Consider the initial-value problem
\[ u' = u(1-u), \quad u(0) = u_0. \quad (1) \]
(i) First, we find the explicit solution \( u(t; u_0) \) with \( u_0 \in (0, 1) \) for Problem (1).

The function \( f(u) = u(1-u) \) is in \( C^1(\mathbb{R}) \). Thus, it is locally Lipschitz. By
Picard–Lindelöf theorem, Problem (1) has a unique local solution. Consequently, it has unique solution on the maximal interval of existence. If \( u_0 = 1 \), we can check that the constant function \( u(t) \equiv 1 \) solves Problem (1).

It is the unique solution. Now consider the case \( u_0 \in (0, 1) \).

The solution \( u \) is continuous at \( t = 0 \), so \( u(t) \) remains in \((0,1)\) if \( t \) is close to 0. For this reason, we can assume \( u(t) \in (0,1) \) while manipulating the differential equation. We have
\[
\frac{\dot{u}}{u(1-u)} = 1.
\]
Taking the antiderivative both sides with respect to \( t \), we get
\[
\int \frac{du}{u(1-u)} = \int dt.
\]
Thus,
\[
\log \frac{u}{1-u} = t + C_1.
\]
Taking the exponential of both sides, we get
\[
\frac{u}{1-u} = C_1 e^t, \quad \text{for some constant } C_1 > 0.
\]
Thus \( u(t) = \frac{C_1 e^t}{1 + C_1 e^t} \). The condition \( u(0) = u_0 \) gives \( C_1 = \frac{u_0}{1-u_0} \). Hence,
\[ u(t) = \frac{u_0 e^t}{1 - u_0 + u_0 e^t} \] \quad (2)

We see that \( u(t) \) remains in \((0,1)\) for all \(t \in \mathbb{R} \). Thus, \( u \) given by (2) is the unique solution to Problem (1). Because of the uniqueness of solutions, we can view \( u \) as a function of \( t \) and \( u_0 \).

\[ u(t; u_0) = \frac{e^t}{C + e^t}, \quad \text{where} \quad C = \frac{1 - u_0}{u_0}. \] \quad (3)

Next, we compute the partial derivatives of \( u \).

\[ \partial_{u_0} u(t; u_0) = \frac{-e^t \partial_{u_0} C}{(C + e^t)^2} = \frac{e^t}{u_0^2(C + e^t)^2} = \frac{e^t}{(1 - u_0 + u_0 e^t)^2} \] \quad (4)

\[ \partial_t u(t; u_0) = \frac{e^t(C + e^t) - e^t e^t}{(C + e^t)^2} = \frac{C e^t}{(C + e^t)^2}. \] \quad (5)

(ii) The equations (1) can be written as

\[ \partial_t u(t; u_0) = u(t; u_0) - u(t; u_0)^2, \quad u(0; u_0) = u_0. \] \quad (6)

We only consider the case \( u_0 \in (0,1) \). Differentiating both sides with respect to \( u_0 \), we get

\[ \partial_{u_0} \partial_t u(t; u_0) = \partial_{u_0} u(t; u_0) (1 - 2u(t; u_0)). \]

Let \( \dot{v}(t) = \partial_{u_0} u(t; u_0) \). We get \( \frac{d}{dt} \partial_t u(t; u_0) = \dot{v}(t) (1 - 2u(t; u_0)) \). Because \( u(0; u_0) = u_0 \) for all \( u_0 \in (0,1) \), \( \partial_{u_0} u(0; u_0) = 1 \). Thus, \( \dot{v}(0) = 1 \). The initial-value problem that \( v \) solves is

\[ \dot{v} = (1 - 2u) v, \quad v(0) = 1, \] \quad (7)

where \( u \) is the function given by (3).

Differentiating both sides of (6) with respect to \( t \), we get

\[ \partial_{tt} u(t; u_0) = \partial_u u(t; u_0) (1 - 2u(t; u_0)) = \partial_{uu} u(t; u_0) (1 - 2u(t; u_0)) \]
\[ u(t; u_0) (1 - u(t; u_0)) (1 - 2 u(t; u_0)) = u(t; u_0) (1 - u(t; u_0)) (1 - 2 u(t; u_0)). \]

Put \( w(t) = \frac{1}{2} u(t; u_0) \). We get \( \frac{dw}{dt}(t) = u(t; u_0) (1 - u(t; u_0)) (1 - 2 u(t; u_0)). \) Because \( \frac{\partial u}{\partial u}(0; u_0) = u(0; u_0) (1 - u(0; u_0)) = u_0 (1 - u_0) \), \( w \) solves the initial-value problem

\[ \dot{w} = u(1 - u)(1 - 2u), \quad w(0) = u_0 (1 - u_0), \quad (8) \]

where \( u \) is the function given by (3).

We now solve (7). Because \( v(0) = 1 \), \( v(t) \) remains positive when \( t \) is close to 0. Thus,

\[ \frac{d}{dt} \frac{v}{v} = 1 - 2u = 1 - \frac{2e^t}{C + e^t} = \frac{C - e^t}{C + e^t}. \]

Taking the antiderivative both sides with respect to \( t \), we get

\[ \log v = \int \frac{C - e^t}{C + e^t} \, dt \quad \frac{y = C + e^t}{y = C + e^t} \int \frac{2(C - y)}{y} \, dy = \int \left( \frac{2}{y} + \frac{1}{y - C} \right) dy \]

\[ = \log \frac{y - C}{y^2} + \text{const}. \]

Thus, \( v(t) = C_2 \frac{y - C}{y^2} = \frac{C_2 e^t}{(C + e^t)^2} \). The condition \( v(0) = 1 \) yields \( C_2 = (C + 1)^2 \).

Thus,

\[ v(t) = \frac{(C + 1)^2 e^t}{(C + e^t)^2} = \frac{e^t}{(1 - u_0 + u_0 e^t)^2}. \]

This confirms the formula (4).

Now we solve (8), by (3),

\[ u(1 - u)(1 - 2u) = \frac{e^t}{C + e^t} \cdot \frac{C + e^t}{C + e^t} = \frac{C e^t (C - e^t)}{(C + e^t)^3}. \]

Thus,

\[ w(t) = \int u(1 - u)(1 - 2u) \, dt = \int \frac{C e^t (C - e^t)}{(C + e^t)^3} \, dt \]

\[ \frac{y = C + e^t}{y = C + e^t} \int \frac{C(y - C)(2C - y)}{y^3} \, dy = \int \frac{C(y - C)}{y^2} \, dy = \frac{C(y - C)}{y^2} + \text{const} \]
\[ \frac{C e^t}{(C+e^t)^2} \]

The condition \( w(0) = u_0(t-u_0) = \frac{C}{(C+1)^2} \) yields \( w(t) = \frac{C e^t}{(C+e^t)^2} \). This confirms formula (5).

(2) Let \( f : \mathbb{R}^d \rightarrow \mathbb{R}^d \) be a continuous function such that \( f(x) \cdot x \leq 0 \) for all \( x \) with \( |x| \) sufficiently large. We show that the initial-value problem

\[ \begin{aligned}
  \dot{x} &= f(x), \\
  x(0) &= x_0
\end{aligned} \tag{1} \]

has at least one local solution, and that every local solution can extend to a global solution (i.e., for all \( t \geq 0 \)).

Step 1: Show that Problem (1) has a solution on the interval \([0, 1]\).

There is a number \( R > |x_0| \) such that \( f(x) \cdot x \leq 0 \) for all \( x \in \mathbb{R}^d \), \( |x| \geq R \). For \( 0 < \varepsilon < 1 \), we define \( g(x) = f(x) - \varepsilon x \). Let \( \psi : \mathbb{R}^d \rightarrow \mathbb{R} \) be a function such that

\[
\begin{cases}
  \psi \in C^\infty(\mathbb{R}^d) \\
  \psi(x) \geq 0 \quad \forall x \in \mathbb{R}^d \\
  \text{supp } \psi \subset B_1(0), \\
  \int_{\mathbb{R}^d} \psi \, dx = 1.
\end{cases}
\]

Such a function exists, for example \( \psi(x) = \begin{cases} C \exp\left(\frac{1}{|x|^2 - 1}\right), & |x| < 1 \\
0, & |x| \geq 1. \end{cases} \)

For each \( n \in \mathbb{N} \), we define \( g_n(x) = n^d \psi(nx) \). Then the sequence \((g_n)\) is an approximate identity on \( \mathbb{R}^d \). Let \( g_n = \psi_n \ast g \). Then \( g_n \in C^\infty(\mathbb{R}^d, \mathbb{R}^d) \) and \((g_n)\) converges to \( g \) uniformly on every bounded subset of \( \mathbb{R}^d \). There exists \( N \in \mathbb{N} \) such that

\[ |g_n(x) - g(x)| < \frac{\varepsilon}{2} \quad \forall x \in \mathbb{R}^d, \quad 1 < 2R, \quad \forall n \geq N \]
By shifting the indices of the sequence \( g_n(x) \) if necessary, we can assume \( N = 1 \).

Thus,

\[
\begin{align*}
g_n(x) - y_n(x) &= (g_n(x) - y_n(x)) \cdot x + g_n(x) - y_n(x) \\
&\leq |g_n(x) - y_n(x)| |x| + f(x) \cdot x - \varepsilon |x|^2 \\
&\leq \frac{\varepsilon}{2} |x| - \varepsilon |x|^2 = \varepsilon |x| \left( \frac{R}{2} - |x| \right) < 0 \quad \forall x \in \mathbb{R}^d, \quad R \leq |x| < 2R. \tag{2}
\end{align*}
\]

We first show that the initial-value problem

\[
x' = g_n(x), \quad x(0) = x_0, \tag{3}
\]

has a global solution (i.e., for all \( t \geq 0 \)). Because \( g_n \in C^1(\mathbb{R}^d, \mathbb{R}^d) \), it is locally Lipschitz. By Picard-Lindelöf theorem, Problem (3) has a unique local solution.

Let \( x \) be the unique solution in a minimal interval \( t \in [0, \alpha) \) for \( 0 < \alpha \leq \infty \).

Suppose by contradiction that \( \alpha < \infty \). We show that \( |x(t)| \leq R \) for all \( t \in [0, \alpha) \).

\[
p := \inf \{ s \in [0, \alpha) : |x(s)| \leq R \quad \forall t \in [0, s] \}.
\]

Suppose that \( p < \alpha \). Then \( |x(p)| = R \) because of the continuity of \( x \). For every \( k \in \mathbb{N} \), there exists \( t_k \) such that \( p < t_k < \min \{ p, p + \frac{1}{k} \} \) and \( |x(t_k)| > R \). Then

\[
\frac{d|x|^2}{dt}(p) = \lim_{k \to \infty} \frac{|x(t_k)|^2 - |x(p)|^2}{t_k - p} = \lim_{k \to \infty} \frac{|x(t_k)|^2 - R^2}{t_k - p} \geq 0.
\]

On the other hand, by taking the dot product by \( x \) both sides of (3), we get\( x \cdot x = g_n(x) \cdot x \). Thus,

\[
\frac{1}{2} \frac{d|x|^2}{dt}(p) = g_n(x(p)) \cdot x(p) \leq 0. \tag{2}
\]

This is a contradiction. Hence, \( \alpha = \infty \) and \( |x(t)| \leq R \) for all \( t \in [0, \alpha) \).

Let \( (t_k) \) be a sequence in \( [0, \alpha) \) that converges to \( \alpha \). Because \( (x(t_k)) \) is
bounded in $\mathbb{R}^d$, it has a converging subsequence. By replacing $(t_k)$ by this subsequence, we can assume $(x(t_k))$ converges to some $a \in \mathbb{R}^d$. We show that

$$\lim_{s \to a^-} x(s) = a.$$ 

Denote $\tilde{M} = \max_{|x| \leq R} |g_n(x)|$.

$$x(s) = x(t_k) + x(s, x(t_k) (s-t_k) = x(t_k) + g_n(x(t_k))(s-t_k),$$

for some $\tilde{x}_{s,k}$ lying between $s$ and $t_k$. Thus,

$$|x(s) - x(t_k)| = |g_n(x(t_k))| |s-t_k| \leq M|s-t_k| \quad \forall k \in \mathbb{N}.$$ 

Letting $k \to \infty$, we get $|x(s) - a| \leq M|s-a|$. Hence, $\lim_{s \to a^-} x(s) = a$. In addition,

$$\dot{x}(s) = g_n(x(s)) \to g_n(a) \quad \text{as} \quad s \to a^-.$$ 

Thus, $x$ and $\dot{x}$ extend continuously to $a$. The initial-value problem

$$\dot{x} = g_n(x), \quad x(0) = a$$

has a unique local solution in some interval $(a-s, a+s)$ by Picard-Lindeberg's theorem. It leads to a continuation of the solution to Problem (3) beyond $a$.

This is a contradiction. Thus, $a = \infty$. In other words, Problem (3) has a unique global solution, say $x_0$. We also showed that

$$|x_0(t)| \leq R \quad \forall t \geq 0 \quad (4)$$

Next, we show that the initial-value problem

$$\dot{x} = g(x), \quad x(0) = x_0 \quad (5)$$

has a solution on $[0, 1]$. Denote $\overline{B_R} = \{x \in \mathbb{R}^d : |x| \leq R\}$. Recall that $C(\overline{B_R})$ is a normed space with $\|u\|_{\overline{B_R}} = \sup_{x \in \overline{B_R}} |u(x)|$. Viewing $g_n, g$ and $f$ as
functions on $\overline{B_R}$, we have
\[ \|g_n\|_{B_R} = \|g_n \cdot g\|_{B_R} \leq \|g\|_{B_R} = \|f - \epsilon \sigma\|_{B_{R+1}} \leq \|f\|_{B_{R+1}} + \epsilon (R+1). \] (6)

Put $M = \max_{1 \leq k \leq R+1} |f(x)| + R+1$. Then
\[ |g_n(x)| \leq M \quad \forall x \in \overline{B_R}, n \in \mathbb{N}. \]
Then, $|x_n(t) - x_m(t)| = |g_n(x)(t)| \leq M |t-s| \quad \forall n \in \mathbb{N}.$ (7)

For $0 \leq s, t \leq 1$,
\[ |x_n(t) - x_n(s)| = |g_n(x)(t-s)| \leq M |t-s| \quad \forall n \in \mathbb{N}. \]

If we view $x_n$ as its restriction on $[0,1]$, $x_n \in C([0,1])$. By (4), $(x_n)$ is bounded. By (7), $(x_n)$ is equicontinuous. By Arzelà-Ascoli theorem, $(x_n)$ has a convergent subsequence. By replacing $(x_n)$ by this subsequence, we can assume $(x_n)$ converges to some $x_e \in C([0,1])$.

Recall that $C([0,1])$ is a normed space with $\|\cdot\|_{[0,1]} = \sup_{t \in [0,1]} |x(t)|$. We have
\[ \|g_n(x_n) - g(x_e)\|_{[0,1]} \leq \|g_n(x_n) - g(x_n)\|_{[0,1]} + \|g(x_n) - g(x_e)\|_{[0,1]} \]
\[ \leq \|g_n - g\|_{\overline{B_R}} + \|g(x_n) - g(x_e)\|_{[0,1]}. \] (8)

Because $g$ is continuous on the compact set $\overline{B_R}$, it is uniformly continuous there. For each $s > 0$, there exists $s' > 0$ such that $|g(x) - g(y)| < s$ for all $x, y \in \overline{B_R}$, $|x-y| < s'$. There exists $N \in \mathbb{N}$ such that $\|x_n - x_e\|_{[0,1]} < s'$ for all $n > N$. Thus,
\[ \|g(x_n) - g(x_e)\|_{[0,1]} < s \quad \forall n \in \mathbb{N}, n > N. \]

Then (8) implies $\|g_n(x_n) - g(x_e)\|_{[0,1]} < \|g_n - g\|_{\overline{B_R}} + s \quad \forall n > N$.

Because $\|g_n - g\|_{\overline{B_R}} \to 0$ as $n \to \infty$, $g_n(x_n) \to g(x_e)$ in $C([0,1])$. Thus, $x_n \to g(x_e)$.
in $C([0,1])$.

$$x_n(t) = x_0 + \int_0^t \dot{x}_n(s) \, ds \to x_0 + \int_0^t g(x(s)) \, ds.$$ 

Thus,

$$x(t) = x_0 + \int_0^t g(x(s)) \, ds.$$ 

This implies $x$ is a solution to Problem (5). We also showed that

$$\|x_\varepsilon\|_{[0,1]} \leq R \quad \forall \varepsilon \in (0,1).$$

Next, we show that Problem (1) has a solution on $[0,1]$. We have

$$|x_\varepsilon(t) - x(s)| = |\dot{x}_\varepsilon(\xi)| |t-s| = |g(x_\varepsilon(\xi))| |t-s| \leq M |t-s| \quad \forall \xi \in [t,s], \forall \varepsilon \in (0,1).$$

Thus, the set $\{x_\varepsilon : \varepsilon \in (0,1)\}$ is bounded and equiconvergent in $C([0,1])$. By Arzelà-Ascoli theorem, there is a convergent sequence $(x_n)$ in $C([0,1])$. Denote $x_\varepsilon \to x_n$. Then

$$\|g(x_\varepsilon) - f(x_n)\|_{[0,1]} = \varepsilon \|x_\varepsilon\|_{[0,1]} \to 0 \quad \text{as} \quad n \to \infty.$$

Because $f$ is uniformly continuous on $\overline{B_R}$, we have

$$\|f(x_\varepsilon) - f(x_n)\|_{[0,1]} \to 0 \quad \text{as} \quad n \to \infty.$$ 

Thus, $(\|g(x_\varepsilon) - f(x)\|_{[0,1]} \to 0 \quad \text{as} \quad n \to \infty$. Equivalently, $\|x_\varepsilon - f(x)\|_{[0,1]} \to 0 \quad \text{as} \quad n \to \infty$.

$$x_\varepsilon(t) = x_0 + \int_0^t x_\varepsilon(s) \, ds \to x_0 + \int_0^t f(x(s)) \, ds, \quad \forall t \in [0,1]$$

Hence,

$$x(t) = x_0 + \int_0^t f(x(s)) \, ds, \quad \forall t \in [0,1].$$

This means $x$ is a solution on the interval $[0,1]$ to Problem (1).

**Step 2:** Suppose $x$ is a solution on an interval $[0,\delta], \delta > 0$, to Problem (4). We show that $x$ can extend to a solution on $[0,\infty)$.

To do so, it suffices to show that $x$ can extend to a solution on $[0,\delta + 1]$. ...
Indeed, by applying this result repetitively, we can extend \( x \) to a solution on \([0, \delta + 2]\), then \([0, \delta + 3]\), and so on. By Step 1, there is a solution \( y \) to the initial value problem
\[
y = f(y), \quad y(\delta) = x(\delta)
\]
on the interval \([0, \delta]\). By defining \( z(t) := y(t-\delta) \) for \( t \in [\delta, \delta + 1] \), we get \( z \in C([0, \delta + 1]) \) and \( z = f(z) \) \( \forall t \in [0, \delta + 1] \setminus \{\delta\} \).

Because \( \lim_{t \to \delta} z(t) = \lim_{t \to \delta} f(x(t)) = f(x(\delta)) \), \( z \) is differentiable at \( t = \delta \) and \( z(\delta) = f(x(\delta)) \). Therefore, \( z = f(z) \) for all \( t \in [0, \delta + 1] \).

(3) Theorem 7.6 in Amann “Ordinary Differential Equations” 1990 (page 100) gives us a criterion for global existence of solutions to an ODE. We state a consequence which is enough for our purpose.

Let \( f: \mathbb{R}^d \to \mathbb{R}^d \) be a locally Lipschitz function and \( u_0 \in \mathbb{R}^d \). Let \( (\alpha, \beta) \subset \mathbb{R} \) be the maximal interval of existence for the initial-value problem
\[
\dot{u} = f(u), \quad u(0) = u_0.
\]
If \( \alpha > -\infty \) then \( \lim_{t \to \alpha^+} |u(t)| = \infty \). If \( \beta < \infty \) then \( \lim_{t \to \beta^-} |u(t)| = \infty \).

In the following, we study whether each given initial-value problem has a global solution (i.e. for all \( t \in \mathbb{R} \)). Denote by \((\alpha, \beta)\) the maximal interval of existence. Because initial conditions are specified at \( t = 0 \), \( \alpha < 0 < \beta \).

(i) \( \dot{x} = \sin x \), \( x(0) = x_0 \). \hfill (1)

In this case, \( f: \mathbb{R} \to \mathbb{R} \), \( f(x) = \sin x \). Because \( f \in C^1(\mathbb{R}) \), it is locally Lipschitz.
\[
x(t) = x(0) + t \dot{x}(0) = x_0 + t \sin x_0 \quad \forall t \in (\alpha, \beta).
\]
Thus, $\|x(t)\| \leq |x_0| + |t|$ for all $t \in (x, \beta)$. This implies that $x$ does not blow up in finite time. Therefore, Problem (1) has a global solution.

(ii) $x = x - x^3$, $x(0) = 2$. \hfill (2)

$f(x) = x - x^3$ is locally Lipschitz because it is in $C^1(\mathbb{R})$. Suppose there is $t_0 \in (x, \beta)$ such that $x(t_0) = 1$. Then the constant function $y_1(t) = 1$ solves the problem $\dot{y} = y - y^3$, $y(0) = x(t_0)$. On the other hand, the function $y_2(t) = x(t_0)$ also solves this problem. By the uniqueness of solutions, $x(t + t_0) = 1$ for all $t$. This is a contradiction because $x(0) = 2$. Therefore, $x(t) \neq 1$ for all $t \in (x, \beta)$.

Because $x$ is continuous on $(x, \beta)$ and $x(0) > 1$, $x(t) > 1$ for all $t \in (x, \beta)$. Then $\dot{x}(t) = x(t) - x(t)^3 < 0$. Thus, $x$ is decreasing on $(x, \beta)$. Then $x(t) \geq x(0) = 2$ for all $t \in (x, 0)$. Then
\[
\dot{x}(t) = x(t) - x(t)^3 \leq x(t)^2 - x(t)^3 = -x(t)^2 + x(t)^2 (2 - x(t)) \leq -x(t)^2 + x(t)^2 = 0
\]

Thus,
\[
\int_0^t \frac{x(s)}{x(s)^2} \, ds \leq \int_0^t -1 \, ds \quad \forall t \in (x, 0),
\]

which gives
\[
-\frac{1}{x(0)} + \frac{1}{x(t)} \leq t \quad \forall t \in (x, 0).
\]

Because $x(0) = 2$, $\frac{1}{x(t)} \leq t + \frac{1}{2}$ \quad \forall t \in (x, 0). \hfill (3)

If $x < -\frac{1}{2}$ then $-\frac{1}{2} \in (x, 0)$. Substituting $t = -\frac{1}{2}$ into (3), we get $x(-\frac{1}{2}) < 0$. This is a contradiction. Therefore, $x \geq -\frac{1}{2}$. It implies that the local solution to Problem (2) cannot extend beyond $t = -\frac{1}{2}$.

(iii) $x = x - x^3$, $x(0) = \frac{1}{2}$. \hfill (4)
The function $f$ is the same as in Problem (2). By the same argument as in part (ii), we can show that $x(t) \notin \{0, 1\}$ for all $t \in (\alpha, \beta)$. Because $x$ is continuous on $(\alpha, \beta)$ and $x(0) \in (0, 1)$, $x(t) \in (0, 1)$ for all $t \in (\alpha, \beta)$. The boundedness of $x(t)$ implies $\alpha = -\infty$ and $\beta = +\infty$. In other words, Problem (4) has a global solution.

(iv) \[ \dot{x} = \frac{4x^3 + y}{x^2 + y^2 + 1}, \quad \dot{y} = \frac{y^5 - x^5}{x^4 + y^4 + 1}, \quad x(0) = x_0, \quad y(0) = y_0. \quad (5) \]

In this case, $f: \mathbb{R}^2 \to \mathbb{R}^2$, $f(x, y) = \left( \frac{4x^3 + y}{x^2 + y^2 + 1}, \frac{y^5 - x^5}{x^4 + y^4 + 1} \right)$.

$f$ is locally Lipschitz because it is in $C^1(\mathbb{R}^2, \mathbb{R}^2)$. Denote the maximal interval of existence by $\mathcal{I}$ (instead of $(\alpha, \beta)$ to avoid confusion of notations). Put $u = (x, y)$ and $u_0 = (x_0, y_0)$. Problem (5) is equivalent to

\[ \dot{u} = f(u), \quad u(0) = u_0. \quad (6) \]

Then

\[ u(t) = u_0 + \int_0^t \dot{u}(s) \, ds = u_0 + \int_0^t f(u(s)) \, ds \quad \forall t \in \mathcal{I}. \]

Thus

\[ \|u(t)\| \leq \|u_0\| + \left| \int_0^t \|f(u(s))\| \, ds \right| \quad \forall t \in \mathcal{I}. \quad (7) \]

We now estimate $\|f(u)\|$.

\[ |4x^3 + y| \leq 4|x|^3 + |y| \leq 4\|u\|^3 + \|u\| = \|u\| (4\|u\|^2 + 1) \leq 4\|u\| (\|u\|^2 + 1) \]

\[ |y^5 - x^5| \leq |y|^5 + |x|^5 \leq 2\|u\|^5, \]

\[ x^4 + y^4 + 1 \geq \frac{1}{2} (x^2 + y^2)^2 + 1 = \frac{1}{2} \|u\|^4 + 1. \]

Hence

\[ \|f(u)\|^2 = \frac{|4x^3 + y|^2}{(x^2 + y^2 + 1)^2} + \frac{|y^5 - x^5|^2}{(x^4 + y^4 + 1)^2} \leq \frac{16\|u\|^8 (\|u\|^2 + 1)^2}{(\|u\|^4 + 1)^2} + \frac{4\|u\|^{10}}{(\frac{1}{2} \|u\|^4 + 1)^2} \]
\[
\leq 161u^2 + \frac{161u^2 \left( \frac{1}{2} 16u^4 \right)^2}{\left( \frac{1}{2} u^4 + 1 \right)^2} \leq 161u^2 + 161u^2.
\]

Thus, \(|f(u)| \leq 4\sqrt{2}u_1u_1\). Substituting this estimate into (7), we get

\[
|u(t)| \leq |u_0| + \left| \int_0^t 4\sqrt{2} |u(s)| \, ds \right| \quad \forall t \in J.
\]

Gronwall's inequality then gives us an explicit estimate for \(|u(t)|\):

\[
|u(t)| \leq |u_0| + 4\sqrt{2} |u_0| \int_0^t e^{4\sqrt{2} (t-s)} \, ds \quad \forall t \in J.
\]

This implies that \(u(t)\) does not blow up in finite time. Therefore, Problem (6) has a global solution (i.e. for all \(t \in \mathbb{R}\)).