(1) Consider
\[ A' = B, \quad B' = -A + A^2 \bar{A}, \quad A, B \in \mathbb{C}, \]
which possesses a family of periodic orbits parameterized by \( k \in (-1, 1), \)
\[ A_*(t) = Re^{ikt}, \quad B_*(t) = i k Re^{ikt}, \quad R = \sqrt{1 - k^2}. \]
Our goal is to compute the Floquet multipliers for the linearization at \((A_*, B_*)(t)\).

(i) Convince yourself that the linearization can be written in the form
\[ A' = B, \quad B' = -A + 2R^2 A + R^2 e^{2ikt} \bar{A}, \]
\[ \bar{A}' = \bar{B}, \quad \bar{B}' = -\bar{A} + 2R^2 \bar{A} + R^2 e^{-2ikt} A. \]
Verify that \((A'_*, B'_*, \bar{A}'_*, \bar{B}'_*)\) are solutions to this linear equation.

(ii) Show that the change of coordinates \((A, B, \bar{A}, \bar{B}) = (e^{ikt}a, e^{ikt}b, e^{-ikt}\bar{a}, e^{-ikt}\bar{b})\)
transforms the system into a constant-coefficient equation. Interpret this change
of coordinates in terms of Floquet theory, that is, find \(Q(t)\) and \(B\).

(iii) Find the eigenvalues of \(B\) and their multiplicities for \(k^2 < 1/3, k^2 > 1/3,\) and \(k^2 = 1/3.\)

(2) Compute the Floquet exponents of the periodic orbit in the van der Pol oscillator
numerically using Euler’s method. Therefore, consider
\[ x' = \frac{1}{\varepsilon} (x - x^3 - y), \quad y' = x - y \]
For the following, \(\varepsilon = 0.3\) and \(dt = 0.001\) are reasonable choices.

(i) Write a simple Euler scheme starting with \(x = 0.1, y = 0.1,\) say, and iterate
for a long time \((T = 100,\) say); verify that the trajectory “looks” periodic,
eventually.

(ii) With initial condition taken as the final value from the previous step, integrate
until \(y = 0\) and \(x > 0\) (for instance use a while loop to check for sign change
of \(y\) during iteration).

(iii) Compute one further loop to compute the periodic orbit for precisely one more
period; test the dependence of the period on \(\varepsilon\) and \(dt.\)
(iv) To this last loop over one period, add the Euler time stepping for the linear variational equation with initial conditions \((1, 0)^T\) and \((0, 1)^T\) (this is two separate Euler steppings!). The final values put together as a matrix give \(\Phi_{T,0}\) (why?), the eigenvalues are the Floquet multipliers. Also compute the Floquet exponents (for checking: \(\lambda_1 \sim 0, \lambda_2 \sim -1.7\)).

*Due December 10 in class. Choose one of the two exercises or two from the last homework (that you did not already complete).*