(1) Consider \( \dot{u} = f(u) + \varepsilon g(t,u) \) with \( f, g \in C^1, f(0) = 0, \) and \( A = Df(0) \) hyperbolic, for \( \varepsilon \) small.

(i) Show that the linearization \( \frac{d}{dt} - A \) is invertible as an operator from \( C^1(\mathbb{R}, \mathbb{R}^n) \) into \( C^0(\mathbb{R}, \mathbb{R}^n) \).

(ii) Show that there exists a small neighborhood of the origin \( U(0) \) so that for all \( \varepsilon \) sufficiently small, there exists a unique solution \( u(t) \) such that \( u(t) \in U(0) \) for all \( t \in \mathbb{R} \).

(iii) Show that \( u(t) \) is 1-periodic if \( g(t,u) \) is 1-periodic in \( t \).

(2) Consider
\[
\begin{align*}
x' &= -x + ay^2 \\
y' &= -2y + bx^2.
\end{align*}
\]

(i) Draw the phase portrait of the linearization and express trajectories as graphs \( x = h(y) \) or \( y = h(x) \).

(ii) Find the Taylor jet of the (smooth) strong stable manifold, \( x = h(y) \) for the nonlinear system up to order two.

(iii) Try to find a “weak stable” manifold tangent to \( \{y = 0\} \), by calculating a quadratic Taylor jet of \( y = h(x) \) — what goes wrong?

(iv) Set \( a = 0 \) and compute the solutions explicitly. Express \( y \) as a function of \( x \). Show that there are many invariant manifolds \( y = h(x) \) but none is \( C^2 \) because of terms of the form \( x^2 \log x \).

(3) Consider the linear equation \( \dot{x} = Ax, A = \text{diag} (\lambda_j), \lambda_1 > \lambda_2 > \ldots > \lambda_n \).

(i) Derive an equation for the projectivized flow, that is, write \( x = u \cdot |x| \) and derive an equation for \( u \in S^{n-1} \). Find all equilibria of this flow on the sphere.

(ii) Show that the Rayleigh quotient \( V(u) = -\frac{1}{2} \langle Au, u \rangle \) is a strict Lyapunov function, that is, strictly decreasing for non-equilibrium stolutions. Which equilibria are stable?

(iii) Conclude that all trajectories are heteroclinic and describe heteroclinic orbits.

(iv) Describe equilibria and heteroclinic orbits for the (non-self-adjoint) \( A = \begin{pmatrix} 0 & 1 \\ \mu & 0 \end{pmatrix} \) for all \( \mu \in \mathbb{R} \)?
(v) *Alternative to the above:* Create a phase portrait on $S^2$ numerically when $\lambda_j = -j$.

(4) Implement classical Runge-Kutta for the competing species problem studied in class and demonstrate numerically that the method is of order 4.

(5) Study error propagation numerically in the Lorenz model,

\[
\begin{align*}
    x' &= \sigma(y - x) \\
    y' &= x(\rho - z) - y \\
    z' &= xy - \beta z,
\end{align*}
\]

$\sigma = 10$, $\beta = 8/3$, $\rho = 28$. Therefore, start with $x = y = z = 0.1$ and integrate using Euler and/or Matlab’s RK solver. Compare the solutions at various time intervals and for various step sizes: when do solutions differ qualitatively? Now study the difference between the solutions with initial conditions $x = y = z = 0.1 + \varepsilon$, $\varepsilon$ small. Demonstrate numerically that the difference grows exponentially for a certain time.

(6) Find (complex) stability regions for

\[
\begin{align*}
    u_{n+1} &= u_n + hf(u_n + \frac{h}{2} f(u_n)), \\
    u_{n+1} &= u_n + \frac{h}{2} (f(u_n) + f(u_{n+1})).
\end{align*}
\]

*Alternative/Optional:* Find the (real) stability boundaries numerically.

*Homework is due on Wednesday, November 26, in class. Choose three, or more for extra credit.*