty.\end{cases}
\end{align*}
\]

Thus \( p_1 \) is the only zero of \( \pi_1 \), and \( p_2 \) is the only pole of \( \pi_1 \). Since \( y \) is a coordinate of \( X \) around \( p_1 \), the coordinate representation of \( \pi_1 \) is
\[ \Phi(y) \rightarrow x = \tau(y) \]. We have

\[ y^2 = x^5 - x = \tau(y)^5 - \tau(y). \]

Differentiate both sides with respect to \( y \), we get

\[ 2y = 5 \tau^4 \frac{dx}{dy} - \frac{dx}{dy} = (5 \tau^4 - 1) \frac{dx}{dy} \]

Thus \[ \frac{dx}{dy} = \frac{2y}{4 \tau^5 - 1} \] (*)

As a consequence, \[ \frac{dx}{dy}(0) = 0 \]. Differentiate both sides of (*) w.r.t. \( y \),

\[ \frac{d^2x}{dy^2} = \frac{\tau(4 \tau^5 - 1) - 2y \cdot 20 \tau^4 \frac{dx}{dy}}{(4 \tau^5 - 1)} \]

Thus, \[ \frac{d^2x}{dy^2}(0) = \frac{-2}{4} = -\frac{1}{2} \]

Thus means \( \tau(y) = -y^2 + O(y^3) \). Thus, \( \Phi(y) = \tau(y) = -y^2 + \text{higher orders} \)

Therefore, \[ \text{ord}_2(\Phi) = 2 \].

Since \( w \) is a coordinate of \( X \) around \( p_c \), the coordinate representation of \( \Phi \) is \( \Phi_w : w \rightarrow x = \frac{1}{w} = \frac{1}{\tau(w)} \). We have

\[ w^2 = -\tau^5 + 2 = -\tau(w)^5 + \tau(w) \]

Taking derivative both sides w.r.t. \( y \), we get

\[ 2w = -5 \tau^4 \frac{dz}{dw} + \frac{dz}{dw} = (1 - 5 \tau^4) \frac{dz}{dw} \]

Then \[ \frac{dz}{dw} = \frac{2w}{1 - 5 \tau^4} \] (**) \]

Consequently, \[ \frac{dz}{dw}(0) = 0 \]. Now take the derivative w.r.t. \( w \) both
Sides of (**) : \[
\frac{d^2 z}{dw^2} = \frac{2 \left(1 - 5 \bar{z}^4 \right) - 2 \bar{w} (-20) \bar{z}^3 \frac{d \bar{w}}{dz}}{(1 - 5 \bar{z}^4)^2}.
\]

Thus, \[
\frac{dz}{dw}(0) = \frac{2}{1} = 2 \neq 0.
\]

This means \( z(w) = w^2 + O(w^3) \). Thus
\[
\pi(z) = \frac{1}{z(w)} = w^2 + \text{higher order terms}.
\]

Thus \( \text{ord}_{p_2}(\pi_1) = -2 \). Therefore, \( \text{div}(\pi_1) = 2p_1 - 2p_2 \).

Consider \( \pi_2 \)
\[
\pi_2(x, y) = 0 \iff y = 0 \iff \begin{cases} x^5 - y = 0 \iff (x, y) \in \{(0, 0), (5, 0), (5^3, 0), (5^4, 0)\}
\end{cases}
\]

where \( 5 = i \). Let \( q_j \in X \) be the point in \( X \) having coordinate \( (x, y) = (5^j, 0) \)
for \( 1 \leq j \leq 4 \).

\[
\pi_2(x, y) = \infty \iff y = \infty \iff \begin{cases} x = \infty
\end{cases}
\]

Thus, \( p_1, q_1, q_2, q_3, q_4 \) are all zeros of \( \pi_2 \), and \( p_2 \) is the only pole of \( \pi_2 \). Since \( y \) is a coordinate of \( X \) around \( p_1 \), the coordinate representation of \( \pi_2 \) is \( \pi_2 : y \to y \). Thus \( \text{ord}_{p_1}(\pi_2) = 1 \).

For each \( j = 1, 2, 3, 4 \), we have \( \frac{\partial y_j}{\partial x}(q_j) = \frac{\partial y}{\partial x}(5^j, 0) = -5(5^j)^4 \bar{w} = -4t_0 \).
Thus \( y \) is a coordinate of \( X \) around \( q_j \). Then the coordinate representation of \( \pi_2 \) is \( \pi_2 : y \to y \). Thus \( \text{ord}_{q_j}(\pi_2) = 1 \).

Since \( w \) is a coordinate of \( X \) around \( p_2 \), the coordinate representation...
the relation \( \tilde{w} : w \mapsto y = w x^3 = \frac{w}{x^3} \).

We showed earlier in the case of \( \Phi_1 \) that \( \pi_1(w) = w + O(w^3) \) Thus,

\[ \tilde{\pi}_2(y) = \frac{w}{x^3} = \frac{w}{w^6 + O(w^7)} = w^{-5} + \text{higher-order terms} \]

Thus, \( \text{ord}_{\tilde{\pi}_2}(w) = -5 \). Therefore,

\[ \text{div}(\pi_2) = 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 - 5 - 5 - p_2 \]

Consider a homogeneous polynomial of 3 complex variables

\[ F(x_1, y_1, z_1) = x_1^3 + y_1^3 + z_1^3 \]

To show that \( F \) determines a smooth projective plane curve which is also a Riemann surface, we need to show that \( F \) is irreducible in \( \mathbb{C}[x_1, y_1, z_1] \) and nonsingular on its locus of roots. We can write

\[ F(x_1, y_1, z_1) = \tilde{F}(x) = z x^3 + y z^3 + y z^3 \in (\mathbb{C}[y_1, z_1])[x] \]

We know that \( y \) is a prime in \( \mathbb{C}[y_1, z_1] \) since it's irreducible. Also, \( y_1 y_1^3, y_1^2 y_1^3, y_1 z_1 \). Then by the Eisenstein's criterion, \( \tilde{F}(x) \) is irreducible over the field of fractions of \( \mathbb{C}[y_1, z_1] \). On the other hand,

\[ \gcd(z_1, y_1^3, y_1^3) = 1 \]

Thus \( \tilde{F}(x) \) is a primitive polynomial. Thus it is also irreducible over \( \mathbb{C}[y_1, z_1] \). Thus, \( F(x_1, y_1, z_1) \) is irreducible in \( \mathbb{C}[x_1, y_1, z_1] \).

Next, suppose by contradiction that \( F \) is singular at a point \( (x_0, y_0, z_0) \neq (0, 0, 0) \) on its locus of roots. Then
\[ F(x_0, y_0, z_0) = \frac{\partial F}{\partial x} (x_0, y_0, z_0) = \frac{\partial F}{\partial y} (x_0, y_0, z_0) = \frac{\partial F}{\partial z} (x_0, y_0, z_0) = 0. \]

Equivalently,
\[
\begin{align*}
x_0 y_0^3 + y_0 z_0^3 + z_0 x_0^3 &= 0 \\
y_0^3 + 3 x_0 z_0^2 &= 0 \\
x_0^3 + 3 y_0 x_0^2 &= 0 \\
x_0^3 + 3 y_0 z_0^2 &= 0
\end{align*}
\]

\[ \implies \begin{align*}
x_0 y_0^3 + y_0 z_0^3 + z_0 x_0^3 &= 0 \quad (1) \\
y_0^3 = -3 z_0 x_0^2 \quad (2) \\
x_0^3 = -3 x_0 y_0^2 \quad (3) \\
x_0^3 = -3 y_0 z_0^2 \quad (4)
\end{align*} \]

Multiplying (2), (3), (4) together, we get \( x_0 y_0^3 z_0^3 = -27 x_0^3 y_0^3 z_0^3 \). Thus one of \( x_0, y_0, z_0 \) must be zero. We can assume that \( x_0 = 0 \). Then (2) and (3) lead to \( y_0 = z_0 = 0 \). Thus \( (x_0, y_0, z_0) = (0, 0, 0) \), which is a contradiction.

Therefore, \( F \) determines a smooth projective plane curve, which is also a Riemann surface \( X = \{ [x : y : z] \mid F(x, y, z) = 0 \} \). Also, as a closed subset of \( \mathbb{P}^2 \), which is compact, \( X \) is compact. Since \( F \) is a polynomial of degree 4, by Plücker's formula, we get the genus of \( X \),
\[ g(X) = \frac{(4-1)(4-2)}{2} = 3. \]

Then by Hurwitz's theorem, \( |\text{Aut}(X)| \leq 84 (g(X)-1) = 84 \times 2 = 168 \).

Note that \( 168 = 3 \times 7 \times 2^3 \). We'll show that \( |\text{Aut}(X)| = 168 \) by showing that \( \text{Aut}(X) \) has an element of order 3, an element of order 7, and a subgroup of order 8. We have to do some preparation to prove this.
We recall that the projective plane is \( \mathbb{P}^2 = \{[x:y:z] \mid (x,y,z) \in \mathbb{C}^3 \setminus \{0\} \} \)

where \([x:y:z]\) stands for the orbit of \((x,y,z)\) under the action of \(\mathbb{C}^*\) upon \(\mathbb{C}^3 \setminus \{0\}\). As a matter of notation, we now write an element in \(\mathbb{C}^3 \setminus \{0\}\) as a column vector \(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\), and the orbit of its under \(\mathbb{C}^*\) as \(\{[x:y:z]\}\).

For each matrix \(A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \in \text{GL}(3, \mathbb{C})\), and \(\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{C}^3 \setminus \{0\}\), we have

\[
A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 x + a_2 y + a_3 z \\ b_1 x + b_2 y + b_3 z \\ c_1 x + c_2 y + c_3 z \end{pmatrix} \neq 0.
\]

This allows us to define a map \(\pi_A : \mathbb{P}^2 \to \mathbb{P}^2\), \(\pi_A\)\{\begin{pmatrix} x \\ y \\ z \end{pmatrix}\} = A\{\begin{pmatrix} x \\ y \\ z \end{pmatrix}\} = \{\begin{pmatrix} x \\ y \\ z \end{pmatrix}\}.

For \(A, B \in \text{GL}(3, \mathbb{C})\) and \(\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{C}^3 \setminus \{0\}\), we have

\[
\pi_A \circ \pi_B \{\begin{pmatrix} x \\ y \\ z \end{pmatrix}\} = \pi_A \{B\{\begin{pmatrix} x \\ y \\ z \end{pmatrix}\}\} = \{A(B\{\begin{pmatrix} x \\ y \\ z \end{pmatrix}\})\} = \pi_{AB} \{\begin{pmatrix} x \\ y \\ z \end{pmatrix}\}.
\]

Thus \(\pi_A \circ \pi_B = \pi_{AB}\). In particular, \(\pi_A\) is bijective and \((\pi_A)^{-1} = \pi_A^{-1}\).

Now suppose that we have a matrix \(A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \in \text{GL}(3, \mathbb{C})\)
such that \(\pi_A(X) \subset X\) and \(\pi_A^{-1}(X) \subset X\). We will show that \(\pi_A : \pi_A^{-1}(X) \to X\) is in \(\text{Aut}(X)\) - the group of all holomorphic automorphisms of \(X\).
Indeed, since $\pi_A(X) \subset X$ and $\pi_{A-1}(X) \subset X$, $\pi_A(X) = X$. Thus $\pi_A$ is a bijection from $X$ to itself. The inverse of $\pi_A$ is therefore $\pi_{A-1}$. Thus, we only need to show that $\pi_A$ is holomorphic.

Recall that $\mathbb{P}^2$ has 3 complex charts:

$$U_1 = \left\{ \left\{ \frac{x}{y} \in \mathbb{P}^2 \mid x \neq 0 \right\} \right\},$$

$$U_2 = \left\{ \left\{ \frac{y}{z} \in \mathbb{P}^2 \mid y \neq 0 \right\} \right\},$$

$$U_3 = \left\{ \left\{ \frac{x}{z} \in \mathbb{P}^2 \mid z \neq 0 \right\} \right\}.$$

Put $X_i = X \cap U_i$ for $i = 1, 2, 3$. Then $\{X_1, X_2, X_3\}$ is an open cover of $X$.

$$X_1 = \left\{ \left\{ \frac{a}{b} \right\} \mid F(a, a_1, b) = 0 \right\},$$

$$X_2 = \left\{ \left\{ \frac{a}{b} \right\} \mid F(a_1, b, b) = 0 \right\},$$

$$X_3 = \left\{ \left\{ \frac{a}{b} \right\} \mid F(a, b, b) = 0 \right\}.$$

Thus $X_1, X_2, X_3$ are smooth affine curves. Now take $p = \left\{ \frac{x_0}{y_0} \right\} \in X$.

We'll show that $\pi_A$ is smooth at $p$. We will prove this for the cases $p \in X_1$, $\pi_A(p) \in X_1$, and $p \in X_1$, $\pi_A(p) \in X_2$ only. Other cases will be done similarly. In both cases, on some neighborhood $U$ of $p$, either $u = \frac{y}{x}$ or $v = \frac{z}{x}$ is a coordinate.
\[ \tilde{\pi}_A(p) \in X_1 \]  

This means  
\[ \tilde{\pi}_A(p) = \left\{ \frac{a_1 x_0 + a_2 y_0 + a_3 z_0}{b_1 x_0 + b_2 y_0 + b_3 z_0}, \frac{c_1 x_0 + c_2 y_0 + c_3 z_0}{b_1 x_0 + b_2 y_0 + b_3 z_0} \right\} \subseteq X_1. \]

Then on some neighborhood \( V \) of \( \tilde{\pi}_A(p) \), either 

\[ w = \frac{b_1 x + b_2 y + b_3 z}{a_1 x + a_2 y + a_3 z} = \frac{b_1 + b_2 u + b_3 v}{a_1 + a_2 u + a_3 v} \]  
or  

\[ t = \frac{c_1 x + c_2 y + c_3 z}{a_1 x + a_2 y + a_3 z} = \frac{c_1 + c_2 u + c_3 v}{a_1 + a_2 u + a_3 v} \]  
is a coordinate.

Thus the coordinate representation of \( \tilde{\pi}_A \) on \( U \) is one of the following types: \( u \mapsto w, u \mapsto t, v \mapsto w, v \mapsto t. \) If \( u \) is a coordinate on \( U \) then \( w \) is a holomorphic function of \( u. \) Then \( w \) and \( t \) are holomorphic functions of \( u. \) If \( v \) is a coordinate on \( U, \) then \( u \) is a holomorphic function of \( v. \) Then \( w \) and \( t \) are holomorphic functions of \( v. \) Thus, in both cases, \( \tilde{\pi}_A \) is always holomorphic at \( p. \)

\[ \tilde{\pi}_A(p) \in X_2 \]  

This means  
\[ \tilde{\pi}_A(p) = \left\{ \frac{a_1 x_0 + a_2 y_0 + a_3 z_0}{b_1 x_0 + b_2 y_0 + b_3 z_0}, \frac{c_1 x_0 + c_2 y_0 + c_3 z_0}{b_1 x_0 + b_2 y_0 + b_3 z_0} \right\} \subseteq X_2. \]

Then on some neighborhood \( V \) of \( \tilde{\pi}_A(p) \), either 

\[ w = \frac{a_1 x + a_2 y + a_3 z}{b_1 x + b_2 y + b_3 z} = \frac{a_1 + a_2 u + a_3 v}{b_1 + b_2 u + b_3 v} \]  
or  

\[ t = \frac{c_1 x + c_2 y + c_3 z}{b_1 x + b_2 y + b_3 z} = \frac{c_1 + c_2 u + c_3 v}{b_1 + b_2 u + b_3 v} \]
is a coordinate. Thus, the coordinate representation of $\tilde{\pi}_A$ on $U$ is one of the following types: $u \mapsto w, u \mapsto t, v \mapsto w, u \mapsto t$. If $u$ is a coordinate on $U$ then $v$ is a holomorphic function of $u$. Then $w$ and $t$ are holomorphic functions of $u$. If $v$ is a coordinate on $U$ then $u$ is a holomorphic function of $v$. Then $w$ and $t$ are holomorphic functions of $v$.

In both cases, $\tilde{\pi}_A$ is always holomorphic at $p$.

So far, we have proved that if $\tilde{\pi}_A(X) \subset X$ and $\tilde{\pi}_A^{-1}(X) \subset X$ then $\tilde{\pi}_A \in \text{Aut}(X)$. This gives us a way to construct an element in $\text{Aut}(X)$. We want to relate the order of $\tilde{\pi}_A$ to the order of $A$. Put $E = \{ \lambda I_3 \mid \lambda \in \mathbb{C} \} \subset \text{GL}(3, \mathbb{C})$. Suppose $n \in \mathbb{N}$ is the smallest number such that $A^n \in E$. We'll show that $\text{ord}(\tilde{\pi}_A) = n$.

First, for each $\{ \frac{x}{z} \} \in X$, we have

$$\tilde{\pi}_A^n \{ \frac{x}{z} \} = \{ A^n \left( \frac{x}{z} \right) \} = \{ \frac{A^n(x)}{z} \} = \{ \frac{x}{z} \}.$$

Thus $\tilde{\pi}_A^n = \text{id}_X$. Thus, $\tilde{\pi}_A^{-1} = \text{id}_X$. Suppose by contradiction that there is $1 \leq m < n$ such that $\tilde{\pi}_A^m = \text{id}_X$. Then $\tilde{\pi}_A^m = \text{id}_X$. Then

$$\{ A^m \left( \frac{x}{z} \right) \} = \{ \frac{x}{z} \} \quad \forall \{ \frac{x}{z} \} \in X.$$

In other words, for any $\left( \frac{x}{z} \right) \in \mathbb{C}^3 \setminus \{0 \}$ that satisfies
\[ xy^3 + yz^3 + z^3 x^3 = 0, \quad (*) \]

there exists \( \lambda_{x,y,z} \in \mathbb{C}^* \) such that \( A^m \left( \frac{x}{y} \right) = \lambda_{x,y,z} \left( \frac{x}{y} \right) \).

Note that \( \left( \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right), \left( \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right), \left( \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right) \) satisfy (\( \ast \)). Put \( \lambda_1 = \lambda_{1,0,0}, \lambda_2 = \lambda_{0,1,0}, \lambda_3 = \lambda_{0,0,1} \).

We have
\[
A^m \left( \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right) = \lambda_1 \left( \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right), \quad A^m \left( \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right) = \lambda_2 \left( \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right), \quad A^m \left( \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right) = \lambda_3 \left( \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right).
\]

Thus
\[
A^m = \left( \begin{array}{ccc} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{array} \right).
\]

For \( x = y = 1 \), (\( \ast \)) becomes \( x^3 + x + 1 = 0 \), which has a root \( z_0 \in \mathbb{C}^* \). Thus
\[
\left( \begin{array}{c} 1 \\ 1 \\ z_0 \end{array} \right) \text{satisfies (\( \ast \)). Thus } A^m \left( \begin{array}{c} 1 \\ 1 \\ z_0 \end{array} \right) = \lambda_{1,1,z_0} \left( \begin{array}{c} 1 \\ 1 \\ z_0 \end{array} \right).
\]

Then \( \lambda_1 = \lambda', \lambda_2 = \lambda', \lambda_3 z_0 = \lambda' z_0 \). Thus \( \lambda_1 = \lambda_2 = \lambda_3 \). Therefore \( A \in E \).

This contradicts the minimality of \( n \). Thus \( n = \text{ord}(\tilde{\pi}_A) \).

So far, we showed that if \( A \in \text{GL}(3, \mathbb{C}) \) and \( n \in \mathbb{N} \) are such that \( \tilde{\pi}_A(X) \subset X, \tilde{\pi}_{A^{-1}}(X) \subset X \) and \( n \) is the smallest number such that \( A^n \in E \) is diagonalized, then \( \tilde{\pi}_A \in \text{Aut}(X) \) and \( \text{ord}(\tilde{\pi}_A) = n \). We will use this method to get some elements of \( \text{Aut}(X) \) together with their orders. Put
\[
A = \left( \begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{array} \right) \in \text{GL}(3, \mathbb{C}).
\]
Then \( A^2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad A^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3 \).

Thus \( 3 \) is the smallest \( n \) such that \( A^n \) is in \( E \). Moreover, \( A^{-1} = A^2 \).

For \( \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right\} \in X, \) we have \( \left\{ A \left( \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right) \right\} = \left\{ \begin{pmatrix} x \\ y \frac{3z}{2} + z^3 + xyz \end{pmatrix} \right\} \in X \) because \( y^2 + z^3 + 2x^2 + 3xyz = 0 \).

Thus \( \pi_A(X) \subset X \). Moreover, \( \pi_{A^{-1}}(X) = \pi_A^2(X) = \pi_A \left( \pi_A(X) \right) \subset \pi_A(X) \subset X \).

Therefore \( \pi_A \in \text{Aut}(X) \) and \( \text{ord}(\pi_A) = 3 \).

Put \( B = \begin{pmatrix} 1 & r & r^3 \\ & r & r^3 \end{pmatrix} \in \text{GL}(3, \mathbb{C}) \), where \( r \) is a primitive \( 7 \)-th root of unity.

Then \( B^j = \begin{pmatrix} 1 & r^j & r^{3j} \\ & r^j & r^{3j} \end{pmatrix} \) for any \( j \in \mathbb{N} \). Then \( B^j \in E \) if and only if \( r^j = r^{3j} = 1 \), which occurs only if \( j \equiv 0 \pmod{7} \). Thus the smallest \( j \in \mathbb{N} \) such that \( B^j \in E \) is \( 7 \). Moreover, \( B^7 = I_3 \) and so \( B^{-7} = B^6 \).

For \( \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right\} \in X, \) we have \( \left\{ B \left( \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right) \right\} = \left\{ \begin{pmatrix} x \\ y \frac{3z}{2} + z^3 +xyz \end{pmatrix} \right\} \in X \) because

\[
x(ry)^3 + (ry)(r^3z)^3 + (r^3z)(2z^3) = r^3(xy^3 + yz^3 + 2x^2z) = 0.
\]

Thus \( \pi_B(X) \subset X \). Moreover, \( \pi_B^{-1}(X) = \pi_B^6(X) = \pi_B \circ \pi_B \circ \cdots \circ \pi_B(X) \subset X \).

Therefore, \( \pi_B \in \text{Aut}(X) \) and \( \text{ord}(\pi_B) = 7 \).

Put \( c_r = \frac{i}{\sqrt{7}} \begin{pmatrix} t_1 & t_4 & t_2 \\ t_4 & t_2 & t_1 \\ t_2 & t_1 & t_4 \end{pmatrix} \) where \( t_j = r^j - r^{-j} \) and \( r = \exp \left( \frac{2 \pi i}{7} \right) \).
\[
\det(C) = \left(\frac{i}{\sqrt{7}}\right)^3 \left[ 3 \ t_1 t_2 t_4 - (t_1^3 + t_2^3 + t_4^3) \right] \quad (1)
\]

We will do some calculations about \( t_1, t_2, t_4 \) as follows.

\[
t_1^2 = (r-r^{-1})^2 = r^2 + r^{-2} - 2,
\]
\[
t_2^2 = (r^2-r^{-2})^2 = r^4 + r^{-4} - 2,
\]
\[
t_4^2 = (r^4-r^{-4})^2 = r^8 + r^{-8} - 2.
\]

Thus, \( t_1^2 + t_2^2 + t_4^2 = (r+r^2+\ldots+r^6)-6 \). Since \( 1, r, \ldots, r^6 \) are all roots of \( X^7 - 1 \), their sum is 0 by Viète's theorem. Thus,

\[
t_1^2 + t_2^2 + t_4^2 = -7 \quad (2)
\]

We have

\[
t_1 t_2 = (r-r^{-1})(r^2-r^{-2}) = (r^4+r^{-4})-(r+r^{-1}),
\]
\[
t_2 t_4 = (r^2-r^{-2})(r^4-r^{-4}) = (r+r^{-1})-(r^2+r^{-2}),
\]
\[
t_4 t_1 = (r^4-r^{-4})(r-r^{-1}) = (r^2+r^{-2})-(r^4+r^{-4}).
\]

Thus, \( t_1 t_2 + t_2 t_4 + t_4 t_1 = 0 \quad (3) \)

We have

\[
\text{Im}(t_j) = \text{Im}(r^j) - \text{Im}(r^{-j}) = \sin\left(\frac{2\pi j}{7}\right) - \sin\left(-\frac{2\pi j}{7}\right)
\]
\[
= 2 \sin\left(\frac{2\pi j}{7}\right)
\]

Thus, \( \text{Im}(t_1 t_2 + t_2 t_4 + t_4 t_1) = 2 \left( \sin\frac{2\pi}{7} + \sin\frac{4\pi}{7} + \sin\frac{8\pi}{7} \right) > 0. \quad (4) \)

We have

\[
(t_1 t_2 + t_2 t_4)^2 = (t_1 t_2 + t_2 t_4) + 2(t_1 t_2 t_4)
\]

\[
(2) \text{ and } (3) \Rightarrow -7.
\]

Then because of (4), \( t_1 t_2 + t_2 t_4 = i\sqrt{7} \quad (5) \)
We have \[ t_1 t_2 t_4 = (r-r^{-1})(r^2-r^{-2})(r^4-r^{-4}) = r^7 + r - r^5 - r^3 - r^{-3} - r^2 + r^{-2} + r^{-5} = (r-r^{-1}) + (r^2 - r^{-2}) + (r^4 - r^{-4}) = t_1 + t_2 + t_4, \] \[ t_1^3 = (r-r^{-1})^3 = r^3 - r^3 - 3 r r^{-1} (r-r^{-1}) = - t_4 - 3 t_1, \]
\[ t_2^3 = (r^2 - r^{-2})^3 = r^6 - r^6 - 3 r^2 r^{-2} (r^2 - r^{-2}) = - t_4 - 3 t_2, \]
\[ t_4^3 = (r^4 - r^{-4})^3 = r^{12} - r^{12} - 3 r^4 r^{-4} (r^4 - r^{-4}) = - t_4 - 3 t_4. \]
Thus, \[ t_1^3 + t_2^3 + t_4^3 = - 4 (t_1 + t_2 + t_4) \] \[ \text{Thus, } \det(C) \overset{\text{(6)}\text{and(7)}}{=} \left( \frac{c}{f^2} \right)^3 \frac{7}{f} (t_1 + t_2 + t_4) = - \frac{c}{f^2} \frac{7}{f} (i \sqrt{7}) = 1. \]
Thus, \( C \in \mathcal{E} \). Moreover, by (2) and (3), we have \( C^2 = I_3 \).
Thus 2 is the smallest number in such that \( C^n \in \mathcal{E} \). Moreover, \( C^4 = C \).

Now we'll only need to show that \( \pi_C(X) \subseteq Y \). We have
\[ \{C(y)\}_{y} = \left\{ \begin{array}{l} t_1 x + t_2 y + t_4 z \\ t_4 x + t_2 y + t_4 z \\ t_2 x + t_4 y + t_4 z \end{array} \right\} \]
Given that \((x,y,z) \in C^3 \setminus \{(0,0,0)\} \) satisfies
\[ ny^3 + y z^3 + z x^3 = 0, \quad \text{(**) \text{(*)}} \]
we'll show that \( x^3 + y^3 + z^3 + x x^3 = 0 \), where
\[ \alpha = t_1 x + t_4 y + t_2 z, \]
\[ \beta = t_4 x + t_2 y + t_1 z, \]
\[ \gamma = t_2 x + t_4 y + t_4 z. \]

We have
\[ t_1 t_2 = (r^3 - r^{-1})(r^3 + r^{-3} - 2), \]
\[ = r^5 + r^{-3} - 2r - r^{-1} - r^3 + 2r^{-1} \]
\[ = -2t_1 + t_2 + t_4, \]
\[ t_1^2 t_2 = (r^3 + r^{-1} - 2)(r^3 - r^{-3} - 2), \]
\[ = t_1 - 2t_2, \]
\[ t_2 t_4 = (r^3 + r^{-3} - 2)(r^3 + r^{-1} - 2), \]
\[ = t_2 - 2t_4, \]
\[ t_2^2 t_4 = (r^3 + r^{-1} - 2)(r^3 + r^{-3} - 2), \]
\[ = t_4 - 2t_1 - t_4, \]
\[ t_4^2 t_2 = (r^3 + r^{-3} - 2)(r^3 + r^{-1} - 2), \]
\[ = t_2 - 2t_4, \]
\[ t_4^2 t_4 = (r^3 + r^{-1} - 2)(r^3 + r^{-3} - 2), \]
\[ = -t_1 + t_2 - 2t_4, \]
\[ t_2^2 t_4 = (r^3 + r^{-3} - 2)(r^3 + r^{-1} - 2), \]
\[ = t_4 - 2t_1 - t_4, \]
\[ t_4^2 t_4 = (r^3 + r^{-1} - 2)(r^3 + r^{-3} - 2), \]
\[ = -2t_1 + t_2. \]

Also, we computed earlier that
\[ t_1^3 = -t_4 - 3t_1, \]
\[ t_2^3 = -t_1 - 3t_2, \]
\[ t_4^3 = -t_2 - 3t_4, \]
\[ t_1 t_2 t_4 = t_1 + t_2 + t_4. \]
With these identities, we now can compute $x \beta^3 + y \gamma^3 + z \delta^3$.

\[
\alpha^3 = (t_4 x^3 + t_4 y^3 + t_4 z^3) + 3 t_4 t_2 x^2 y + 3 t_4 t_1 t_2 x z + 3 t_4 t_1^2 x z + 3 t_4 t_2^2 x z + 3 t_2 t_4 x^2 y + 3 t_2 t_4 z^2 + 6 t_1 t_2 t_4 x y z = (-t_4 - 3 t_1) x^3 + (-t_2 - 5 t_4) y^3 + (-t_4 - 3 t_2) z^3 + 3 (-t_4 + t_2 - 2 t_4) x^2 y + 3 (-2 t_4 + t_1) x y^2 + 3 (-2 t_1 - t_2 + t_4) x z^2 + 3 (t_4 - 2 t_2) z^2 + 3 (t_1 - 2 t_2 - t_4) x^2 y + 3 (t_4 - 2 t_4) x y^2 + 6 (t_1 + t_2 + t_4) x y z.
\]

Then

\[
\gamma^3 = (-t_2 t_4 - 3 t_2 t_4) x^4 + (-t_2 t_1 - 3 t_1 t_4) y^4 + (-3 t_2 t_4 - t_1 t_4) z^4 + (-t_4 - 3 t_2 t_4 - 6 t_2 t_4 + 3 t_2 t_4) x^3 y + (3 t_1 t_4 - 6 t_1 t_4 - t_2 - 3 t_2 t_4) y^3 z + (3 t_2 t_4 - 6 t_1 t_1 - 3 t_2 t_4) x^2 z + (6 t_2 t_4 - t_1 t_4 - 3 t_2 t_4 - 3 t_4 + 3 t_2) x y z + (-3 t_2 t_4 + 3 t_1 t_4 + 3 t_4 - 3 t_2 t_4) y z^2 + (-3 t_2 t_4 + 3 t_1 - 3 t_4 - 6 t_1 t_4 + 3 t_2 - t_4) x z^2 + (-3 t_2 t_4 + 3 t_1 - 3 t_4 - 6 t_1 t_4) x^2 y + (-3 t_4 + 3 t_4 - 6 t_1 t_4 - 3 t_4) y^2 z + (-6 t_2 t_4 - 3 t_2 t_4 + 3 t_2 t_4) x^2 z + (6 t_2 t_4 - 3 t_4 - 3 t_2 t_4 + 3 t_2 t_4) x y z + (9 t_2 t_4 - 6 t_4 + 6 t_2 t_4) x y z + (-6 t_2 t_4 + 6 t_2 + 9 t_1 t_4) x y z.
\]

The formulae for $x \beta^3$ and $y \gamma^3$ are then obtained by replacing the triple $(t_1, t_2, t_4)$ by $(t_1, t_4, t_1)$ and $(t_4, t_4, t_2)$ respectively in the
formula of $8x^3$. Using the fact that $6t_2t_3t_4 + 4t_4^2 = 0$, we get
\[ 2p^3 + 8x_3^2 + 8x_3^3 = (-7t_4^2 - 7t_2^2 - 7t_3^2)x_3 + (-7t_4^2 - 7t_2^2 - 7t_3^2)y_3^2 \\
+ (-7t_4^2 - 7t_2^2 - 7t_3^2)z_3 \\
\]
by (2)
\[ 4g(x_3 + y_3^2 + z_3^2) = 0. \]

We have proved that $\pi_c(X) \subset X$. Thus, $\pi_c \in \text{Aut}(X)$ and $\text{ord}(\pi_c) = 2$. So far, we have found 3 elements of $\text{Aut}(X)$: $\pi_A$ with order 3, $\pi_B$ with order 7, and $\pi_C$ with order 2.

\[ \pi_A \left\{ \begin{array}{c} x \\ x \\ x \end{array} \right\} = \left\{ \begin{array}{c} y \\ z \\ x \end{array} \right\}, \]

\[ \pi_B \left\{ \begin{array}{c} x \\ y \\ z \end{array} \right\} = \left\{ \begin{array}{c} x \\ r^2 y \\ z \end{array} \right\} \text{ with } r = \exp \left( \frac{2\pi i}{7} \right), \]

\[ \pi_C \left\{ \begin{array}{c} x \\ y \\ z \end{array} \right\} = \left\{ \begin{array}{c} t_4 x + t_2 y + t_4 z \\ t_2 x + t_4 y + t_4 z \\ 6z + t_4 y + t_4 z \end{array} \right\} \text{ with } t_4 = r^2 - r^4. \]

Therefore, $|\text{Aut}(X)|$ is divisible by $3 \times 7 \times 2 = 42$. Since $|\text{Aut}(X)| \leq 168$, there are only 4 possibilities, namely $|\text{Aut}(X)| \in \{ 42, 84, 126, 168 \}$. We will show that only the case $|\text{Aut}(X)| = 168$ can happen. Denote
\[ G = \text{Aut}(X). \]
In all of these cases, $7 \mid |G|$ but $7^2 \not\mid |G|$. Thus all $7$-Sylow subgroups of $G$ are of order $7$. Moreover, by Sylow's theorem, all of these $7$-subgroups are conjugate and the number of them is congruent to $1$ (mod $7$). Denote $H = \langle \pi_b \rangle$ - the cyclic group generated by $\pi_b$. Let $N_H$ be the normalizer of $H$ in $G$. Then we also know that the number of subgroups of $G$ that are conjugate to $H$ is $(G:N_H)$.

Thus $(G:N_H) \equiv 1$ (mod $7$). Moreover, since $H \subseteq N_H$, $(G:N_H)$ divides $(G:H) = \frac{|G|}{7}$. Thus we have

$$
\left\{ \begin{array}{l}
(G:N_H) \equiv 1 \pmod{7} \\
(G:N_H) \mid \frac{|G|}{7} 
\end{array} \right.
$$

If $|G| = 42$ then

$$
\left( \Rightarrow \right) \left\{ \begin{array}{l}
(G:N_H) \equiv 1 \pmod{7} \\
(G:N_H) \mid 6 \\
\Rightarrow (G:N_H) = 1 \\
\Rightarrow H \text{ is normal in } G.
\end{array} \right.
$$

If $|G| = 84$ then

$$
\left( \Rightarrow \right) \left\{ \begin{array}{l}
(G:N_H) \equiv 1 \pmod{7} \\
(G:N_H) \mid 12 \\
\Rightarrow (G:N_H) = 1 \\
\Rightarrow H \text{ is normal in } G.
\end{array} \right.
$$
If $|G| = 126$ then 

\[(**+) \iff (G : N_4) \equiv 1 \pmod{7} \]
\[
(G : N_4) \mid 18
\]
\[
\iff (G : N_4) = 1
\]

\[\iff H \text{ is a normal subgroup of } G.\]

Therefore, if $|G| \neq 168$ then $H = \langle \pi_b \rangle$ must be a normal subgroup of $G$. However, we'll show that $H$ is actually not normal in $G$.

Therefore, if $|G| \neq 168$ then $H = \langle \pi_b \rangle$ must be a normal subgroup of $G$. However, we'll show that $H$ is actually not normal in $G$.

Suppose that $H$ is normal in $G$. Then $\pi_c \pi_b \pi_c \in H$, i.e. $\pi_c \pi_b \pi_c \in H$.

Then there exists $k \in K$ such that $\pi_c \pi_b = \pi_b k$. Thus $\pi_c \pi_b$ and $\pi_b$ differ by a factor $k \in K^*$. In particular, $\pi_c \pi_b$ must be diagonal because $\pi_b$ is diagonal. We have

\[
\pi_c \pi_b = \frac{1}{r} \begin{pmatrix}
    t_1 & t_4 & t_2 \\
    t_4 & t_2 & t_1 \\
    t_2 & t_1 & t_4
\end{pmatrix}
\begin{pmatrix}
    1 & 0 & 0 \\
    0 & r & 0 \\
    0 & 0 & r^3
\end{pmatrix}
\begin{pmatrix}
    t_1 & t_4 & t_2 \\
    t_2 & t_1 & t_4 \\
    t_2 & t_1 & t_4
\end{pmatrix}
\]

\[
\pi_c \pi_b = \frac{1}{r} \begin{pmatrix}
    t_1 & rt_4 & r^3t_2 \\
    t_4 & rt_2 & r^3t_1 \\
    t_2 & rt_1 & r^3t_4
\end{pmatrix}
\begin{pmatrix}
    t_1 & t_4 & t_2 \\
    t_4 & t_2 & t_1 \\
    t_2 & t_1 & t_4
\end{pmatrix}
\]

The coefficient at row 2, column 1, is supposed to be zero. However,

\[
t_1 t_4 + rt_2 t_4 + r^3 t_1 t_4 = (r - r^{-1})(r^4 - r^{-4}) + r(r^2 - r^{-2})(r^2 - r^{-4})
\]
\[+ r^3(r - r^{-1})(r^2 - r^{-2})\]
\[
\begin{align*}
&= \left( r^2 + r^2 - r^4 - r^4 \right) + r \left( r + r^{-1} - r^2 - r^{-2} \right) + r^3 \left( r + r^{-3} - r - r^{-1} \right) \\
&= r^2 + r^2 + 2 - 2r^4 - 2r^{-4} \\
&= r^4 \left( r^6 + 2r^4 + r^2 - 2r - 2 \right) \quad (8)
\end{align*}
\]

Since \( \Phi_7(X) = X^6 + X^5 + X^4 + X^3 + X^2 + X + 1 \) is the irreducible polynomial of \( r \) over \( \mathbb{Q} \), \( r \) is not a root of \( X^6 + 2X^4 + X^2 - 2X - 2 \). Thus (8) implies \( t_1 t_4 + r t_1 t_4 + r^3 t_1 t_2 \neq 0 \). This is a contradiction.

The above arguments also show that every subgroup of \( G \) containing \( \overrightarrow{A}, \overrightarrow{B}, \overrightarrow{C} \) must be of order 168, and thus must be \( G \). Therefore, \( G = \langle \overrightarrow{A}, \overrightarrow{B}, \overrightarrow{C} \rangle \), which implies

\[ G \simeq \frac{\langle A, B, C \rangle}{E} \]

Subgroup of \( GL(3, \mathbb{C}) \)

(recall that \( E = \{ \lambda I_3 : \lambda \in \mathbb{C}^* \} \))