Let $X$ be a separable Hilbert space over $K$ (either $\mathbb{R}$ or $\mathbb{C}$) and let $L(X)$ be the space of continuous linear operators on $X$. For $R > 0$, denote $B_R(X) = \{ T \in L(X) : \| T \| \leq R \}$. Let $\sigma$ be the strong operator topology on $L(X)$, i.e. the topology defined by the system of open neighborhoods (of the zero operator),

\[
O_{x_1, \ldots, x_n, \varepsilon_1, \ldots, \varepsilon_n} = \{ T \in L(X) : \| Tx_i \| < \varepsilon_i, \ldots, \| Tx_n \| < \varepsilon_n \},
\]

for $n \in \mathbb{N}$, $x_1, \ldots, x_n \in X$, $\varepsilon_1, \ldots, \varepsilon_n > 0$. Denote by $B_\sigma$ this system of neighborhoods.

Let $\tau$ be the weak operator topology on $L(X)$, i.e. the topology defined by the system of neighborhoods

\[
O_{x_1, y_1, \ldots, x_n, y_n, \varepsilon_1, \ldots, \varepsilon_n} = \{ T \in L(X) : |(T x_i, y_i)| < \varepsilon_i, \ldots, |(T x_n, y_n)| < \varepsilon_n \},
\]

for $n \in \mathbb{N}$, $x_1, y_1, \ldots, x_n, y_n \in X$, $\varepsilon_1, \ldots, \varepsilon_n > 0$. Denote by $B_\tau$ this system of neighborhoods.

(i) Assume $X$ is infinite dimensional. We show that $(L(X), \sigma)$ and $(L(X), \tau)$ are not metrizable. The following lemma is needed.

[Lemma 1: Let $X$ be an infinite dimensional Hilbert space over $K$ (either $\mathbb{R}$ or $\mathbb{C}$). Then $X$ does not have a countable Hamel basis.]
A consequence is that $X$ is not equal to the linear span of any countable subset.

Proof of the lemma

Suppose otherwise: $X$ has a countable Hamel basis $\{u_1, u_2, u_3, ...\}$.

Then we can construct an orthonormal Hamel basis $\{e_1, e_2, e_3, ...\}$ for $X$ by the Gram-Schmidt process.

$$v_1 = u_1,$$
$$v_2 = u_2 - \frac{(u_2, v_1)}{(v_1, v_1)} v_1,$$
$$v_3 = u_3 - \frac{(u_3, v_1)}{(v_1, v_1)} v_1 - \frac{(u_3, v_2)}{(v_2, v_2)} v_2,$$
$$\vdots$$
$$v_n = u_n - \frac{(u_n, v_1)}{(v_1, v_1)} v_1 - \frac{(u_n, v_2)}{(v_2, v_2)} v_2 - \cdots - \frac{(u_n, v_{n-1})}{(v_{n-1}, v_{n-1})} v_{n-1},$$

$$\vdots$$

Set $e_k = \frac{v_k}{\|v_k\|}$ for all $k \in \mathbb{N}$.

Because $X$ is a Hilbert space and $\sum_{k=1}^{\infty} \frac{1}{k} < \infty$, the series $\sum_{k=1}^{\infty} \frac{1}{k} e_k$ converges in $X$. Then it is a linear combination of $\{e_1, e_2, e_3, ...\}$. In other words, there exist $\alpha_1, \alpha_2, ..., \alpha_m \in \mathbb{K}$ such that

$$\sum_{k=1}^{\infty} \frac{1}{k} e_k = \sum_{k=1}^{m} \alpha_k e_k.$$

Then

$$\sum_{k=1}^{m} \left( \frac{1}{k} - \alpha_k \right) e_k + \sum_{k=m+1}^{\infty} \frac{1}{k} e_k = 0.$$
Taking the norm of both sides, we get
\[
\sqrt{\frac{m}{\delta_k} \left| \frac{1}{\mu_k} - x_k \right|^2 + \sum_{k=m+1}^{n} \frac{1}{\delta_k}} = 0.
\]
This is a contradiction.

Return to the problem. Suppose by contradiction that \((L(X), 0)\) is metrizable. Then \((L(X), 0)\) has a countable system of open neighborhoods (of the zero operator), namely \(\{B_1, B_2, B_3, \ldots\}\). Each \(B_j\) must contain an element of \(B_0\). Then for each \(j \in \mathbb{N}\), there exist \(x_1^{(j)}, \ldots, x_n^{(j)} \in X, \varepsilon_1^{(j)}, \ldots, \varepsilon_n^{(j)}>0\) such that \(O_{x_1^{(j)}, \ldots, x_n^{(j)}; \varepsilon_1^{(j)}, \ldots, \varepsilon_n^{(j)}} \subseteq B_j\). Let
\[
A = \bigcap_{j=1}^{\infty} \{x_1^{(j)}, \ldots, x_n^{(j)}\} \subseteq X.
\]
Then \(A\) is a countable subset of \(X\). By Lemma 1, there exists \(x_0 \in X\) that does not belong to the linear span of \(A\). Because the set
\[
O_{x_0} = \{T \in L(X): \|Tx_0\| < 1\}
\]
is open in \((L(X), 0)\), there exists \(j \in \mathbb{N}\) such that \(B_j \subseteq O_{x_0}^{1}\). This implies
\[
O_{x_1^{(j)}, \ldots, x_n^{(j)}; \varepsilon_1^{(j)}, \ldots, \varepsilon_n^{(j)}} \subseteq O_{x_0}^{1} \quad (1)
\]
For convenience, we simply denote \(y_1^{(j)}, \ldots, y_n^{(j)}, \varepsilon_1^{(j)}, \ldots, \varepsilon_n^{(j)}\) as \(x_1, \ldots, x_n, \varepsilon_1, \ldots, \varepsilon_n\) respectively. Then \(x_0\) does not belong to the linear span of \(Y := \text{linear span of } \{x_1, x_2, \ldots, x_n\}\). Denote by \(Y^*\) the linear span of \(\{x_0\} \cup Y\). We
define a map \( S: Y \to K \{ z_0 \} \), \( S(tz_0 + y) = tz_0 \) for all \( t \in K \), \( y \in Y \).

Because \( S \) is a linear map between two finite dimensional normed spaces, it is continuous. By Hahn–Banach theorem, \( S \) extends to a continuous linear map (still denoted by \( S \)) from \( (X, \| \cdot \|) \) to \( K \{ z_0 \} \). Then \( SE_L(X) \).

We have \( Sz_0 = 1 \) and \( Sz_1 = Sz_2 = \ldots = Sx_n = 0 \). Hence,

\[
S \in O_{\varepsilon_1, \ldots, \varepsilon_n; \alpha_1, \ldots, \alpha_n \setminus \{ z_0 ; 1 \}}.
\]

This contradicts (1).

Next, suppose that \( (L(X), \| \cdot \|) \) is metrizable. We seek a contradiction by the same method as was done for \( (X, \| \cdot \|) \). \( (L(X), \| \cdot \|) \) has a countable system of open neighborhoods (of the zero operator), namely \( \{ B_t, B_0, B_0, \ldots \} \).

Each \( B_t \) must contain an element of \( B_t \). Then for each \( j \in \mathbb{N} \), there exist \( x^{(i)}_1, y^{(i)}_1, \ldots, x^{(i)}_j, y^{(i)}_j \in X, \varepsilon^{(i)}_1, \ldots, \varepsilon^{(i)}_j > 0 \) such that \( O_{\varepsilon^{(i)}_1, \ldots, \varepsilon^{(i)}_j} \subset B_t \).

But

\[
A = \bigcup_{j=1}^{\infty} \{ x^{(i)}_1, \ldots, x^{(i)}_j \} \subset X.
\]

Then \( A \) is a countable subset of \( X \). By Lemma 1, there exists \( x_0 \in X \) that does not belong to the linear span of \( A \).

Put \( y_0 = \frac{x_0}{\| x_0 \|} \). Because the set

\[
O_{x_0, y_0; 1} = \{ T \in L(X) : \| (x_0, y_0) \| < 1 \}
\]

is open in \( (L(X), \| \cdot \|) \), there exists \( j \in \mathbb{N} \) such that \( B_t \subset O_{x_0, y_0; 1} \).
This implies
\[ O_{x_1, \ldots, x_n; y_1, \ldots, y_n; \varepsilon_1, \ldots, \varepsilon_n} \subset O_{x_0, y_0; 1}. \]

For convenience, we simply denote \( x_1, y_1, \ldots, x_n, y_n, \varepsilon_1, \ldots, \varepsilon_n \) as \( x_1, y_1, \ldots, x_n, y_n, \varepsilon_1, \ldots, \varepsilon_n \) respectively. Then \( x_0 \) does not belong to \( Y := \text{linear span of } \{x_1, x_2, \ldots, x_n\} \). Denote by \( Y' \) the linear span of \( \{y_0\} \cup Y \). We define a map \( S : Y' \to K\{x_0\} \), \( S(tx_0 + y) = tx_0 \) for all \( t \in K \), \( y \in Y \). Because \( S \) is a linear map between two finite dimensional normed spaces, it is continuous. By Hahn–Banach theorem, \( S \) extends to a continuous linear map (still denoted by \( S \)) from \( (X, \| \cdot \|) \) to \( K\{x_0\} \). Then \( S \in \mathcal{L}(X) \).

We have
\[ (Sx_0, y_0) = (x_0, y_0) = (x_0, \frac{x_0}{\|x_0\|}) = 1, \]
\[ Sx_1 = Sx_2 = \ldots = Sx_n = 0. \]
Hence, \( S \in O_{x_1, y_1, \ldots, x_n, y_n; \varepsilon_1, \ldots, \varepsilon_n} \setminus O_{x_0, y_0; 1} \). This contradicts (2).

In case \( X \) is finite dimensional, \( \sigma \) and \( \tau \) are the same as the topology induced by the norm on \( L(X) \), and thus are metrizable. We prove this claim as follows. Let \( Y \) be the topology on \( L(X) \) that is induced by the norm. Then \( \sigma, \tau \subset Y \). For any \( x_1, y_1, \ldots, x_n, y_n \in X \), \( \varepsilon_1, \ldots, \varepsilon_n > 0 \), put \( M = \max \{\|y_1\|, \|y_2\|, \ldots, \|y_n\|\} + 1 \), and
\[ \varepsilon'_1 = \frac{\varepsilon_1}{M}, \ldots, \varepsilon'_n = \frac{\varepsilon_n}{M}. \]
For \( T \in \mathcal{O}_{e_1, \ldots, e_n; e_1', \ldots, e_n'} \),

\[
| (T_{e_j}, y_i) | \leq \| T_{e_j} \| \| y_i \| \leq \varepsilon_i \| y_i \| < \frac{\varepsilon_i}{m} M = \varepsilon_j \quad \forall 1 \leq j \leq n.
\]

Thus, \( T \in \mathcal{O}_{e_1, y_1; e_n(y_i); e_1', \ldots, e_n'} \). This implies \( \mathcal{O}_{e_1, \ldots, e_n; e_1', \ldots, e_n'} \subseteq \mathcal{O}_{e_1, y_1; e_n(y_i); e_1', \ldots, e_n'} \).

Then each open neighborhood of \( O \) in \( (\mathcal{L}(X), \tau) \) contains an open neighborhood of \( O \) in \( (\mathcal{L}(X), \delta) \). Hence, \( \tau \subseteq \delta \). It now suffices to show \( \delta \subseteq \tau \). We know that \( \delta \) has a system of open neighborhoods (of the zero operator) consisting of

\[
U_\varepsilon = \{ T \in \mathcal{L}(X) : \| T \| < \varepsilon \} \quad \forall \varepsilon > 0.
\]

Fix \( \varepsilon > 0 \). We need to show that \( U_\varepsilon \) contains a neighborhood of \( O \) in \( (\mathcal{L}(X), \delta) \). Let \( \{ e_1, e_2, \ldots, e_m \} \) be an orthonormal basis of \( X \). Put

\[
\delta = \frac{\varepsilon}{2m} > 0. \quad \text{For each } T \in \mathcal{O}_{e_1, e_2, \ldots, e_m; e_1', e_2', \ldots, e_m'} \text{, we have } (T_{e_j}, e_k) < \delta \quad \forall 1 \leq j, k \leq m.
\]

Then

\[
\| T e_j \| = \sum_{k=1}^{m} \| (T e_j, e_k) \| \leq m \delta^2 \quad \forall 1 \leq j \leq m.
\]

For each \( x \in X, \| x \| \leq 1 \), we write \( x = c_1 e_1 + \ldots + c_m e_m \) where \( |c_1|^2 + \ldots + |c_m|^2 \leq 1 \).

Then

\[
\| T x \| = \| c_1 T e_1 + \ldots + c_m T e_m \| \leq (|c_1| \| T e_1 \| + \ldots + |c_m| \| T e_m \|)^2 \leq (|c_1|^2 + \ldots + |c_m|^2)(\| T e_1 \|^2 + \ldots + \| T e_m \|^2) \leq 1 \cdot (m \delta^2 + \ldots + m \delta^2) = m^2 \delta^2 = \left( \frac{\varepsilon}{2} \right)^2.
\]
Thus, \[ \|T\| = \sup_{\|x\| \leq 1} \|Tx\| \leq \frac{\varepsilon}{2} < \varepsilon. \]

Then \( T \in U_{\varepsilon} \). This shows that \( \bigcup_{e_i, e_{i+1}, \ldots, e_{i+j} \in \mathbb{E}} \mathbb{N}_j \subseteq U_{\varepsilon} \).

(ii) Let \( R > 0 \). If \( X = \{0\} \) then the topology on \( B_{R}(X) = \{0\} \) is the trivial topology, which is metrizable and compact. Consider the case \( X \neq \{0\} \). Because \( X \) is separable, it has a dense countable subset \( \{e_k : k \in \mathbb{N}\} \). We can assume that this set does not contain 0.

We show that \( (B_{R}(X), d) \) is metrizable. Define a map \( d_{\varepsilon} : B_{R}(X) \times B_{R}(X) \rightarrow \mathbb{R} \),

\[
d_{\varepsilon}(T, S) = \sum_{k=1}^{\infty} \frac{\|T-S\|e_k\|}{\|e_k\|} 2^{-k} \quad \forall T, S \in B_{R}(X). 
\] (3)

First, we verify that \( d_{\varepsilon} \) is well-defined.

\[
\frac{\|T-S\|e_k\|}{\|e_k\|} \leq \frac{\|T-S\|\|e_k\|}{\|e_k\|} \leq \|T\| + \|S\| \leq 2R.
\]

Thus,

\[
d_{\varepsilon}(T, S) \leq \sum_{k=1}^{\infty} (2R)2^{-k} = 2R < \infty.
\]

Next, we verify that \( d_{\varepsilon} \) is a metric on \( B_{R}(X) \). It is clear that

\[
d_{\varepsilon}(T, S) > 0, \quad d_{\varepsilon}(T, S) = d_{\varepsilon}(S, T) \quad \forall T, S \in B_{R}(X).
\]

For \( T_1, T_2, T_3 \in B_{R}(X) \),

\[
d_{\varepsilon}(T_1, T_2) + d_{\varepsilon}(T_2, T_3) = \sum_{k=1}^{\infty} \frac{\|T_1 e_k - T_2 e_k\| + \|T_2 e_k - T_3 e_k\|}{\|e_k\|} 2^{-k} \geq
\]
\[
\sum_{k=1}^{\infty} \frac{\| (T_1 e_k - T_2 e_k) + (T_2 e_k - T_3 e_k) \|}{\| a_k \|} 2^{-k} = \sum_{k=1}^{\infty} \frac{\| (T_k - T) e_k \|}{\| a_k \|} 2^{-k} = d_f(T_k, T) .
\]

Suppose \( d_f(T, S) = 0 \). Then \( T a_k = S a_k \) for every \( k \in \mathbb{N} \). Take \( x \in X \). There exists a sequence \( \{a_n \} \) in \( \mathcal{E}(a) \) such that \( a_n \rightarrow x \). We have \( T a_n = S a_n \) for all \( n \in \mathbb{N} \). Then \( T x = \lim_{n \to \infty} T a_n = \lim_{n \to \infty} S a_n = S x \). Hence, \( T = S \). We have showed that \( d_f \) is a metric on \( B_{\mathbb{K}}(X) \).

Next, we show that the topology on \( B_{\mathbb{K}}(X) \) induced by \( d_f \) is the same as \( (B_{\mathbb{K}}(X), \sigma) \). Denote

\[
B_{\mathbb{K}}(S, \varepsilon) = \{ T \in B_{\mathbb{K}}(X) : d_f(T, S) < \varepsilon \} \quad \forall S \in B_{\mathbb{K}}(X), \forall \varepsilon > 0 .
\]

To show that the topology \( (B_{\mathbb{K}}(X), d_f) \) is coarser than \( (B_{\mathbb{K}}(X), \sigma) \), we show that \( B_{\mathbb{K}}(S, \varepsilon) \) contains a neighborhood of \( S \) in \( (B_{\mathbb{K}}(X), \sigma) \).

\[
B_{\mathbb{K}}(S, \varepsilon) = \{ T \in L(X) : \| T \| \leq R, \quad \sum_{k=1}^{\infty} \frac{\| (T - S) a_k \|}{\| a_k \|} 2^{-k} \leq \varepsilon \} .
\]

There exists \( n \in \mathbb{N} \) such that

\[
\sum_{k=m+1}^{\infty} 2^{-k} < \frac{\varepsilon}{4R} .
\]

Put \( \varepsilon' = \frac{\varepsilon}{2} \left( \sum_{k=1}^{m} \frac{1}{\| a_k \|} \right)^{-1} > 0 \). For every \( T \in \mathcal{C}(S + \sum_{k=1}^{m} a_k \varepsilon, a_{m+1}, \ldots, a_n \varepsilon', \ldots, a_k \varepsilon) \cap B_{\mathbb{K}}(X) \), we have \( \| T \| \leq R \) and \( \| (T - S) a_k \|, \ldots, \| (T - S) a_n \| \leq \varepsilon' \). Then
\[
\sum_{k=1}^{m} \frac{\| (T-S) a_k \|}{\| a_k \|} 2^{-k} < \varepsilon', \quad \sum_{k=1}^{m} \frac{1}{\| a_k \|} 2^{-k} < \varepsilon', \quad \sum_{k=1}^{m} \frac{1}{\| a_k \|} = \frac{\varepsilon}{2},
\]
\[
\sum_{k=m+1}^{\infty} \frac{\| (T-S) a_k \|}{\| a_k \|} 2^{-k} \leq \sum_{k=m+1}^{\infty} \frac{\| T-S \|}{\| a_k \|} 2^{-k} \leq \sum_{k=m+1}^{\infty} (2R) 2^{-k} < 2R \frac{\varepsilon}{4R} = \frac{\varepsilon}{2}.
\]

Then \[
\sum_{k=1}^{\infty} \frac{\| (T-S) a_k \|}{\| a_k \|} 2^{-k} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

This shows that \((S + O_{a_1, \ldots, a_m} ; \varepsilon', \ldots, \varepsilon') \cap B_R(X) \subset B_0(S, \varepsilon)\). Thus, \(B_0(S, \varepsilon)\) contains a neighborhood of \(S\) in \((B_R(X), \sigma)\).

To show that the topology \((B_R(X), \sigma)\) is coarser than \((B_R(X), d_R)\), we show that each neighborhood \(U\) of \(S\) in \((B_R(X), \sigma)\) contains a ball \(B_R(S, \varepsilon)\). We can assume \(U = (S + O_{x_1, \ldots, x_m} ; \varepsilon_1, \ldots, \varepsilon_n) \cap B_R(X)\) for some \(x_1, x_2, \ldots, x_m \in X, \varepsilon_1, \ldots, \varepsilon_n > 0\). Because the set \(\{ a_m : m \in \mathbb{N} \}\) is dense in \(X\), there exist \(a_1, a_2, \ldots, a_n\) such that
\[
\| a_k - x_k \| < \frac{\varepsilon_k}{4R} \quad \forall 1 \leq k \leq n.
\]

Put \(\varepsilon = \min \left\{ \frac{\varepsilon_k}{2\| a_k \|} : 1 \leq k \leq n \right\} > 0\). For every \(T \in B_0(S, \varepsilon)\), we have \(\| T \| \leq R\) and
\[
\sum_{k=1}^{\infty} \frac{\| (T-S) a_k \|}{\| a_k \|} 2^{-k} < \varepsilon.
\]

Thus,
\[
\frac{\| (T-S) a_k \|}{\| a_k \|} 2^{-m_k} < \varepsilon \leq \frac{\varepsilon_k}{2\| a_k \|} 2^{-m_k} \quad \forall 1 \leq k \leq n.
\]
Then \[ \| (T - S) a_{mk} \| < \frac{\varepsilon_k}{2} \quad \forall 1 \leq k \leq n. \]

Then
\[
\| (T - S) x_k \| \leq \| (T - S) a_{mk} \| + \| (T - S) (a_{mk} - x_k) \|
\]
\[
< \frac{\varepsilon_k}{2} + \frac{\| T - S \| \| a_{mk} - x_k \|}{8R} < \frac{\varepsilon_k}{4R}
\]
\[
< \frac{\varepsilon_k}{2} + \frac{\varepsilon_k}{2} = \varepsilon_k \quad \forall 1 \leq k \leq n.
\]
Thus, \( T - S \in \mathcal{O}_{\varepsilon_1, \ldots, \varepsilon_n; \varepsilon_1, \ldots, \varepsilon_n} \). This shows \( B_0(S, \varepsilon) \subset U \). We have showed that the topologies on \( (B_0(X), \sigma) \) and on \( (B_0(X), d_0) \) are the same.

Next, we show that \( (B_0(X), \tau) \) is also metrizable. Define a map
\[ d_\tau : B_0(X) \times B_0(X) \to \mathbb{R}, \]
\[ d_\tau(T, S) = \sum_{k,j=1}^{\infty} \frac{|(T - S) a_{kj}|}{\| a_k \| \| a_j \|} 2^{-k-j} \quad \forall T, S \in B_0(X). \tag{4} \]

First, we verify that \( d_\tau \) is well-defined.
\[ \frac{|(T - S) a_{kj}|}{\| a_k \| \| a_j \|} \leq \frac{\| (T - S) a_k \| \| a_j \|}{\| a_k \| \| a_j \|} \leq \| T - S \| \leq 2R \quad \forall T, S \in B_0(X). \]

Thus,
\[ d_\tau(T, S) \leq 2R \sum_{k,j=1}^{\infty} 2^{-k-j} = 2R \left( \sum_{k=1}^{\infty} 2^{-k} \right) \left( \sum_{j=1}^{\infty} 2^{-j} \right) = 2R < \infty. \]

Next, we verify that \( d_\tau \) is a metric on \( B_0(X) \). It is clear that
\[ d_\tau(T, S) > 0, \quad d_\tau(T, S) = d_\tau(S, T) \quad \forall T, S \in B_0(X). \]
For \( T, T_1, T_2 \in B_0(X), \)
\[ d_c(T_1, T_2) + d_c(T_2, T_3) = \sum_{k,j=1}^{\infty} \frac{|(T_1 a_k - T_2 a_k, a_j)| + |(T_2 a_k - T_3 a_k, a_j)|}{\|a_k\| \|a_j\|} 2^{-k-j} \]

\[ \geq \sum_{k,j=1}^{\infty} \frac{|(T_1 a_k - T_2 a_k, a_j)| + |(T_2 a_k - T_3 a_k, a_j)|}{\|a_k\| \|a_j\|} 2^{-k-j} \]

\[ = \sum_{k,j=1}^{\infty} \frac{|(T_1 a_k - T_3 a_k, a_j)|}{\|a_k\| \|a_j\|} 2^{-k-j} = d_c(T_1, T_3). \]

Suppose \( d_c(T, S) = 0 \). Then \( (T a_k, a_j) = (S a_k, a_j) \), for every \( k, j \in \mathbb{N} \). Let \( x, y \in X \). There exist sequences \( (x_n) \) and \( (y_n) \) in \( \{a_k : k \in \mathbb{N}\} \) such that \( x_n \to x \) and \( y_n \to y \). We have \( (T a_n, y_n) = (S a_n, y_n) \) for all \( n \in \mathbb{N} \).

\[ |(T a, y) - (S a, y)| \leq \frac{|(T a_n, y_n) - (T a_n, y)| + |(T a_n, y) - (T a_n, y)| + |(T a_n, y_n) - (S a_n, y_n)|}{\|a\|} \]

\[ \leq |(T a_n, y_n) - (S a_n, y_n)| + |(S a_n, y_n) - (S a_n, y)| \]

\[ \leq |(T a_n, y_n) - (S a_n, y_n)| + |(S a_n, y_n) - y_n| + |y_n - y| \]

\[ \to 0 \text{ as } n \to \infty. \]

Then,

\[ \|T x - x_n\| \to 0 \text{ as } n \to \infty, \]

\[ \|T x - x_n\| \to \|T x\|. \]

Thus, \( (T x, y) = (S x, y) \) for every \( x, y \in X \). Then

\[ \|T x - x_n\| = (T x - S x, y) = (T x, y) - (S x, y) = 0 \quad \forall x \in X. \]

Therefore, \( T = S \). We have showed that \( d_c \) is a metric on \( B_k(X) \).
Next, we show that the topology on $\mathcal{B}_K(X)$ induced by $d_\mathcal{K}$ is the same as 
$(\mathcal{B}_K(X), \tau)$. Denote 
$$
\mathcal{B}_c(S, \varepsilon) = \left\{ T \in \mathcal{B}_K(X) : d_\mathcal{K}(T, S) < \varepsilon \right\} \quad \forall S \in \mathcal{B}_K(X), \forall \varepsilon > 0.
$$
To show that the topology $(\mathcal{B}_K(X), d_\mathcal{K})$ is coarser than $(\mathcal{B}_K(X), \tau)$, we show that $\mathcal{B}_c(S, \varepsilon)$ contains a neighborhood of $S$ in $(\mathcal{B}_K(X), \tau)$.

$$
\mathcal{B}_c(S, \varepsilon) = \left\{ T \in \mathcal{B}_K(X) : \|T\| < R, \sum_{k_j=1}^{\infty} \frac{|(T-S)_{k_j} q_j|}{\|q_j\| \|q_j\|} 2^{-k_j} < \varepsilon \right\}.
$$

There exists $m \in \mathbb{N}$ such that

$$
\sum_{k_j=1}^{m} 2^{-k_j} - \sum_{k_j=1}^{\infty} 2^{-k_j} < \frac{\varepsilon}{4R}.
$$

Put

$$
\varepsilon' = \frac{\varepsilon}{2} \left( \sum_{k_j=1}^{m} \frac{1}{\|q_j\| \|q_j\|} \right)^{-1} > 0.
$$

Let $U = 0 (a_1, a_1, a_1, \ldots, a_1, a_1, a_1, a_1, a_1, \ldots, a_1, a_1, a_1, a_1, \ldots, a_m, a_1, a_1, \ldots, a_m, a_1) \varepsilon', \varepsilon', \varepsilon', \ldots.

For every $T \in (S + U) \cap \mathcal{B}_K(X)$,

$$
\sum_{k_j=1}^{m} \frac{|(T-S)_{k_j} q_j|}{\|q_j\| \|q_j\|} 2^{-k_j} < \sum_{k_j=1}^{m} \frac{\varepsilon'}{\|q_j\| \|q_j\|} 2^{-k_j} < \varepsilon' \sum_{k_j=1}^{m} \frac{1}{\|q_j\| \|q_j\|} = \frac{\varepsilon}{2}.
$$

$$
\sum_{(k_j) \geq m \text{ or } j \geq m} \frac{|(T-S)_{k_j} q_j|}{\|q_j\| \|q_j\|} 2^{-k_j} \leq \sum_{(k_j) \geq m \text{ or } j \geq m} \frac{\|T-S\| \|q_j\| \|q_j\|}{\|q_j\| \|q_j\|} 2^{-k_j} \leq 2R \frac{\varepsilon}{2} 2^{-k_j}.
$$
\[ \leq 2R \left( \sum_{k,j=1}^{\infty} 2^{-k-j} - \sum_{k,j=1}^{m} 2^{-k-j} \right) \times \frac{\frac{\varepsilon}{4}}{2} < \frac{\varepsilon^2}{2} + \frac{\varepsilon}{2} = \varepsilon. \]

This shows \((S+U) \cap B_R(x) \subset B_{\varepsilon}(S, \varepsilon)\). Thus, \(B_{\varepsilon}(S,\varepsilon)\) contains a neighborhood of \(S\) in \((B_R(x), \varepsilon)\).

To show that the topology \((B_R(x), \varepsilon)\) is coarser than \((B_R(x), d_x)\), we show that each neighborhood \(V\) of \(S\) in \((B_R(x), \varepsilon)\) contains a ball \(B_\varepsilon(S, \varepsilon)\). We can assume \(V = (S + \mathbb{Q}_{x_1, y_1, \ldots, x_m, y_m} \mathbb{Q}) \cap B_\varepsilon(V)\) for some \(x_1, y_1, \ldots, x_m, y_m \in X, \varepsilon_1, \ldots, \varepsilon_n > 0\). Because the set \(\{a_m : m \in \mathbb{N}\}\) is dense in \(X\), there exist \(a_1, a_2, \ldots, a_n\) such that

\[ \| x_e - a_m \| < \frac{\varepsilon e}{6R(\|y\|e + 1)} \quad \forall 1 \leq e \leq n. \]  

There exist \(a_1, a_2, \ldots, a_n\) such that

\[ \| y_e - a_e \| < \frac{\varepsilon e}{6R \|a_m\|} \quad \forall 1 \leq e \leq n. \]  

Let \(\varepsilon = \min \left\{ \frac{\varepsilon e}{3}, \frac{2^{-m-e-2}}{\|a_m\| \|a_e\|} : 1 \leq e \leq n \right\} > 0.\)
For every $T \in B_2(S, \varepsilon)$, we have $\|T\| \leq \varepsilon$ and
\[
\sum_{k=1}^{2k} \frac{|(T-S)(q_k, q_k)|}{||q_k|| ||q_k||} 2^{-k-j} < \varepsilon.
\]
Thus,
\[
\frac{|(T-S)(q_k, q_k)|}{||q_k|| ||q_k||} 2^{-m_k-2} \leq \frac{\varepsilon}{3} \leq \frac{2^{-m_k-2}}{||q_k|| ||q_k||}.
\]
Then
\[
|(T-S)(q_k, q_k)| < \frac{\varepsilon}{3} + 1 \leq \ell \leq n.
\]
Then
\[
|(T-S)(x, y)| \leq |(T-S)(x-a_m, y-c)| + |(T-S)a_m, y-c-a_c)|
\]
\[
< \|T-S\| \|x-a_m\| \|y-c\| + \|T-S\| \|a_m\| \|y-c-a_c\| + \varepsilon \leq 2R \|x-a_m\| \|y-c\| + 2R \|a_m\| \|y-c-a_c\| + \frac{\varepsilon}{3}
\]
\[
\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \quad \forall 1 \leq \ell \leq n.
\]
Thus, $T-S \in \bigcap_{i=1}^{n} B_{\varepsilon}(x, \varepsilon), \ldots, B_{\varepsilon}(x, \varepsilon), \ldots, B_{\varepsilon}(x, \varepsilon)$. This shows $B_{\varepsilon}(S, \varepsilon) \subseteq V$. We have showed that the topology on $(B_2(X), \varepsilon)$ and the topology $(B_r(X), d_2)$ are the same.

By the definition of $d_2$ at (3) and $d_\varepsilon$ at (4), we deduce that for any $T, T_1, T_2, T_3, \ldots \in B_r(X)$,
(a) $T_n \rightarrow T$ in $(B_r(X), d_0)$ if and only if $\lim_{n \to \infty} T_n x_k = T x_k$ for all $k \in \mathbb{N}$.

(b) $T_n \rightarrow T$ in $(B_r(X), d_c)$ if and only if $\lim_{n \to \infty} (T_n x_k, a_j) = (T x_k, a_j)$ for all $k, j \in \mathbb{N}$.

Next, we show that $(B_r(X), c)$ is compact. That is to show $(B_r(X), d_c)$ is a compact metric space. Let $(T_n)$ be a sequence in $(B_r(X), d_c)$. We show that it has a convergent subsequence. Let $\{x_k : k \in \mathbb{N}\}$ be a dense subset of $X$. For each $k, j \in \mathbb{N}$,

$$|(T_n x_k, a_j)| \leq \|T_n x_k\| \|a_j\| \leq \|T_n\| \|x_k\| \|a_j\| \leq R \|x_k\| \|a_j\| \quad \text{for all } n \in \mathbb{N}.$$ 

Thus, the sequence $\{(T_n x_k, a_j)\}_{n \in \mathbb{N}}$ is bounded in $K$, which is either $\mathbb{R}$ or $\mathbb{C}$. Then it has a convergent subsequence. Choose a way to enumerate the elements of $\mathbb{N} \times \mathbb{N}$, for example as in the following diagram.

```
(1,1) → (1,2) → (1,3) → (1,4) → (1,5) → ... 
(2,1) → (2,2) → (2,3) → (2,4) → (2,5) → ... 
(3,1) → (3,2) → (3,3) → (3,4) → (3,5) → ... 
(4,1) → (4,2) → (4,3) → (4,4) → (4,5) → ... 
```

Let $(T_n^{(1,1)})$ be a subsequence of $(T_n)$ such that $\{(T_n^{(1,1)} x_1, a_1)\}$ converges.

Let $(T_n^{(1,2)})$ be a subsequence of $(T_n^{(1,1)})$ such that $\{(T_n^{(1,2)} x_1, a_2)\}$ converges.

Let $(T_n^{(2,2)})$ be a subsequence of $(T_n^{(1,2)})$ such that $\{(T_n^{(2,2)} x_2, a_2)\}$ converges.
by Cantor's diagonal method, there is a subsequence of \((T_n)\), called \((T'_{n})\),
such that \(\{(T'_{n}, x, y)\}_{n\in\mathbb{N}}\) converges for all \(x, y \in \mathbb{N}\). For an explicit expression
for \((T'_{n})\), we can take \((T'_{n})_{n\in\mathbb{N}} = (T_{n}^{(k, j)})_{n\in\mathbb{N} \times \mathbb{N}}\) where the order
in \(\mathbb{N} \times \mathbb{N}\) is indicated as in the above diagram. By replacing \((T_n)_{n\in\mathbb{N}}\)
with \((T'_{n})_{n\in\mathbb{N}}\), we can assume \(\{(T'_{n}, x, y)\}_{n\in\mathbb{N}}\) converges for all \(x, y \in \mathbb{N}\).
Let \(V\) be the linear span of \(\{x_k : k \in \mathbb{N}\}\). Then \(\{T'_{n}(x, y)\}_{n\in\mathbb{N}}\) converges
for all \(x, y \in \mathbb{C}\). Denote
\[
(T_{x, y}) = \lim_{n \to \infty} (T_{n}, x, y) \quad \forall x, y \in \mathbb{C}. \quad (7)
\]
Then for every \(x \in \mathbb{C}\), \(T_x\) can be viewed as a linear map from \(\mathbb{C}\) to \(\mathbb{C}\).
\[
|T_{x}(y)| \leq \|T_x\| \|x\| \|y\| \leq R \|x\| \|y\| \quad \forall x, y \in \mathbb{C}.
\]
Thus, \(|T_{x}(y)| \leq R \|x\| \|y\|\) for all \(y \in \mathbb{C}\). This implies \(T_x : \mathbb{C} \to \mathbb{C}\) is a
continuous linear map. Since \(V = \mathbb{C}\), \(T_x\) can be extended to a continuous
linear map from \(\mathbb{C}\) to \(\mathbb{C}\). Because \(\mathbb{C}\) is a Hilbert space, \(T_x\) can be
identified with an element of \(\mathbb{C}\). Then \(T : \mathbb{C} \to \mathbb{C}\) is a linear map.
\[
\|T\| = \sup_{y \in \mathbb{C}, \|y\| = 1} |T(y)| = \sup_{y \in \mathbb{C}, \|y\| = 1} R \|y\| \quad \forall x \in \mathbb{C}. \quad (8)
\]
This implies \(T\) is continuous. Because \(V = \mathbb{C}\), \(T\) can extend to a continuous
linear map from \(\mathbb{C}\) to \(\mathbb{C}\). Then \(T \in \mathcal{L}(\mathbb{C})\). The estimate (8) implies \(\|T\| \leq R\).
Then $TEB_k(X)$. By (I) and statement (b), we conclude that $(t_n)$ converges to $T$ in $(B_k(X), d_T)$. Thus, $(B_k(X), d_T)$ is a compact metric space.

In case $X$ is finite dimensional, $B_k(X)$ is a compact subset of $B_k(X)$. We showed on pages 5-6 that $(B_k(X), d_T)$ is the same as the topology induced by the norm. Thus, $(B_k(X), d_T)$ is compact. Now assume $X$ is infinite dimensional. We show that $(B_k(X), d_T)$ is not compact. Specifically, we construct a sequence $(t_n)$ in $(B_k(X), d_T)$ that has no convergent subsequence.

Let $\{e_i : i \in \mathbb{N}\}$ be a Hilbert basis of $X$. By replacing the set $\{e_i : i \in \mathbb{N}\}$ with $\{e_i : i \in \mathbb{N}\} \cup \{e_3\}$, we can assume that $e_2 \in \{e_i : i \in \mathbb{N}\}$. Let $S$ be the $1$-translation operator of $X$, i.e. $S \in L(X)$ and $Se_k = e_{k+1}$ for every $k \in \mathbb{N}$.

Then $\|S\| = 1$. Denote $S^n = S \circ S \circ \cdots \circ S$ ($n$ times). Then

$$\|S^n\| = \underbrace{\|S\| \cdot \|S\| \cdots \|S\|}_{n \text{ times}} \leq 1.$$ 

Let $T_n = R \cdot S^n$ for every $n \in \mathbb{N}$. Then $T_n \in L(X)$ and $\|T_n\| \leq R$.

$$T_n e_i = R \cdot S^n e_i = Re_{n+1} \quad \forall n \in \mathbb{N}.$$ 

Then

$$\|T_n e_i - T_m e_i\| = R \|e_{n+i} - e_{m+i}\| = R \sqrt{2} \quad \forall m, n \in \mathbb{N}, m \neq n.$$ 

This implies the sequence $(T_n e_i)$ has no convergent subsequence. By statement (a), the sequence $(T_n)$ has no convergent subsequence.
(iii) Suppose $X$ is infinite dimensional. We show that the composition $(S, T) \mapsto S \circ T$ is not continuous as a map from $(\mathcal{L}(X), \sigma) \times (\mathcal{L}(X), \sigma)$ to $(\mathcal{L}(X), \sigma)$.

Suppose otherwise. Take any $x_0 \in X \setminus \{0\}$. Then the preimage of

$$O_{x_0, 1} = \{ L \in \mathcal{L}(X) : \| L x_0 \| < 1 \}$$

under the decomposition map is a neighborhood of $(0, 0)$ in $(\mathcal{L}(X), \sigma) \times (\mathcal{L}(X), \sigma)$. There exist $x_1, x_2, ..., x_n \in X$, $x'_1, ..., x'_m \in X$, $\varepsilon_1, ..., \varepsilon_n > 0$, $\varepsilon'_1, ..., \varepsilon'_m > 0$ such that $S \circ T \in O_{x_0, 1}$ for all $S \in O_{x_1, \varepsilon_1} \times ... \times O_{x_n, \varepsilon_n}$ and $T \in O_{x'_1, \varepsilon'_1} \times ... \times O_{x'_m, \varepsilon'_m}$.

In other words, for any $S, T \in \mathcal{L}(X)$ such that

$$\| S x_j \| < \varepsilon_j \quad \forall 1 \leq j \leq n,$$

$$\| T x'_k \| < \varepsilon'_k \quad \forall 1 \leq k \leq m$$

we have $\| S(T x_0) \| < 1$. Put

$$\varepsilon = \frac{\min \{ \varepsilon'_1, \varepsilon'_2, ..., \varepsilon'_m \}}{\max \{ 1, (\| x'_1 \|, \| x'_2 \|, ..., \| x'_m \|, \| x_0 \|) \} + 1},$$

$$Y = \text{linear span} \{ x_1, x_2, ..., x_n \},$$

$$Z = \text{linear span} \{ x'_1, x'_2, ..., x'_m \}.$$

Because $X$ is infinite dimensional, there exists $y_0 \in X \setminus Y$. By rescaling $y_0$ if necessary, we can assume $\| y_0 \| = \varepsilon$. We show that there exists $T \in O_{x'_1, \varepsilon'_1} \times ... \times O_{x'_m, \varepsilon'_m}$ such that $B x_0 = y_0$. Consider two following cases.
Let \( \{x_0, x_1, x_2, \ldots, x_k\} \) be an orthogonal basis of \( X \). Define a linear map

\[
B : X \to K \{y_0\},
\]

\[
B x_0 = y_0, \quad B x_1 = B x_2 = \ldots = B x_k = 0.
\]

By Hahn-Banach theorem, \( B \) can extend to a continuous linear map from \( X \) to \( K \{y_0\} \), which is still denoted by \( B \). We can regard \( B \in \mathcal{L}(X) \). Then

\[
B x_j' = \frac{(x_0, x_j')}{\|x_0\|^2} y_0 + \frac{(x_1, x_j')}{\|x_1\|^2} B x_1 + \ldots + \frac{(x_k, x_j')}{\|x_k\|^2} B x_k = y_0.
\]

Then

\[
\|B x_j'\| = \|x_0\| \frac{\|y_0\|}{\|x_0\|^2} = \frac{|(x_0, x_j')|}{\|x_0\|^2} \|y_0\| < \|y_0\|, \quad \forall 1 \leq j \leq m.
\]

Thus, \( B \in O_{x_0, \ldots, x_m; y_0, \ldots, y_m} \).

Since \( x_0 \notin 2 \) :

Define a linear map

\[
B : \mathbb{Z} \oplus K \{y_0\} \to K \{y_0\},
\]

\[
B x_0 = y_0, \quad B 1 = 0.
\]

By Hahn-Banach theorem, \( B \) can extend to a continuous linear map from \( \mathbb{Z} \oplus K \{y_0\} \) to \( K \{y_0\} \). We can regard \( B \in \mathcal{L}(X) \). Since \( B 1 = 0 \), \( B x_1' = B x_2' = \ldots = B x_k' = 0 \).

Then \( B \in O_{x_0, \ldots, x_m; y_0, \ldots, y_m} \). The existence of \( B \) is proved.

Next, define a linear map

\[
A : y \oplus K \{y_0\} \to K \{y_0\},
\]

\[
A y_0 = \frac{y_0}{\varepsilon}, \quad A 1 = 0.
\]
By Hahn-Banach theorem, $A$ can extend to a continuous linear map from $X$ to $K\{y\}$. We can regard $A \in \mathcal{L}(X)$. Because $A_{x_1} = A_{x_2} = \ldots = A_{x_n} = 0$, $A \in \mathcal{B}_{\{x_1, \ldots, x_n\}; \epsilon_1, \ldots, \epsilon_n}$.

$$A(\delta \cdot y_0) = A \cdot y_0 = \frac{y_0}{\epsilon}.$$  

Then $\|A(\delta \cdot y_0)\| = \frac{\|y_0\|}{\epsilon} = 1$, which implies $A \cdot 0 = 0$. This is a contradiction.

(iii) We show that the composition $S \circ T$ is continuous as a map from $(\mathcal{B}_{r_1}(X), \delta) \times (\mathcal{B}_{r_2}(X), \delta)$ to $(\mathcal{B}_{r_1 r_2}(X), \delta)$.

First, this map is well-defined because $\|S \circ T\| \leq \|S\| \|T\| \leq r_1 r_2$. Because $(\mathcal{B}_{r}(X), \delta)$ is metrizable for every $R > 0$, it suffices to show that the composition map is sequentially continuous. Let $(s_n)$ be a sequence in $(\mathcal{B}_{r_1}(X), \delta)$ that converges to $s \in \mathcal{B}_{r_1}(X)$. Let $(t_n)$ be a sequence in $(\mathcal{B}_{r_2}(X), \delta)$ that converges to $t \in \mathcal{B}_{r_2}(X)$. We show that $(s_n \circ t_n)$ converges to $s \circ t$ in $(\mathcal{B}_{r_1 r_2}(X), \delta)$. By statement (ii), on page 156, it suffices to show that $s_n \circ t_n(y) \rightarrow s \circ t(y)$ for every $y \in X$.

Fix $y \in X$. By Because $t_n \rightarrow t$ in $(\mathcal{B}_{r_2}(X), \delta)$, $t_n y \rightarrow t y$. But $ty = z$. Because $s_n \rightarrow s$ in $(\mathcal{B}_{r_1}(X), \delta)$, $s_n z \rightarrow s z$. Then

$$\|s_n \circ t_n(y) - s \circ t(y)\| = \|s_n(t_n y - t y) + (s_n - s) \circ t(y)\|$$

$$\leq \|s_n\| \|t_n y - t y\| + \|s_n z - s z\|$$

$$\leq R_1 \|t_n y - t y\| + \|s_n z - s z\| \quad \text{as } n \to \infty.$$
Therefore, \( \lim_{n \to \infty} S_n \circ T_n(y) = S \circ T(y) \).

(v) We show that the composition \((S \circ T) \mapsto S \circ T\) is continuous as a map from \((B_{2R_1}(X), \tau) \times (B_{2R_2}(X), \sigma) \) to \((B_{R_4 \cdot \tau}(X), \tau)\).

Because \((B_{2R}(X), \tau)\) and \((B_{2R_2}(X), \sigma)\) are metrizable for every \(R > 0\), it suffices to show that the composition map is sequentially continuous. Let

\[
S_n \to S \quad \text{in} \quad (B_{2R_1}(X), \tau),
\]

\[
T_n \to T \quad \text{in} \quad (B_{2R_2}(X), \sigma).
\]

We show that \(S_n \circ T_n \to S \circ T\) in \((B_{R_4 \cdot \tau}(X), \tau)\). By statement (b) on page 15, it suffices to show that \((S_n \circ T_n(x), y) \to (S \circ T(x), y)\) for all \(x, y \in X\).

Fix \(x, y \in X\). Because \(T_n \to T\) in \((B_{2R_2}(X), \sigma)\), \(T_n x \to T x\). Put \(z = T x\). Because \(S_n \to S\) in \((B_{2R_1}(X), \tau)\), \((S_n z, y) \to (S z, y)\). We have

\[
| (S_n \circ T_n(x), y) - (S \circ T(x), y) | = | (S_n (T_n x) - S(T x), y) |
\]

\[
= | (S_n (T_n x - T x), y) + (S_n (T x), y) - (S(T x), y) |
\]

\[
\leq | (S_n (T_n x - T x), y) | + | (S_n (T x), y) - (S(T x), y) |
\]

\[
\leq \| S_n \| \| T_n x - T x \| \| y \| + | (S_n z, y) - (S z, y) |
\]

\[
\leq R_4 \| T_n x - T x \| \| y \| + \underbrace{| (S_n z, y) - (S z, y) |}_{\to 0} \quad \text{as} \quad n \to \infty.
\]

Therefore, \( \lim_{n \to \infty} (S_n \circ T_n(x), y) = (S \circ T(x), y) \).
(vi) Suppose $X$ is infinite dimensional. We show that the composition $(S,T) \rightarrow S \circ T$ is not continuous as a map from $(\mathcal{B}_{c_0}(X), \sigma) \times (\mathcal{B}_c(X), \tau)$ to $(\mathcal{B}_{c_0}(X), \tau)$.

Let $(e_n)$ be a Hilbert basis of $X$. For each $x \in X$, we have the Parseval's identity

$$\|x\|^2 = \sum_{n=1}^{\infty} |(x, e_n)|^2.$$ 

Thus, $\lim_{n \to \infty} (x, e_n) = 0$. Consider the linear maps $S_n, T_n : X \to X$,

$$S_n x = R_1(x, e_n) e_1 \quad \forall x \in X,$$

$$T_n x = R_2(x, e_1) e_n \quad \forall x \in X.$$ 

Then

$$\|S_n x\| = R_1 \left| (x, e_n) \right| \leq R_1 \|x\| \|e_n\| = R_1 \|x\|,$$

$$\|T_n x\| = R_2 \left| (x, e_1) \right| \leq R_2 \|x\| \|e_1\| = R_2 \|x\|.$$ 

Thus, $S_n, T_n \in L(X)$ and $\|S_n\| \leq R_1$, $\|T_n\| \leq R_2$. For each $x \in X$, $S_n x \to 0$ because $(x, e_n) \to 0$ as $n \to \infty$. Thus $S_n \to 0$ in $(\mathcal{B}_{c_0}(X), \sigma)$ as $n \to \infty$. For $x, y \in X$,

$$(T_n x, y) = (R_2(x, e_1) e_n, y) = R_2(x, e_1)(e_n, y) \to 0 \quad \text{as} \quad n \to \infty.$$ 

Thus, $T_n \to 0$ in $(\mathcal{B}_c(X), \tau)$ as $n \to \infty$.

$$S_n \circ T_n (x) = \sum_{n=1}^{\infty} (R_2(x, e_1) e_n, e_1) e_1 = R_2(x, e_1) e_n$$

$$= R_2(x, e_1) R_1(e_n, e_1) e_1 = R_1 R_2(x, e_1) e_1 \quad \forall x \in X.$$ 

Then
\((S_n \circ T_n(e_i), e_i) = (R_1 R_2(e_i, e_i) e_i, e_i) = R_1 R_2 > 0 \quad \forall n \in \mathbb{N}. \)

This shows that \(S_n \circ T_n\) does not tend to \(0\) in \((B_K, \tau)\).

(vii) Suppose \(X\) is infinite dimensional. We show that the involution \(S \mapsto S^*\) is not continuous as a map from \((B_K(X), \sigma)\) to \((B_K(X), \sigma)\).

Let \((e_n)\) be a Hilbert basis of \(X\) and \(S_n \in B_K(X)\) be the map as in Part (vi):

\[ S_n x = R(x, e_n) e_1 \quad \forall x \in X. \]

We show that \(S_n \to 0\) in \((B_K(X), \sigma)\). For \(x, y \in X\),

\[ (S_n x, y) = (R(x, e_n) e_1, y) = R(x, e_n) (e_1, y) = (x, R(\overline{e_1 y}) e_n) \]

\[ = (x, R(y, e_1) e_n). \]

Thus, \(S_n^* y = R(y, e_1) e_n\). Then \(S_n^* e_1 = R(e_1, e_1) e_n = R e_n\). Since \(\|S_n^* e_1\| = R\), \(S_n^* e_1 \to 0\) in \(X\) as \(n \to \infty\). This shows that \(S_n^* \to 0\) in \((B_K(X), \sigma)\).

(viii) We show that the involution \(T \mapsto T^*\) is continuous as a map from \((B_K(X), \tau)\) to \((B_K(X), \tau)\).

Let \((T_n)\) be a sequence in \((B_K(X), \tau)\) that converges to \(T\). For every \(x, y \in X\),

\[ (T_n^* x, y) = (x, T_n y) = \overline{(T_n y, x)} \quad \forall n \in \mathbb{N}. \]

Taking the limit both sides, we get

\[ \lim_{n \to \infty} (T_n^* x, y) = \overline{(T y, x)} = (x, T y) = (T^* x, y). \]

This shows that \((T_n^*)\) converges to \(T^*\) in \((B_K(X), \tau)\).
(ix) We show that the involution $T \mapsto T^*$ is continuous as a map from $(\mathcal{L}(X), \tau)$ to $(\mathcal{L}(X), \tau)$. Because the map is linear, it suffices to show that it is continuous at 0. Let $U$ be a neighborhood of 0 in $(\mathcal{L}(X), \tau)$. There exist $x_1, y_1, \ldots, x_n, y_n \in X$ and $\varepsilon_1, \ldots, \varepsilon_n > 0$ such that

$$O_{x_1, y_1, \ldots, x_n, y_n; \varepsilon_1, \ldots, \varepsilon_n} \subset U.$$ 

Put $V = O_{x_1, y_1, \ldots, x_n, y_n; \varepsilon_1, \ldots, \varepsilon_n}$. For each $T \in V$, $|(T_{xy}, x_j)| < \varepsilon_j$ for all $1 \leq j \leq n$. Then

$$|(T^*_{xy}, y_j)| = |(T_{xy}, y_j)| = |(T_{xy}, x_j)| < \varepsilon_j, \quad \forall 1 \leq j \leq n.$$ 

Thus, $T^* \in O_{x_1, y_1, \ldots, x_n, y_n; \varepsilon_1, \ldots, \varepsilon_n} \subset U$. We conclude that the involution map is continuous at 0.

(2) Let $S^1 \subset \mathbb{C}$ be the unit circle centered at 0, and $\lambda$ be the Lebesgue measure on $S^1$ normalized such that $\lambda(S^1) = 2\pi$. (It is the measure of angle). Let $X = L^2(S^1, \lambda)$, which is a Hilbert space over $\mathbb{C}$. Define a linear map $T : X \to X$, $Tf(z) = zf(z)$ for $f \in X$, $z \in S^1$, where $z = x + iy$, $x, y \in \mathbb{R}$.

$$\|Tf\|_X = \left(\int_{S^1} |f(z)|^2 d\lambda \right)^{1/2} \leq \left(\int_{S^1} |f(z)|^2 d\lambda \right)^{1/2} = \|f\|_X \quad \forall f \in X.$$ 

Thus, $T$ is continuous and $\|T\| \leq 1$. We compute $T^*$. For $f, g \in X$,

$$\langle Tf, g \rangle = \int_{S^1} Tf(z) \overline{g(z)} d\lambda = \int_{S^1} zf(z) \overline{g(z)} d\lambda = \int_{S^1} \overline{zf(z)} \overline{g(z)} d\lambda = \int_{S^1} \overline{f(z)} \overline{g(z)} d\lambda = (f, \overline{g}).$$
Thus, \( T^*g(z) = x\bar{g(z)} \). Then
\[
T(T^*g)(z) = T(z \mapsto x\bar{g(z)}) = x^2\bar{g(z)},
\]
\[
T^*(Tg)(z) = T^*(z \mapsto xg(z))(z) = x^2\bar{g(z)} = x^2g(z).
\]
Then \( TT^* = T^*T \), i.e., \( T \) is a normal operator of \( X \).

We compute the spectrum of \( T \). For \( x \in \mathbb{C} \),
\[
(\alpha \text{Id}_X - T)f(z) = \alpha f(z) - T f(z) = (\alpha - x)f(z) \quad \forall f \in X, \forall z \in S^1.
\]
If \( \alpha \notin [-1,1] \) then \( \alpha \text{Id}_X - T \) has a continuous inverse, which is
\[
g(z) \mapsto \frac{g(z)}{\alpha - x} \quad \forall g \in X, \forall z \in S^1.
\]
Consider the case \( \alpha \in [-1,1] \). Suppose there exists \( f_0 \in X \) such that \((\alpha \text{Id}_X - T)f_0 = 0\).

Then
\[
f_0(z) = \frac{1}{\alpha - x} \quad \text{a.e. } z \in S^1.
\]

Then
\[
\|f_0\|_X^2 = \int_{S^1} |f_0(z)|^2 d\lambda = \int_0^{2\pi} |f_0(\text{e}^{i\theta})|^2 d\theta = \int_0^{2\pi} \frac{1}{(\alpha - \cos \theta)^2} d\theta. \tag{1}
\]

Since \( \alpha \in [-1,1] \), there exists \( \theta_0 \in [\theta, 2\pi] \) such that \( \alpha = \cos \theta_0 \). Then
\[
|\alpha - \cos \theta| = |\cos \theta_0 - \cos \theta| = |\theta_0 - \theta| \sin \delta \quad (\text{for some } \delta \text{ between } \theta_0 \text{ and } \theta)
\][
\leq |\theta_0 - \theta|.
\]

Then (1) implies
\[
\|f_0\|_X^2 \geq \frac{2\pi}{(\theta_0 - \theta)^2} \int_0^{2\pi} \frac{1}{(\theta_0 - \theta)^2} d\theta = \infty.
\]

This is a contradiction. Therefore, there is no \( f_0 \in X \) such that...
\((x \text{Id}_X - T)_{\phi_0} = 1\). Then \(x \text{Id}_X - T \notin \mathcal{L}^\infty(X)\). We conclude that the spectrum of \(T\) is 
\(\sigma(T) = [-1, 1]\).

Next, we compute \(T\) as an algebra action of \(C([-1, 1])\) on \(X\). For each polynomial \(P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0\), we denote
\[
P(T) = a_n T^n + a_{n-1} T^{n-1} + \ldots + a_1 T + a_0 \text{Id}_X \in \mathcal{L}(X).
\]
Then
\[
P(T)f(z) = a_n T^n f(z) + a_{n-1} T^{n-1} f(z) + \ldots + a_1 T f(z) + a_0 f(z)
\]
\[
= a_n x^n f(z) + a_{n-1} x^{n-1} f(z) + \ldots + a_1 x f(z) + a_0 f(z)
\]
\[
= P(x) f(z) \quad \forall f \in X, x \in S^1.
\]
Denote \(T(P) \in \mathcal{L}(X)\) to be the map
\[
T(P).f(z) = P(x) f(z) \quad \forall f \in X, x \in S^1.
\]
Then \(T\) can be regarded as an algebra action of \(C([-1, 1])\) on \(X\) as follows.
\[
T : C([-1, 1]) \rightarrow \mathcal{L}(X),
\]
\[
T(h)f(z) = h(x)f(z) \quad \forall h \in C([-1, 1]), f \in X, x \in S^1.
\]
Next, we decompose \(X\) into invariant cyclic subspaces under the action \(T\).

Put \(\xi_1, \xi_2 \in X\), \(\xi_1(z) \equiv 1, \xi_2(z) = y\),
\[
X_1 = \{ T(g) \xi_1 : g \in C([-1, 1]) \} \subset X,
\]
\[
X_2 = \{ T(g) \xi_2 : g \in C([-1, 1]) \} \subset X.
\]
Write
\[ Y_1 = \{ Tg \xi_1 : g \in C([-1,1]) \} = \{ z \mapsto g(x) \xi_1(x) : g \in C([-1,1]) \} = \{ z \mapsto g(x) : g \in C([-1,1]) \}, \]

\[ Y_2 = \{ T(h) \xi_2 : h \in C([-1,1]) \} = \{ z \mapsto h(x) \xi_2(x) : h \in C([-1,1]) \} = \{ z \mapsto h(x) y : h \in C([-1,1]) \}. \]

Denote by \((\cdot,\cdot)\) the scalar product in \(X\). We have

\[
(\mapsto g(x), \mapsto h(x) y) = \int g(x) \overline{h(x) y} \, dx = \int g(x) \overline{h(x)} y \, dx
\]

\[
= \int_0^{2\pi} g(\cos \theta) \overline{h(\cos \theta)} \sin \theta \, d\theta \quad \text{(where } x = \cos \theta, y = \sin \theta) \]

\[
= \int_0^{\pi} g(\cos \theta) \overline{h(\cos \theta)} \sin \theta \, d\theta + \int_{\pi}^{2\pi} g(\cos \theta) \overline{h(\cos \theta)} \sin \theta \, d\theta
\]

\[
= \int_0^{\pi} g(\cos \theta) \overline{h(\cos \theta)} \sin \theta \, d\theta - \int_0^{\pi} g(\cos \theta) \overline{h(\cos \theta)} (\sin \gamma) \, d\gamma
\]

\[ = 0. \]

This shows that \(Y_1\) and \(Y_2\) are perpendicular. Then

\[ Y_1 \subseteq Y_2^\perp = \overline{Y_2} = X_2^\perp. \]

Since \(X_2^\perp\) is closed in \(X\), \(X = \overline{Y_1} \subseteq X_2^\perp\). Thus, \(X_1\) and \(X_2\) are perpendicular.

Now we show that \(X = X_1 + X_2\). Because \(X_1 + X_2\) is a direct sum of two closed subspaces of \(X\), it is closed in \(X\). It suffices to show that \(X_1 + X_2\) is dense in \(X\). Because the polynomials are dense in \(X = L^2(S^1, \mathbb{R})\), for
each $f \in X$ and $\varepsilon > 0$ there exists a polynomial $P = P(z)$ such that $\|f - P\|_X < \varepsilon$. Write

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_2 z + a_0$$

$$= a_n (z + iy)^n + a_{n-1} (z + iy)^{n-1} + \cdots + a_2 (z + iy) + a_0$$

$$= \sum_{0 \leq k \leq n} c_k (z + iy)^k,$$

where $c_k$'s are complex coefficients. Because $z \in S^1$, $|z| = |z + iy| = 1$. Then

$$y = 1 - x^2.$$  Then

$$P(z) = \sum_{0 \leq k \leq n} c_k z^k y^m + \sum_{0 \leq k \leq n} c_k (1 - x^2)^m$$

$$= \sum_{0 \leq k \leq n} c_k z^k (1 - x^2)^m$$

$$= \sum_{0 \leq k \leq n} c_k z^k (1 - x^2)^m$$

$$= \sum_{0 \leq k \leq n} c_k z^k (1 - x^2)^m$$

$$= \sum_{0 \leq k \leq n} c_k z^k (1 - x^2)^m$$

Because $(z \mapsto g(z)) \in X_1$ and $(z \mapsto h(z)y) \in X_2$, we get $(z \mapsto P(z)) \in X_1 + X_2$.

Hence, $X_1 + X_2$ is dense in $X$. We have showed that $X = X_1 \oplus X_2$.

Let $f \in C([-1,1])$. An element $\xi \in Y_1$ is of the form $\xi(x) = g(x)$ for some $g \in C([-1,1])$.

$$T(f)\xi(z) = f(z) \xi(z) = f(z) g(z).$$

Then $T(f)\xi \in Y_1$. Thus, $T(f)(Y_1) \subseteq Y_1$. Because $T(f) \in L(X)$ and $X_1 = Y_1$, 

we also have \( T(f)(X_1) \subset X_1 \).

An element \( \gamma \in X_2 \) is of the form \( \gamma(x) = y_h(x) \) for some \( h \in C([-1,1]) \).

\[ T(f) \gamma(x) = f(x) \gamma(x) = f(x) y_h(x). \]

Then \( T(f) \gamma \in X_2 \). Thus, \( T(f)(X_2) \subset X_2 \). Because \( T(f) \in L(X) \) and \( X_2 = \overline{X} \), we get \( T(f)(X_2) = X_2 \).

Therefore, \( X = X_1 \oplus X_2 \) is a decomposition of \( X \) into invariant cyclic subspaces under the algebra action \( T \). Next, we determine the measures \( \mu_j \) on \([-1,1]\) associate with the cyclic action \( T : C([-1,1]) \to L(X_j) \) for \( j \in \{1,2\} \). By definition, \( \mu_j \) is the measure on \([-1,1]\) such that

\[
(T(f) \xi_j, \xi_j) = \int_{[-1,1]} f(x) \, d\mu_j(x) \quad \forall f \in C([-1,1]). \tag{2}
\]

For \( j = 1 \),

\[
\text{LHS}(2) = \int_{[-1,1]} T(f) \xi_1(x) \overline{\xi_1(x)} \, d\lambda = \int_{[-1,1]} f(x) \xi_1(x) \overline{\xi_1(x)} \, d\lambda
\]

\[
= \int_{[-1,1]} f(x) \, d\lambda = \int_0^{2\pi} f(\cos \theta) \, d\theta
\]

\[
= \int_0^{\pi} f(\cos \theta) \, d\theta + \int_{\pi}^{2\pi} f(\cos \theta) \, d\theta = \int_0^{\pi} f(\cos \theta) \, d\theta + \int_0^{\pi} f(\cos \theta) \, d\theta = 2 \int_0^{\pi} f(\cos \theta) \, d\theta
\]

\[
= 2 \int_0^{\pi} f(\cos \theta) \, d\theta
\]

\[
= \int_{-1}^1 \frac{2f(x)}{1-x^2} \, dx.
\]
This shows that $\mu$ is the measure on $[-1,1]$ whose Radon-Nikodym derivative with respect to the Lebesgue measure is

$$\frac{d\mu}{dx} = \frac{2}{\sqrt{1-x^2}}.$$

In other words,

$$\mu(B) = \int_B \frac{2}{\sqrt{1-x^2}} \, dx \quad \# \text{Borel set } B \subseteq [-1,1].$$

For $j=2$,

$$\text{LHS} = \int_{S^1} \left( f_j(x) \overline{\xi_j(x)} \right) \, d\Lambda = \int_{S^1} (f_j(x) y) \, d\Lambda$$

$$= \int_0^{2\pi} f(\cos \theta) \sin^2 \theta \, d\theta = \int_0^{2\pi} f(\cos \theta) (1-\cos^2 \theta) \, d\theta$$

$$= \int_0^{2\pi} f(\cos \theta) (1-\cos^2 \theta) \, d\theta + \int_0^{2\pi} f(\cos \theta) (1-\cos^2 \theta) \, d\theta$$

$$= \int_0^{2\pi} f(\cos \theta) (1-\cos^2 \theta) \, d\theta$$

$$= 2 \int_0^{\pi} f(\cos \theta) (1-\cos^2 \theta) \, d\theta$$

$$= 2 \int_{-1}^{1} f(x) (1-x^2) \frac{1}{\sqrt{1-x^2}} \, dx = \int_{-1}^{1} 2\sqrt{1-x^2} f(x) \, dx.$$

This shows that $\mu$ is the measure on $[-1,1]$ whose Radon-Nikodym derivative with respect to the Lebesgue measure is

$$\frac{d\mu}{dx} = 2\sqrt{1-x^2}.$$

More explicitly,

$$\mu(\mathcal{B}) = \int_{\mathcal{B}} 2\sqrt{1-x^2} \, dx \quad \# \text{Borel set } \mathcal{B} \subseteq [-1,1].$$
We established in Lecture 04/03/2015 that there is an isomorphism (of Banach spaces) \( \varphi_j : X_j \rightarrow L^2([-1,1], \mu_j) \) satisfying
\[
\varphi_j((Tg)\xi_j) = g \quad \forall g \in C([-1,1]).
\]
For each \( f \in C([-1,1]) \), define the maps
\[
\begin{align*}
\Psi_j(f) & : L^2([-1,1], \mu_j) \rightarrow L^2([-1,1], \mu_j) \\
\Psi_j(f)g(x) &= g(x)f(x),
\end{align*}
\]
\[
\begin{align*}
\Psi_2(f) & : L^2([-1,1], \mu_j) \rightarrow L^2([-1,1], \mu_j) \\
\Psi_2(f)g(x) &= g(x)f(x).
\end{align*}
\]
The following diagram commutes (as established in the same lecture).

\[
\begin{array}{ccc}
X_j & \xrightarrow{T(\cdot)} & X_j \\
\varphi_j \downarrow & \circ & \downarrow \varphi_j \\
L^2(K, \mu_j) & \xrightarrow{\Psi_j(\cdot)} & L^2(K, \mu_j)
\end{array}
\]

where \( K = [-1,1] \).

Then the following diagram also commutes.

\[
\begin{array}{ccc}
X = X_1 \oplus X_2 & \xrightarrow{T(\cdot)} & X = X_1 \oplus X_2 \\
\Phi_1 \oplus \Phi_2 \downarrow & \circ & \downarrow \Phi_1 \oplus \Phi_2 \\
L^2(K, \mu_j) \oplus L^2(K, \mu_j) & \rightarrow & L^2(K, \mu_j) \oplus L^2(K, \mu_j)
\end{array}
\]

Denote \( \Phi = \Phi_1 \oplus \Phi_2 \), \( \Phi_1(f) = \Phi_1(f) \oplus \Phi_2(f) \), \( Y_j = L^2([-1,1], \mu_j) \).
Next, we find a measure \( \mu \) on \([-1, 1]\) such that \( Y_1 \oplus Y_2 \) is isomorphic (as a Banach space) to \( Y \oplus Y \) where \( Y = L^2([-1, 1], \mu) \). Define the maps

\[
\begin{align*}
\varphi_1 : [-1, 1] &\to [0, 0], \quad \varphi_1(x) = \frac{x-1}{2}, \\
\varphi_2 : [-1, 1] &\to [0, 1], \quad \varphi_2(x) = \frac{x+1}{2}.
\end{align*}
\]

Then \( \mu \) induces a measure \( \tilde{\mu}_1 \) on \([-1, 0]\) as

\[
\tilde{\mu}_1(B) = \mu(\varphi_1^{-1}(B)) \quad \forall \text{ Borel set } B \subset [-1, 0].
\]

Similarly, \( \mu \) induces a measure \( \tilde{\mu}_2 \) on \([0, 1]\) as

\[
\tilde{\mu}_2(B) = \mu(\varphi_2^{-1}(B)) \quad \forall \text{ Borel set } B \subset [0, 1].
\]

Define a measure \( \nu \) on \([-1, 1]\) as follows.

\[
\nu(B) = \tilde{\mu}_1([-1, 0] \cap B) + \tilde{\mu}_2([0, 1] \cap B) \quad \forall \text{ Borel set } B \subset [-1, 1].
\]

Define a map \( \gamma : Y_1 \oplus Y_2 \to Y \oplus Y \),

\[
\gamma(g_1, g_2) = (g_1 \circ \varphi_1^1) X_{[-1, 0]} \oplus (g_2 \circ \varphi_2^1) X_{[0, 1]} \quad \forall g_1 \in Y_1, g_2 \in Y_2.
\]

This is a linear map.

\[
\| \gamma(g_1, g_2) \|_{Y \oplus Y}^2 = \int_{[-1, 1]} \left( (g_1 \circ \varphi_1^1(X_{[-1, 0]}) + (g_2 \circ \varphi_2^1(X_{[0, 1]}) \right)^2 d\mu
\]

\[
= \int_{[-1, 0]} |g_1 \circ \varphi_1^1|^2 d\mu + \int_{[0, 1]} |g_2 \circ \varphi_2^1|^2 d\mu
\]

\[
= \int_{[-1, 0]} |g_1(\varphi_1^1(x))|^2 d\tilde{\mu}_1(x) + \int_{[0, 1]} |g_2(\varphi_2(x))|^2 d\tilde{\mu}_2(x)
\]
$$\{1\} \quad \frac{y = s_{1t}(x)}{\int \left| f_1(y) \right|^2 \, d\mu_1(y)} = \|g_1\|_{\mu_1}^2,$$

$$\{2\} \quad \frac{y = s_{2t}(x)}{\int \left| f_2(y) \right|^2 \, d\mu_2(y)} = \|g_2\|_{\mu_2}^2.$$

Then \( \|\varphi(g_1, g_2)\|_{\mu_1 \otimes \mu_2}^2 = \|g_1\|_{\mu_1}^2 + \|g_2\|_{\mu_2}^2 = \|\varphi(g_1, g_2)\|_{\mu_1 \otimes \mu_2}^2 \). Thus, \( \varphi \) is a continuous map.

Unfortunately \( \varphi \) is not an isomorphism because it is not surjective. A different way to define \( \varphi \) is needed.