1. Identify the circle $S^1$ with $[0,1] / \sim_1$. Let $x \in (0,1) \setminus \mathbb{Q}$, $J_0 = [0, 1 - x)$, $J_1 = (-x, 1)$. Let $R_x : S^1 \to S^1$, $R_x(x) = (x + \alpha) \mod 1$ be the rigid circle rotation.

(i) For $x \in J_1$, we determine the smallest positive integer $k$ such that $R_x^k(x) \in J_1$. For $k \in \mathbb{N}$, write $x + k\alpha = m_k + \beta_k$ where $m_k = \lfloor x + k\alpha \rfloor$ is the integer part of $x + k\alpha$. Then

$$k = \frac{\beta_k - x + m_k}{\alpha} \in \mathbb{N}$$

If $m_k = 0$ then $k = \frac{\beta_k - x}{\alpha} < \frac{1 - (1 - x)}{\alpha} = 1$.

If $m_k = 1$ then $k = \frac{\beta_k - x + 1}{\alpha}$. To have $\beta_k \in J_1$, we require

$$\frac{x - 1}{\alpha} - 1 = \frac{(1 - x) - x + 1}{\alpha} \leq k < \frac{1 - x + 1}{\alpha} = \frac{2 - x}{\alpha}.$$  

The interval $[\frac{2 - x}{\alpha} - 1, \frac{2 x}{\alpha}]$ is of length 1 and $\frac{2 - x}{\alpha} > \frac{1}{\alpha} > 1$. Thus, it contains exactly one positive integer, which is $\lfloor \frac{2 - x}{\alpha} \rfloor + 1$ if $\frac{2 - x}{\alpha} \notin \mathbb{Z}$, or $\lfloor \frac{2 - x}{\alpha} \rfloor - 1$ if $\frac{2 - x}{\alpha} \in \mathbb{Z}$.

If $m_k > 2$ then $k = \frac{\beta_k - x + m_k}{\alpha} > \frac{x - 2}{\alpha} = \frac{2 - x}{\alpha}$.

Therefore, the smallest $k \in \mathbb{N}$ such that $\beta_k \in J_1$ is found only in the case $m_k = 1$. We conclude
\[ k(x) = \begin{cases} \left\lceil \frac{2-x}{\alpha} \right\rceil & \text{if } \frac{2-x}{\alpha} \notin \mathbb{Z}, \\ \left\lceil \frac{2-x}{\alpha} \right\rceil - 1 & \text{if } \frac{2-x}{\alpha} \in \mathbb{Z}. \end{cases} \tag{1} \]

Put \( l = \left\lceil \frac{1}{\alpha} \right\rceil \in \mathbb{Z}. \) We show that \( k(x) \) given by (1) assumes one of two values \( l \) or \( l+1 \). According to (1),

\[
k(x) < \frac{2-x}{\alpha} < \frac{2-(1-x)}{\alpha} = \frac{1}{\alpha} + 1, \]

\[
k(x) > \frac{2-x}{\alpha} - 1 > \frac{2-1}{\alpha} - 1 = \frac{1}{\alpha} - 1.
\]

Thus, \( k(x) \in \left( \frac{1}{\alpha} - 1, \frac{1}{\alpha} + 1 \right) \). Because \( \frac{1}{\alpha} \notin \mathbb{Q} \), the interval contains only two integers: \( \left\lceil \frac{1}{\alpha} \right\rceil \) and \( \left\lceil \frac{1}{\alpha} \right\rceil + 1 \). Therefore, \( k(x) \in \{l, l+1\} \).

With \( k(x) \) given at (1), we define the return map \( \phi : J_1 \to J_1 \),

\[
\phi(x) = R_{\alpha}^{k(x)}(x) = (x + k(x)x) \mod 1
= x + k(x)x - 1. \tag{2}
\]

(ii) We show that \( \phi(1-x) = \lim_{x \to 1^-} \phi(x) \). Put \( x' = 1-x \).

\[
\frac{2-x'}{\alpha} = \frac{1+x}{\alpha} = \frac{1}{\alpha} + 1 \notin \mathbb{Z}.
\]

Thus,

\[
k(x') = \left\lceil \frac{2-x'}{\alpha} \right\rceil = \left\lceil \frac{1}{\alpha} \right\rceil + 1 = l+1.
\]

By (2),

\[
\phi(x') = x' + k(x')x - 1 = (1-x) + (l+1)x - 1 = lx.
\]

We see that \( \frac{2-1}{\alpha} = \frac{1}{\alpha} \notin \mathbb{Z} \). Because of the discreteness of \( \mathbb{Z} \) in \( \mathbb{R} \), there exists \( \varepsilon \in (0, x) \) such that \( \frac{2-x}{\alpha} \notin \mathbb{Z} \) for all \( x \in (-\varepsilon, 1+\varepsilon) \). Thus \( \left\lceil \frac{2-x}{\alpha} \right\rceil \) is constant in the interval \( x \in (-\varepsilon, 1+\varepsilon) \). It is equal to \( \left\lceil \frac{2-1}{\alpha} \right\rceil = l \).
\[ k(x) = \left\lceil \frac{2-x}{\lambda} \right\rceil = l \quad \forall x \in (1-\varepsilon, 1). \]

Then
\[ \phi(x) = x + k(x)\lambda - 1 = x + \lambda x - 1 \quad \forall x \in (1-\varepsilon, 1). \]

Then \( \lim_{x \to x'} \phi(x) = 1 + \lambda x - 1 = \lambda x = \phi(x'). \)

Next, define an equivalence relation in \( \mathbb{R} : x \sim y \iff \frac{x-y}{\lambda} \in \mathbb{Z}. \)

Denote \( \bar{x} \) the equivalence class of \( x \). Put \( S = \{ \bar{x} : x \in \mathbb{R} \} \). We can identify \( S \cong \mathbb{R}/\sim \cong [1-\varepsilon, 1]/(1-\varepsilon \sim 1) \).

For this reason, \( S \) can be thought as a smaller circle (than \( S' \)). Define a map \( \tilde{\phi} : S \to S \),
\[ \tilde{\phi}(\bar{x}) = \overline{\phi(x)} \quad \forall x \in I_1. \]

It is a well-defined map. We show that it is continuous.
\[ \overline{\phi(x)} = \overline{x + k(x)\lambda - 1} = \overline{x - 1} \quad \forall x \in I_1. \]

Thus, \( \tilde{\phi}(\bar{x}) = \overline{x - 1} \) for all \( x \in I_1 \). We see that
\[ \bar{x} = \bar{y} \iff \frac{x-y}{\lambda} \in \mathbb{Z} \iff \frac{(x-1)-(y-1)}{\lambda} \in \mathbb{Z} \iff \overline{x-1} = \overline{y-1}. \]

Hence, \( \tilde{\phi}(\bar{x}) = \overline{x - 1} \) for all \( x \in \mathbb{R} \). Put \( T : \mathbb{R} \to \mathbb{R}, \ T(x) = x - 1 \). Then
\[ \tilde{\phi}(\bar{x}) = p \circ T(x) \quad \forall x \in \mathbb{R}. \]

Here \( p \) is the map \( p : \mathbb{R} \to S, \ p(x) = \bar{x}. \)

By the characteristic property of quotient topology (Theorem 3.70, John Lee
"Introduction to topological manifolds", 2011), \( \Phi : S \to S \) is continuous if and only if \( \text{Top} : \mathbb{R} \to S \) is continuous. This is the case because both \( T \) and \( p \) are continuous.

(iii) Define a map \( \Psi : S \to S' \),

\[
\Psi(x) = \left( 1 - \frac{x - (1-x)}{x} \right) \mod 1 \quad \forall x \in \mathbb{R}.
\]

It is well-defined because the right-hand side does not depend on the choice of representative in \( x \). We compute the composite map \( \Psi \circ \Phi \circ \Psi^{-1} \).

For \( y \in S' \), the equation \( \Psi(x) = y \) is satisfied if

\[
1 - \frac{x - (1-x)}{x} = y,
\]

which is satisfied if \( x = 1 - xy \). Then \( \Phi(x) = \overline{x} - 1 = -xy \). By the definition of \( \Psi \),

\[
\Psi(-xy) = 1 - \frac{-xy - (1-x)}{x} = 1 + \frac{xy + (1-x)}{x} = y + \frac{1}{x}.
\]

Thus,

\[
\Psi \circ \Phi \circ \Psi^{-1}(y) = \Psi(\Phi(x)) = \Psi(-xy) = \left( y + \frac{1}{x} \right) \mod 1.
\]

Let \( \beta = \frac{1}{x} - 1 = \frac{1}{x} - \left[ \frac{1}{x} \right] \in (0,1) \setminus \mathbb{Q} \). Then

\[
\Psi \circ \Phi \circ \Psi^{-1}(y) = (y + \beta) \mod 1 = R_\beta(y) \quad \forall y \in S'.
\]

Thus, \( \Psi \circ \Phi \circ \Psi^{-1} = R_\beta \).

(iv) By part (iii),

\[
\alpha = \frac{1}{\ell + \beta}.
\]

Labeling \( \alpha \) as \( \alpha_1 \), \( \ell \) as \( \ell_1 \), \( \beta \) as \( \alpha_2 \), we can write
\[ \alpha_i = \frac{1}{l_i + \alpha_{i+1}}. \]

Recall that \( l_i \in \mathbb{N} \) is "almost" the returning time of \([1-\alpha_i, 1)\) under the rigid rotation \( R_{\alpha_i} \). "Almost" is understood as that the returning time of each point of \([1-\alpha_i, 1)\) is either \( l_i \) or \( l_i + 1 \).

Because \( \alpha_i \in (0,1) \setminus \mathbb{Q} \), we can view \( \alpha_2 \) as \( \alpha_1 \) and repeat the process in parts (i) and (ii) and (iii). Then

\[ \alpha_2 = \frac{1}{l_2 + \alpha_3} \]

where \( l_2 \in \mathbb{N} \) is almost the returning time of \([0-\alpha_2, 1)\) under the rigid rotation \( R_{\alpha_2} \), and \( \alpha_3 \in (0,1) \setminus \mathbb{Q} \). Similarly,

\[ \alpha_3 = \frac{1}{l_3 + \alpha_4}, \quad \alpha_4 = \frac{1}{l_4 + \alpha_5}, \ldots \]

Then

\[ \alpha = \alpha_1 = \frac{1}{l_1 + \alpha_2} = \frac{1}{l_1 + \frac{1}{l_2 + \alpha_3}} = \frac{1}{l_1 + \frac{1}{l_2 + \frac{1}{l_3 + \alpha_4}}} = \frac{1}{l_1 + \frac{1}{l_2 + \frac{1}{l_3 + \frac{1}{l_4 + \alpha_5}}}} = \ldots \]

The returning times \( l_1, l_2, l_3, \ldots \) give us a continued fraction expression for \( \alpha \).

(v) Take any \( x \in S^1 \). Consider the orbit

\[ x_j = R_{\alpha}^j(x) = (x + jx) \mod 1, \quad \forall j \in \mathbb{N}. \]

Define a coding sequence
\[ a_j = \begin{cases} 0 & \text{if } x_j \in [0, 1-x), \\ 1 & \text{if } x_j \in [1-x, 1). \end{cases} \]

We show that starting from some index, \((a_j)\) has the pattern: \(l\) or \(l-1\) number 0's followed by exactly one number 1. That is to show

\[
(a_j) = \underbrace{\cdots 0 \cdots 0 1 \cdots 01 \cdots 01} \cdots 01 \cdots
\]

Because the interval \([1-x, 1)\) is of length \(x\), there exists \(j_0 \in \mathbb{N}\) such that \(x_{j_0} \in [1-x, 1)\). We can assume that the sequence \((a_j)\) starts from \(x_{j_0}\), which is equivalent to assuming \(x \in [1-x, 1)\).

In Part (i), we showed that the smallest positive number \(l\) such that \(R_x^l(1) \in [1-x, 1)\) is \(\ell(x) \in \{l, l+1\}\). Thus,

\[
\begin{cases} 
a_1, a_2, \ldots, a_{\ell-1} = 0, \\
a_\ell = 1 \text{ or } (a_\ell = 0 \text{ and } a_{\ell+1} = 1). 
\end{cases}
\]

If \(a_\ell = 1\), we regard \(x_\ell\) as \(x\) which starts a new sequence. Then \(a_\ell\) is followed by \(l-1\) or \(l\) number 0's and then a number 1.

If \(a_{\ell+1} = 1\), we regard \(x_{\ell+1}\) as \(x\) which starts a new sequence. Then \(a_{\ell+1}\) is followed by \((l-1)\) or \(l\) number 0's and then a number 1.

\[
\underbrace{0 \cdots 0 1 \cdots 01}_{\ell-1} \quad \text{or} \quad \ell \quad \text{or} \quad \ell-1
\]

Continue the process of translating the sequence \((a_j)\) by updating the value of \(x\) in \([1-x, 1)\). We conclude that \((a_j)\) has the pattern: \(l\) or \(l-1\) consecutive
number 0's followed by exactly one number 1.

\( l \) is about the time it takes to go (via \( \mathcal{K}_\ell \)) from a point in \([1-\alpha, 1)\) back to the same interval.

(vi) Define a sequence \((b_j)\) obtained from \((a_j)\) by replacing each block \(\overline{0 \cdots 01}\) by number 0, and each block \(\overline{1 \cdots 01}\) by number 1. We show that \((b_j)\) has the same pattern as \((a_j)\), except that the "period" may be different from \(l\).

To do so, we show that the process of substituting each block by a single number creates an orbit on the smaller circle \(S\) which is analogous to the orbit \((x_j)\) on \(S'\).

Translating \((x_j)\) by some index if necessary, we can assume \(x \in (1-\alpha, 1)\).

\[ k(x) = l+1 \iff x_j \text{ has not returned to } [1-\alpha, 1) \text{ after } l \text{ times} \]

\[ \iff x_n = x + l\alpha < 1 + (1-\alpha) \]
\[ \iff x < 2 - (l+1)\alpha. \]

Put \(J_0' = [1-\alpha, 2-(l+1)\alpha)\) and \(J_1' = [2-(l+1)\alpha, 1)\). Then

\[ k(x) = \begin{cases} 
  l+1 & \text{if } x \in J_0', \\
  l & \text{if } x \in J_1'. 
\end{cases} \]

The sequence \((a_j)\) starts with block \(\overline{0 \cdots 01}\) if \(x \in J_0'\), and with block \(\overline{1 \cdots 01}\) if \(x \in J_1'\). Recall from Part (vi) that \(S = \mathbb{R}/\mathbb{Z} \cong [1-\alpha, 1)/_{1-\alpha\mathbb{Z}}\).

The point \(\phi^{k(x)}(x+\ell\alpha) \) may not lie in \([1-\alpha, 1)\), but is equivalent to a
unique point in $\mathcal{S}$ thanks to the equivalence relation $\sim$. Then (a) starts with block $0 \ldots 0 \ 1$ if $\phi(x) = x + k(x)x - 1$ is equivalent to a point in $J_0$, and with block $0 \ldots 0 \ 1$ if $\phi(x)$ is equivalent to a point in $J_0'$. Note that $\phi(x)$ is equivalent to $x - 1$. Put $x_0' = x \in \mathcal{S}$.

Then

$$b_1 = \begin{cases} 1 & \text{if } x_0' \in J_0' \\ 0 & \text{if } x_0' \in J_0 \\ \end{cases}$$

$$b_2 = \begin{cases} 1 & \text{if } x_0' - 1 \text{ is equivalent to some } x_0' \in J_0' \\ 0 & \text{if } x_0' - 1 \text{ is equivalent to some } x_0' \in J_0 \\ \end{cases}$$

$$b_3 = \begin{cases} 1 & \text{if } x_0' - 1 \text{ is equivalent to some } x_0' \in J_0' \\ 0 & \text{if } x_0' - 1 \text{ is equivalent to some } x_0' \in J_0 \\ \end{cases}$$

We can view $(x_j')$ as the orbit of $x$ on the small circle $\mathcal{S}$ under the rigid rotation $R_{-1}$. The map $\#^v$ in Part (iii) transforms $(x_j')$ into an orbit on the circle $\mathcal{S}'$. Let

$$x_j'' = \#(x_j') = \left(1 - \frac{x_j' - (1 - x)}{\alpha}\right) \mod 1.$$

Then

$$x_j'' - x_j' = -\frac{1}{\alpha} (x_j' - 1) \mod 1$$

$$= \frac{1}{\alpha} \mod 1$$

$$= 1.$$

Put $y = \#(x) \in \mathcal{S}'$. Then $(x_j''')$ is the orbit of $y$ on the circle $\mathcal{S}''$ under the rigid rotation $R_\beta$. 

\[ \Psi((1-\alpha)^+) = \left(1 - \frac{(1-\alpha)^+ - (1-\alpha)}{\alpha}\right) \mod 1 = 1^- , \]

\[ \Psi(2-(\ell+1)\alpha) = \left(1 - \frac{2-(\ell+1)\alpha - (1-\alpha)}{\alpha}\right) \mod 1 = 1^- , \]

\[ \Psi(1^-) = \left(1 - \frac{1^-(1-\alpha)}{\alpha}\right) \mod 1 = 0^+ . \]

Thus, \( \Psi(J_0^{'}) = [L-\beta, 1] \) and \( \Psi(J_1^{'}) = [0, 1-\beta] \). We have

\[ b_1 = \begin{cases} 1 & \text{if } x'_1 = \Psi(x_1') \in \Psi(J_0^{'}) = [L-\beta, 1] \\ 0 & \text{if } x'_1 = \Psi(x_1') \in \Psi(J_1^{'}) = [0, 1-\beta] \end{cases} \]

\[ b_2 = \begin{cases} 1 & \text{if } x''_2 \in [L-\beta, 1] \\ 0 & \text{if } x''_2 \in [0, 1-\beta] \end{cases} \]

We see that the sequence \( (b_j) \) is formed the same way as is \( (b_j) \), except that \( \alpha \) is replaced by \( \beta \). Thus, \( (b_j) \) also has the pattern

\[ (b_j) = \overbrace{0 \cdots 01 0 \cdots 01 0 \cdots 01 0 \cdots}^{\text{disordered sequence}} \overbrace{0 \cdots}^{\text{portion}} 01 0 \cdots 01 \cdots \]

where \( \ell' = \left[ \frac{1}{\beta} \right] \).

If we continue to replace each block \( 0 \cdots 01 \) with number \( 0 \), each
Block 0...01 with number 1, the resulting sequence still have the same pattern. The new “period” is $l^\circ = \left[ \frac{1}{5} \right]$ where $\gamma = \frac{1}{p} - \left[ \frac{1}{p} \right]$.

The sequence of periods $l, l', l^\circ, \ldots$ is exactly the sequence $l_1, l_2, l_3, \ldots$ in the continued fractional expression of $\alpha$.

\[
\alpha = \frac{1}{l_1 + \frac{1}{l_2 + \frac{1}{l_3 + \cdots}}}
\]

2. We view the circle $S^1$ as a metric subspace of $\mathbb{R}^2$. Then \[ C^0(S^1, \mathbb{R}^2) = \{ h : S^1 \rightarrow \mathbb{R}^2 \text{ continuous} \} \]

is a normed vector space with $\| h \|_{C^0} = \sup_{x \in S^1} |h(x)|$. Each homeomorphism from $S^1$ to $S^1$ is an element of $C^0(S^1, \mathbb{R}^2)$.

Let $\varphi : S^1 \rightarrow S^1$ be a homeomorphism. The topology on $S^1$ is the topology induced by the map $p : \mathbb{R} \rightarrow S^1$, $p(t) = \exp(2\pi i t)$. Both $\varphi \circ p : \mathbb{R} \rightarrow S^1$ and $p : \mathbb{R} \rightarrow S^1$ are covering maps of $S^1$. Because $\mathbb{R}$ is simply connected, there exists a homeomorphism $F : \mathbb{R} \rightarrow \mathbb{R}$ such that the following diagram commutes (see Proposition M.41, page 287, John Lee “Introduction to Topological Manifolds”, 2011).

\[ \begin{array}{ccc}
\mathbb{R} & \xrightarrow{F} & \mathbb{R} \\
p \downarrow & \searrow & \downarrow \varphi \\
S^1 & \xrightarrow{p} & S^1
\end{array} \]

F is called a lift of $S^1$ according to terminology in Definition 14.1, page 103, Devaney “An Introduction to Chaotic Dynamical Systems”, 1989.

Then $\exp(2\pi F(x)) = \varphi(\exp(2\pi x))$ for all $x \in \mathbb{R}$. Then
\[
\exp(i2\pi F(x+1)) = f(\exp(i2\pi (x+1))) = f(\exp(i2\pi (x+1))) = \exp(i2\pi F(x)) \quad \forall x \in \mathbb{R}.
\]
Thus, \(F(x+1) - F(x) \in \mathbb{Z}\). Because \(F(x+1) - F(x)\) is a continuous function, there exists \(l \in \mathbb{Z}\) such that
\[
F(x+1) - F(x) = l \quad \forall x \in \mathbb{R}.
\]
Once \(F(x)\) is known for all \(x \in [0,1]\), the value of \(F\) elsewhere will be known by adding suitable multiples of \(l\). Then \(F\) is uniformly continuous in \(\mathbb{R}\).

Recall that the rotation number of \(f\) is defined by \(\rho(f) = \rho_0(F) \mod 1\), where \(\rho_0(F) = \lim_{n \to \infty} \frac{F^n(0)}{n}\) and \(F^n = F \circ F \circ \ldots \circ F\). We show that \(\rho\) is continuous in \(f\).

Let \((g_r)\) be a sequence of homeomorphisms from \(S^1\) to \(S^1\) such that \(\|g_r - f\|_{C^0} \to 0\) as \(r \to \infty\). We show that \(\rho(g_r) \to \rho(f)\) in modulo 1.

Without loss of generality, we can assume \(\|g_r - f\|_{C^0} < 2\) for all \(r \in \mathbb{N}\).

Let \(G_r : \mathbb{R} \to \mathbb{R}\) be a lift of \(g_r\). Then
\[
\exp(i2\pi G_r(x)) = g_r(\exp(i2\pi x)) \quad \forall x \in \mathbb{R}.
\]

Then
\[
|\exp(i2\pi G_r(x)) - \exp(i2\pi F(x))| = |g_r(\exp(i2\pi x)) - f(\exp(i2\pi x))| < 2 \quad \forall x \in \mathbb{R}.
\]
(1)

In the definition of \(G_r\), we see that \(G_r(0)\) can be determined up to an integer. We can choose \(G(0)\) such that the length of the smaller arc on \(S^1\) which joins \(g_r(1)\) and \(f(1)\) is \(2\pi |G_r(0) - F(0)|\). Because \(G_r\) and \(F\) are continuous and injective maps from \(\mathbb{R}\) to \(\mathbb{R}\), they are monotone.
Because of (1), they must have the same monotonicity. In addition, the length of the smaller arc which joins \( g_r(\exp(i2\pi x)) \) and \( f(\exp(i2\pi x)) \) is \( 2\pi |G(x) - F(x)| \). The length is not longer than \( \frac{\pi}{2} \) times the length of the chord joining these points, which is \( \frac{\pi}{2} |g_r(\exp(i2\pi x)) - f(\exp(i2\pi x))| \). Thus,

\[
|G_r(x) - F(x)| \leq \frac{1}{4} |g_r(\exp(i2\pi x)) - f(\exp(i2\pi x))| \\
\leq \frac{1}{4} \|g_r - f\|_c \quad \forall x \in \mathbb{R}.
\]

(2)

Next, we show by induction in \( n \in \mathbb{N} \) that

\[
\lim_{r \to \infty} \|G_r^n - F^n\|_c = 0. \tag{3}
\]

(3) is true for \( n = 1 \) thanks to (2). Suppose (3) is true for some \( n \in \mathbb{N} \). Then

\[
|G_r^{n+1}(x) - F^{n+1}(x)| \leq |G_r(G_r^n(x)) - F(G_r^n(x))| + |F(G_r^n(x)) - F(F^n(x))| \\
\leq \|G_r - F\|_c + \sup_{\|y\| \leq \|G_r^n - F^n\|_c} \{1 \leq \|G_r^n(x) - F^n(x)\| \}
\]

(4)

\[
\lim_{r \to \infty} E_1 = 0 \quad \text{because of (2). By the induction hypothesis}, \quad \lim_{r \to \infty} \|G_r^n - F^n\|_c = 0.
\]

Then by the uniform continuity of \( F \), \( \lim_{r \to \infty} E_2 = 0 \). Because the estimate (4) holds for every \( x \in \mathbb{R} \), \( \lim_{r \to \infty} \|G_r^{n+1} - F^{n+1}\|_c = 0 \). We have proved (3).
An earlier result is that there exists $l \in \mathbb{Z}$ such that $F(x+1) = F(l)$ for all $x \in \mathbb{R}$. Now we show that $l \in \{0, 1\}$. First, $l \neq 0$ because $F$ is injective. Suppose by contradiction that $l > 1$. Then the function $h : \mathbb{R} \rightarrow \mathbb{R}$, $h(t) = F(t) - F(0)$ is continuous, $h(0) = 0$ and $h(t) > 1$. There exists $c \in (0, 1)$ such that $h(c) = 1$. Then

$$f(\exp(i2\pi c)) = \exp(i2\pi F(c)) = \exp(i2\pi F(0)) = f(\exp(i2\pi 0)).$$

This is a contradiction because $f$ is injective on $S^1$. The case $l < -1$ is dealt similarly. Therefore,

$$F(x+1) - F(x) = \begin{cases} 1 & \text{if } F \text{ is increasing,} \\ -1 & \text{if } F \text{ is decreasing.} \end{cases}$$

We showed earlier that $F$ and each $G_r$ have the same monotonicity. Denote by $\phi : S^1 \rightarrow S^1$, $\phi(\exp(i2\pi x)) = \exp(-i2\pi x)$ the reflection map. $\phi \circ f$ and $\phi \circ g_r$ are homeomorphisms from $S^1$ to $S^1$ and $\|\phi \circ f - \phi \circ g_r\|_{C^0} \rightarrow 0$ as $r \rightarrow \infty$.

A lift of $\phi \circ f$ is $F$ because

$$\exp(-i2\pi F(x)) = \phi(\exp(i2\pi F(x))) = \phi(f(\exp(i2\pi x))).$$

Therefore, replacing $f$ by $\phi \circ f$, each $g_r$ by $\phi \circ g_r$ if necessary, we can assume $F$ (and thus each $G_r$) is increasing. Thus, $F(x+1) = F(x) + 1$ for every $x \in \mathbb{R}$. We now show that

$$|F(x) - F(y)| < 1 \quad \forall x, y \in (0, 1) \quad \forall n \in \mathbb{N}. \quad (5)$$
Take \( x, y \in \mathbb{R} \), \( x < y \). Because \( F \) is increasing, \( F^n \) is too. Since \( x < y < x + 1 \),

\[
F^n(x) < F^n(y) < F^n(x + 1) = F^n(x) + 1.
\]

Then \( 0 < F^n(y) - F^n(x) < 1 \). Consequently,

\[
|F^n(x) - x| < |F^n(y) - y| < |y - x| < 1 + 1 \quad \forall x, y \in (0, 1), n \in \mathbb{N}.
\]

Therefore, for each \( n \in \mathbb{N} \) there exists \( p_n \in \mathbb{Z} \) such that

\[
p_n < F^n(x) - x < p_n + 5 \quad \forall x \in (0, 1), n \in \mathbb{N}.
\]

Because \( F^n(x + m) = F^n(x) + m \) for every \( m \in \mathbb{Z} \),

\[
p_n < F^n(x) - x < p_n + 5 \quad \forall x \in \mathbb{R}. \quad (6)
\]

Because \( \|G_r - F^n\|_{\infty} \to 0 \) as \( r \to \infty \), there exists \( N(n) \in \mathbb{N} \) such that \( \|G_r - F^n\|_{\infty} < 1 \) for all \( r > N(n) \). Then

\[
|G_r^n(x) - x| < |G_r^n(x) - F^n(x)| < 1 \quad \forall x \in \mathbb{R}, n \in \mathbb{N}, r > N(n).
\]

Thus,

\[
p_{n-1} < G_r^n(x) - x < p_n + 6 \quad \forall x \in \mathbb{R}, n \in \mathbb{N}, r > N(n). \quad (7)
\]

For each \( m \in \mathbb{N} \),

\[
F^{mn}(0) = \sum_{k=1}^{m} \left( F^n(F^{(k-1)n}(0)) - F^{(k-1)n}(0) \right).
\]

Applying (6), we get \( m p_n < F^{mn}(0) < m(p_n + 5) \). Divide both sides by \( mn \),

\[
\frac{p_n}{n} < \frac{F^{mn}(0)}{mn} < \frac{p_n + 5}{n}. \quad (8)
\]
Similarly,
\[ G^m_n(0) = \sum_{t=1}^{m} \left( G^m_n(G^k_n(0)) - G^k_n(0) \right). \]

Applying (7), we get \( m(p_n - 1) < G^m_n(0) < m(p_n + 6) \). Divide both sides by \( mn \),
\[ \frac{p_n - 1}{n} < \frac{G^m_n(0)}{mn} < \frac{p_n + 6}{n}. \quad (9) \]

By (8) and (9),
\[ -\frac{6}{n} < \frac{F^m_n(0) - G^m_n(0)}{mn} < \frac{6}{n} \quad \forall m \in \mathbb{N}, r > N(n). \]

Let \( m \to \infty \),
\[ -\frac{6}{n} \leq s_o(F) - s_o(G_r) \leq \frac{6}{n} \quad \forall n \in \mathbb{N}, r > N(n). \]

Let \( n \to \infty \), \( \lim_{r \to \infty} s_o(G_r) = s_o(F) \). Taking modulo 1 both sides, we get
\[ \lim_{r \to \infty} s(G_r) = s(F) \quad (\text{mod } 1). \]

3. Let \( f : \mathbb{R} \to \mathbb{R} \) be a \( C^1 \) function satisfying \( f(x + 1) = f(x) \) for all \( x \in \mathbb{R} \).

Because \( f \) is locally Lipschitz, by Picard-Lindelöf's theorem, the problem
\[ \dot{x} = f(x), \quad x(0) = x_0 \] has a unique local solution. Since \( f \) is continuous and periodic, it is bounded in \( \mathbb{R} \). Then a global solution exists and is unique.

Denote by \( x(t; x_0) \) this solution. Then \( (\Phi_t)_{t \in \mathbb{R}} \), \( \Phi_t(x_0) = x(t; x_0) \) for \( t, x_0 \in \mathbb{R} \), is a flow.

Each \( \Phi_t : \mathbb{R} \to \mathbb{R} \) is a homeomorphism and \( \Phi_{t+s} = \Phi_t \circ \Phi_s \) for all \( t, s \in \mathbb{R} \).

Because \( f \) is \( 1 \)-periodic,
\[ \frac{d}{dt}[x(t; x_0) + 1] = \frac{d}{dt} x(t; x_0) = f(x(t; x_0)) = f(x(t; x_0) + 1). \]
Thus, $x(t;x_0)+1$ is a global solution to the problem
\[
\begin{cases} 
  \dot{x} = f(x), \\
  x(0) = x_0 + 1.
\end{cases}
\]
By the uniqueness of solutions, $x(t;x_0)+1 = x(t;x_0+1)$. In other words,
\[
\phi_t(x_0+1) = \phi_t(x_0) + 1 \quad \forall x_0 \in \mathbb{R}, t \in \mathbb{R}.
\]
Define a map $p : \mathbb{R} \to S^1$, $p(x) = \exp(i2\pi x)$. This is a quotient map, i.e. the topology on $S^1$ is the one induced by $p$. We see that $p \circ \phi_t : \mathbb{R} \to S^1$ is continuous and is constant in each fiber of $p$. Indeed, suppose $p(x) = p(x')$. Then $x = x' + l$ for some $l \in \mathbb{Z}$. Then
\[
p(\phi_t(x)) = p(\phi_t(x' + l)) = p(\phi_t(x) + l) = p(\phi_t(x')).
\]
For this reason, there exists a continuous map $g_t : S^1 \to S^1$ such that the following diagram commutes (Theorem 3.73, page 72, John Lee "Introduction to Topological Manifolds", 2011). We have
\[
g_t(\exp(i2\pi x)) = \exp(i2\pi \phi_t(x)) \quad \forall x \in \mathbb{R}. \quad (1)
\]
Because $\phi_t$ is continuous and injective on $\mathbb{R}$, it is monotone. Since $\phi_t(x+1) = \phi_t(x) + 1$, $\phi_t$ is increasing. Then
\[
\phi_t([0,1]) = [\phi_t(0), \phi_t(1)] = [\phi_t(0), \phi_t(0) + 1).
\]
Then $\phi_t|_{[0,1]} : [0,1] \to [\phi_t(0), \phi_t(0) + 1)$ is bijective. This implies $g_t$ is a
bijection. Because \( g_t : S^1 \rightarrow S^1 \) is continuous, bijective and \( S^1 \) is compact, we conclude that \( g_t \) is a homeomorphism.

Next, we compute the rotation number of \( g_t \). The identity (1) shows that \( \phi_t \) is a lift of \( g_t \).

\[
\phi_t(0) = \lim_{n \rightarrow \infty} \frac{\phi_t^n(0)}{n} = \lim_{n \rightarrow \infty} \frac{\phi_t^n(0)}{n}.
\]

The number \( \phi_t(0) \) is expressed by the same manner.

\[
\phi_t(0) = \lim_{n \rightarrow \infty} \frac{\phi_t^n(0)}{n} = \lim_{n \rightarrow \infty} \frac{\phi_t^n(0)}{n}.
\]

Let \( M = \sup_{x \in K} |f(x)| < \infty \). Then

\[
\phi_t(x_0) - \phi_t(x_0) = \int_0^t \frac{d}{ds} \phi_s(x_0) ds = \int_0^t f(\phi_s(x_0)) ds.
\]

Thus,

\[
|\phi_t(x_0) - \phi_t(x_0)| \leq \left| \int_0^t |f(\phi_s(x_0))| ds \right| \leq Mt - 1 \quad \forall x_0 \in K (4)
\]

By (4) we have

\[
|g_t(\exp(i2\pi x_0)) - g_t(\exp(i2\pi x_0))| = |\exp(i2\pi \phi_t(x_0)) - \exp(i2\pi \phi_t(x_0))|
\]

\[
\leq |\phi_t(x_0) - \phi_t(x_0)| \sup_{y \in K} \left| \frac{d}{dy} \exp(i2\pi y) \right|
\]

\[
= 2\pi |\phi_t(x_0) - \phi_t(x_0)|
\]

\[
\leq 2\pi Mt - 1 \quad \forall x_0 \in K.
\]

Thus,

\[
\|g_t - g_t\|_{C^0(S^1, K^2)} \leq 2\pi Mt - 1.
\]

This implies \( g_t \rightarrow g \) in \( C^0(S^1, K^2) \) as \( t \rightarrow 1 \). By Problem (2), we conclude that \( \phi_t(x_0) \rightarrow \phi_0(x_0) \) and \( f(g_t) \rightarrow f(g_t) \) as \( t \rightarrow 1 \). A direct proof
Can be obtained by using (2) and (3) and (4).

Assume \( f(x) \neq 0 \) for all \( x \in [0,1] \). We now compute \( \phi_n(\phi_k) \), \( t \in \mathbb{R} \), explicitly in terms of \( f \). That is to compute \( \lim_{n \to \infty} \frac{\phi_n(0)}{n} \) in terms of \( f \). Recall that \( \phi_k(0) \) is the solution to the problem
\[
\begin{cases}
    \dot{x} = f(x), \\
    x(0) = 0.
\end{cases}
\]
Because \( f \) is continuous and nonzero on \([0,1]\), it does not change its sign.

There exist \( m, M > 0 \) such that

\[
m \leq |f(x)| \leq M \quad \forall x \in [0,1].
\]

Integrating the equation \( \dot{x} = \frac{x}{f(x)} \) from 0 to \( t \), we get

\[
t = \int_0^t \frac{dx(t)}{f(x(t))} = \int_0^{x(t)} \frac{ds}{f(s)}. \quad (5)
\]

Define a function \( F: \mathbb{R} \to \mathbb{R} \),

\[
F(t) = \int_0^t \frac{ds}{f(s)}.
\]

We show that \( F \) is a bijection.

If \( f \) is positive then \( F \) is increasing. Moreover,

\[
F(t) = \int_0^t \frac{ds}{f(s)} \geq \int_0^t \frac{ds}{M} = \frac{t}{M} \quad \forall t > 0,
\]

\[
F(t) = -\int_t^0 \frac{ds}{f(s)} \leq -\int_t^0 \frac{ds}{M} = -\frac{t}{M} \quad \forall t < 0.
\]

Then \( \lim_{t \to \pm\infty} F(t) = \pm\infty \). Thus, \( F \) is a bijection.

If \( f \) is negative then \( F \) is decreasing. Moreover,

\[
F(t) = \int_0^t \frac{ds}{f(s)} \leq \int_0^t \frac{ds}{-M} = -\frac{t}{M} \quad \forall t > 0,
\]

\[
F(t) = -\int_0^{-t} \frac{ds}{f(s)} \geq -\int_0^{-t} \frac{ds}{M} = \frac{-t}{M} \quad \forall t < 0.
\]
\[ F(t) = - \int_{\frac{t}{M}}^{0} \frac{ds}{f(s)} \geq - \int_{\frac{t}{M}}^{0} \frac{ds}{-M} = \frac{-t}{M} \quad \forall \ t < 0. \]

Then \( \lim_{t \to \pm \infty} F(t) = \pm \infty \). Thus, \( F \) is a bijection.

Now (5) becomes \( t = F(x(t)) \), which gives \( x(t) = F^{-1}(t) \). Then

\[
\lim_{t \to \pm \infty} \frac{x(t)}{t} = \lim_{t \to \pm \infty} \frac{F^{-1}(t)}{t} = \begin{cases} 
\lim_{s \to \pm \infty} \frac{s}{F(s)} & \text{if } f \text{ is positive}, \\
\lim_{s \to \pm \infty} \frac{s}{f(s)} & \text{if } f \text{ is negative}.
\end{cases}
\]

Consider the case \( f \) is positive. For \( s \neq 0 \),

\[
F(s) = \int_{0}^{s} \frac{dt}{f(t)} = \int_{0}^{[s]} \frac{dt}{f(t)} + \int_{[s]}^{s} \frac{dt}{f(t)},
\]

where \([s]\) is the integer part of \( s \). Since \( f \) is \( 1 \)-periodic,

\[
\int_{0}^{[s]} \frac{dt}{f(t)} = [s] \int_{0}^{1} \frac{dt}{f(t)},
\]

\[
\int_{[s]}^{s} \frac{dt}{f(t)} = \int_{0}^{[s]} \frac{dt}{f(t + [s])} = \int_{0}^{[s]} \frac{dt}{f(t)},
\]

where \([s]\) is the fractional part of \( s \). Then (8) becomes

\[
F(s) = [s] \int_{0}^{1} \frac{dt}{f(t)} + \int_{0}^{[s]} \frac{dt}{f(t)}.
\]

Thus,

\[
\frac{F(s)}{s} = \frac{[s]}{s} \int_{0}^{1} \frac{dt}{f(t)} + \frac{1}{s} \int_{0}^{[s]} \frac{dt}{f(t)}.
\]

\[
\lim_{s \to \pm \infty} A = \int_{0}^{1} \frac{dt}{f(t)} \quad \text{because} \quad \lim_{s \to \pm \infty} \frac{[s]}{s} = 1.
\]

\[
\lim_{s \to \pm \infty} B = 0 \quad \text{because} \quad 0 < \int_{0}^{[s]} \frac{dt}{f(t)} \leq \int_{0}^{1} \frac{dt}{f(t)} \quad \text{for all } s \neq 0.
\]
Therefore, \( \lim_{s \to \infty} \frac{F(s)}{s} = \int_0^1 \frac{dt}{f(t)} \). Then (6) gives us

\[
\lim_{t \to \pm \infty} \frac{x(t)}{t} = \left( \int_0^1 \frac{dt}{f(t)} \right)^{-1} \quad \text{(9)}
\]

Consider the case \( f \) is negative. For \( s \neq 0 \),

\[
F(s) = \int_0^s \frac{dt}{f(t)} = \frac{[s]}{s} \int_0^s \frac{dt}{f(t)} + \int_{[s]}^s \frac{dt}{f(t)} = \left[ \frac{t}{s} \right] \int_0^1 \frac{dt}{f(t)} + \int_{[s]}^s \frac{dt}{f(t)}
\]

Then,

\[
\lim_{s \to \pm \infty} \frac{F(s)}{s} = \lim_{s \to \pm \infty} \frac{[s]}{s} \int_0^s \frac{dt}{f(t)} + \lim_{s \to \pm \infty} \frac{1}{s} \int_{[s]}^s \frac{dt}{f(t)} = \int_0^1 \frac{dt}{f(t)}
\]

Substituting this result into (7), we also get (9). Therefore,

\[
\lim_{t \to \pm \infty} \frac{\phi_t(0)}{t} = \left( \int_0^1 \frac{dt}{f(t)} \right)^{-1}
\]

Then,

\[
f_0(\phi_t) = \lim_{n \to \infty} \frac{\phi_t(n)}{n} = t \lim_{n \to \infty} \frac{\phi_n(0)}{n} = \begin{cases} 
\lim_{s \to \infty} \frac{\phi_s(0)}{s} & \text{if } t > 0, \\
\lim_{s \to -\infty} \frac{\phi_s(0)}{s} & \text{if } t < 0,
\end{cases}
\]

\[
= t \left( \int_0^1 \frac{dt}{f(t)} \right)^{-1} \quad \forall t \neq 0.
\]

This formula is also true for \( t = 0 \) because \( \phi_0 \) is the identity map on \( \mathbb{R} \).

We conclude that

\[
f_0(\phi_t) = t \left( \int_0^1 \frac{dt}{f(t)} \right)^{-1} \quad \forall t \in \mathbb{R},
\]

\[
f_0(g_t) = f_0(\phi_t) \mod 1.
\]