Homework Assignment 3  
(due May 14, 1pm)

Do two (or more) of the following six problems.

1. Let $X$ be a separable Hilbert space and let be $\mathcal{L}(X)$ the space of continuous linear operators on $X$. For $R > 0$ we denote by $B_R(X)$ the set $\{T \in \mathcal{L}(X), \|T\| \leq R\}$. 
Recall that the strong operator topology is defined on $\mathcal{L}(X)$ by the system of neighborhoods (of the zero operator) given by
\[ O_{x_1, \ldots, x_n, \epsilon_1, \ldots, \epsilon_n} = \{ T \in \mathcal{L}(X), \|Tx_1\| < \epsilon_1, \ldots, \|Tx_n\| < \epsilon_n \} , \]
where $n$ is any natural number, $x_1, \ldots, x_n \in X$ and $\epsilon_1, \ldots, \epsilon_n$ are strictly positive real numbers. We will denote this topology by $\sigma$. Note that the topology can also be considered on $B_R(X)$. 
The weak operator topology is defined on $\mathcal{L}(X)$ by the system of neighborhoods 
\[ O_{x_1, y_1, \ldots, x_n, y_n, \epsilon_1, \ldots, \epsilon_n} = \{ T \in \mathcal{L}(X), \|(Tx_1, y_1)\| < \epsilon_1, \ldots, \|(Tx_n, y_n)\| < \epsilon_n \} , \]
where $n$ is any natural number, $x_1, y_1, \ldots, x_n, y_n \in X$ and $\epsilon_1, \ldots, \epsilon_n$ are strictly positive real numbers. We will denote this topology by $\tau$. The topology can also be considered on $B_R(X)$.
(i) Show that neither $(\mathcal{L}(X), \sigma)$ nor $(\mathcal{L}(X), \tau)$ are metrizable.
(ii) Show that both $(B_R(X), \sigma)$ and $(B_R(X), \tau)$ are metrizable and, moreover, $(B_R(X), \tau)$ is compact, while $(B_R(X), \sigma)$ is not compact.
(iii) Show that the standard operator product $(S, T) \rightarrow ST$ is not continuous as a map from $(\mathcal{L}(X), \sigma) \times (\mathcal{L}(X), \sigma)$ to $(\mathcal{L}(X), \sigma)$.
(iv) Show that the standard operator product $(S, T) \rightarrow ST$ is continuous as a map from $(B_{R_1}(X), \sigma) \times (B_{R_2}(X), \sigma)$ to $(B_{R_1R_2}(X), \sigma)$.
(v) Decide whether the standard operator product $(S, T) \rightarrow ST$ is continuous as a map from $(B_{R_1}(X), \tau) \times (B_{R_2}(X), \tau)$ to $(B_{R_1R_2}(X), \tau)$.
(vi) Decide whether the standard operator product $(S, T) \rightarrow ST$ is continuous as a map from $(B_{R_1}(X), \sigma) \times (B_{R_2}(X), \sigma)$ to $(B_{R_1R_2}(X), \tau)$.
(vii) Decide whether the map $T \rightarrow T^*$ of taking adjoint is continuous from $(B_R(X), \sigma)$ to $(B_R(X), \sigma)$.
(viii) Decide whether the map $T \rightarrow T^*$ of taking adjoint is continuous from $(B_R(X), \tau)$ to $(B_R(X), \tau)$.
(ix) Decide whether the map $T \rightarrow T^*$ of taking adjoint is continuous from $(\mathcal{L}(X), \tau)$ to $(\mathcal{L}(X), \tau)$.

2. Let $S^1 \subset \mathbb{C}$ be the unit circle centered at 0. Let $\lambda$ be the standard one-dimensional Lebesgue measure on $S^1$ (normalized so that $\lambda(S^1) = 2\pi$). Let us write complex numbers $z = x + iy$. let $X = L^2(S^1, \lambda)$ and let $T: X \rightarrow X$ be defined by $Tf(z) = xf(z)$, or, equivalently, $Tf(z) = (\text{Re } z)f(z)$. Show that for a suitable measure $\mu$ on $(-1, 1)$ and $Y = L^2((-1,1), \mu)$ there exists a bijective isomorphism $U: X \rightarrow Y \oplus Y$ such that we have 
\[ Tf = U^{-1}MUf \] (3)
where the operator $M: Y \oplus Y \rightarrow Y \oplus Y$ is given by $(f_1, f_2) \rightarrow (xf_1, xf_2)$ (with the slight abuse of notation that $xf_1$ means the function $x \rightarrow xf_1(x)$).
3. Let $X$ be a separable Hilbert space and let $T_1, T_2$ be two bounded self-adjoint operators on $X$ which commute with each other, i.e. $T_1 T_2 = T_2 T_1$. Assume moreover that there is a vector $x_0 \in X$ such that the vectors of the form $\{ \sum_{k,l} c_{k,l} T_1^k T_2^l x_0, \quad c_{k,l} \in \mathbb{C} \text{ with only finitely many non-zero } \}$ are dense in $X$. (The last condition means that the natural representation of the algebra generated by $T_1, T_2$ in $X$ is cyclic.) Show that there is a compact set $K \subset \mathbb{R}^2$ and a Borel measure $\mu$ on $K$ such that $X$ can be identified with $L^2(K, \mu)$ and the operator $T_j$ can be identified with the operators $f \rightarrow x_j f$ (multiplication by $x_j$).

As an optional part of the problem, you can formulate a suitable generalization of the statement to the situation when
(a) we have $n$ mutually commuting self-adjoint operators $T_1, \ldots, T_n$, and
(b) the representation of the algebra is not cyclic.

4. Let $X$ be a Hilbert space and let $T \in \mathcal{L}(X)$ be self-adjoint. Prove that $\lambda \in \mathbb{C}$ is in the spectrum of $T$ if and only if there exists a sequence $x_n \in X$ with $\| x_n \| = 1$, such that $\| T x_n - \lambda x_n \| \rightarrow 0$ as $n \rightarrow \infty$. (This statement is due to H. Weyl. You can also try to decide whether it remains true without the assumption that $T$ be self-adjoint.)

5. True or false? (Please give a reason for your answer.)

Assume $u = u(k, t)$ is a real-valued function on $\mathbb{Z} \times (-\infty, \infty)$ which is smooth in $t$ for each $k$, satisfies

$$\sum_k \left( |u(k, t)|^2 + |\partial_t u(k, t)|^2 \right) \leq c$$

for all $t$ for some $c \geq 0$, and solves the equation

$$\frac{\partial^2}{\partial t^2} u(k, t) = u(k + 1, t) - 2u(k, t) + u(k - 1, t)$$

(which is a discrete version of the wave equation). If $u(k, 0)$ and $\frac{\partial}{\partial t} u(k, 0)$ vanish for $|k| > k_0$, then, for each $T > 0$, there exists $m = m(T, k_0) > 0$ such that $u(k, t)$ vanishes for $|t| \leq T$ and $|k| > m(T)$. (In other words, disturbances propagate with finite speed.)

Hint: use the spectral representation of the operator on the right-hand side of the equation.

6. True or false? (Please give a reason for your answer.)

Let $X$ be a separable Hilbert space and let $T \in \mathcal{L}(X)$ be a Fredholm operator. Then the following two conditions are equivalent:

(i) There exist a self-adjoint operator $P \in \mathcal{L}(X)$ and a unitary operator$^1 U \in \mathcal{L}(X)$ such that $T = UP$.

(ii) $\text{Ind} (T) = 0$.

Hint: Note that when $T = UP$ as above then $T^* T = P^2$.

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$^1$Recall that an operator is called unitary if $U^* U = U U^* = I$. 