Hilbert–Schmidt operators and Tensor Product of

two vector spaces

I. Hilbert–Schmidt operators

1. Definition

Let $V$ and $W$ be two separable Hilbert spaces, $T$ a bounded operator from $V$ to $W$, i.e. $T \in B(V,W)$. Let $(\varphi_n)_{n=1}^{\infty}$ be an orthonormal basis of $V$. We shall prove that the series

$$\sum_{n=1}^{\infty} |T\varphi_n|^2$$

is independent of the choice of basis $(\varphi_n)$.

Indeed, let $(\psi_n)$ be an orthonormal basis of $W$. Then

$$T_{\varphi_n} \psi_m = \sum_{m=1}^{\infty} \langle \varphi_n, \psi_m \rangle \psi_m = \sum_{m=1}^{\infty} \langle \psi_m, T^* \varphi_n \rangle \psi_m$$

Thus

$$|T\varphi_n|^2 = \sum_{m=1}^{\infty} |\langle \varphi_n, T\psi_m \rangle|^2$$

and

$$\sum_{n=1}^{\infty} |T\varphi_n|^2 = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |\langle \varphi_n, T\psi_m \rangle|^2$$

Further

$$= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |\langle \varphi_n, T\psi_m \rangle|^2$$

$$= \sum_{m=1}^{\infty} \|W_m\|^2$$
Thus, the series \( \sum_{n=1}^{\infty} |T_n|^2 \) is independent of the choice of orthonormal basis \( \{w_n\} \).

Definition: \( T \in \mathcal{B}(V, W) \) is called Hilbert-Schmidt operator if the sum \( \sum_{n=1}^{\infty} |T_n|^2 \) is finite.

By this definition, and the above calculation, we easily see that \( T \) is Hilbert-Schmidt if and only if \( T^* \) is Hilbert-Schmidt.

(2) If \( T \) is Hilbert-Schmidt (HS) then \( T \) is compact.

Proof: To show that \( T \) is compact, we only need to show that \( T \) is a norm-limit of a sequence of finite rank operators. For each \( v \in V \) with unit norm,

\[
Tv = \sum_{n=1}^{\infty} \langle T v, w_n \rangle w_n = \sum_{n=1}^{\infty} \langle v, T^{*} w_n \rangle w_n
\]

For each \( n \in \mathbb{N} \), we put

\[
T_n v = \sum_{n=1}^{m} \langle v, T^{*} w_n \rangle w_n \quad \forall v \in V
\]

Then

\[
T_m v - T_n v = \sum_{n=m+1}^{\infty} \langle v, T^{*} w_n \rangle w_n
\]

and

\[
|T v - T_n v|^2 = \sum_{n=m+1}^{\infty} |\langle v, T^{*} w_n \rangle|^2 \leq \sum_{n=m+1}^{\infty} |T^{*} w_n|^2
\]

Thus,
$\|T-T_m\|^2 \leq \sum_{n=m+1}^{\infty} |T^m w_n|^2$

We know that $\sum_{n=1}^{\infty} |T^m w_n|^2 = \sum_{n=1}^{\infty} |T v_n|^2 < \infty$. Hence,

$$\lim_{m \to \infty} \|T-T_m\|^2 = 0.$$  

(3) The space of HS operators contains the space of trace class operators.

**Proof.** Let $T$ be a trace class operator, then

$$\sum_{n=1}^{\infty} |T v_n| < \infty$$

Thus, $\lim_{m \to \infty} |T v_n| = 0$. There exists $N \in \mathbb{N}$ such that

$$|T v_n|^2 \leq |T v_n| \quad \forall n \geq N$$

Hence, the series $\sum_{n=1}^{\infty} |T v_n|^2 < \infty$.

(4) Norm on the space of HS operators.

Hereafter, the space of HS operators from Hilbert space $V$ to Hilbert space $W$ is denoted $B_2(V, W)$. We will show that

$$\|T\|_{HS} = \left( \sum_{n=1}^{\infty} |T v_n|^2 \right)^{1/2}$$

is actually a norm on $B_2(V, W)$. We have to check 3 criteria:

* Positive definite: $\|T\|_{HS} > 0$. 

* Homogeneity: $\alpha \|T\|_{HS} = \|\alpha T\|_{HS}$ for all $\alpha$. 

* Triangle inequality: $\|T_1 + T_2\|_{HS} \leq \|T_1\|_{HS} + \|T_2\|_{HS}$. 

...
If \( \|T\|_{HS} = 0 \) then \( T_n = 0 \) \( \forall n \in \mathbb{N} \), then \( T = 0 \).

* Homogeneous: let \( \lambda \in \mathbb{C} \) then

\[
\|\lambda T\|_{HS} = \left( \sum_{n=1}^{\infty} |\lambda T^n|^2 \right)^{1/2} = \left( \sum_{n=1}^{\infty} |\lambda|^2 |T^n|^2 \right)^{1/2} = |\lambda| \left( \sum_{n=1}^{\infty} |T^n|^2 \right)^{1/2} = |\lambda| \|T\|_{HS}.
\]

* Triangle inequality:

Let \( S \) and \( T \) be two HS operators

\[
\|S + T\|_{HS} = \left( \sum_{n=1}^{\infty} |(S + T)(w_n)|^2 \right)^{1/2} = \left( \sum_{n=1}^{\infty} |S w_n + T w_n|^2 \right)^{1/2} \leq \left( \sum_{n=1}^{\infty} |S w_n|^2 \right)^{1/2} + \left( \sum_{n=1}^{\infty} |T w_n|^2 \right)^{1/2} = \|S\|_{HS} + \|T\|_{HS}.
\]

5. \( \|T\|_{HS} = \|T^*\|_{HS} \) and \( \|T\|_{HS} \leq \|T\|_{HS} \)

Proof. By the calculation in point 1, we have

\[
\|T\|_{HS}^2 = \sum_{n=1}^{\infty} |T^n|^2 = \sum_{n=1}^{\infty} |T^* w_n|^2 = \|T^*\|_{HS}^2.
\]

Thus, \( \|T\|_{HS} = \|T^*\|_{HS} \). For each \( v \in V \) with unit norm,

\[
Tv = \sum_{n=1}^{\infty} \langle v, T^* w_n \rangle w_n = \sum_{n=1}^{\infty} \langle v, T w_n \rangle w_n.
\]

Thus,

\[
|Tv|^2 = \sum_{n=1}^{\infty} |\langle v, T w_n \rangle|^2 \leq \sum_{n=1}^{\infty} |T^* w_n|^2 = \|T^*\|_{HS}^2 = \|T\|_{HS}^2
\]

or \( |Tv| \leq \|T\|_{HS} \). Hence

\[
\|T\|_{HS} = \sup_{\|v\|=1} |Tv| \leq \|T\|_{HS}.
\]
(6) The space of finite-rank operators is dense in $B_2(V, W)$.

Proof: Hereafter, the space of finite-rank operators is denoted $B_{fn}(V, W)$. First, we'll show that $B_{fn}(V, W) \subset B_2(V, W)$.

Let $S \in B_{fn}(V, W)$. Then $\text{Im } S$ is finite dimensional, with a finite orthonormal basis $\{w_1, w_2, \ldots, w_k\}$. Let $\{\overline{w}_1, \overline{w}_2, \ldots, \overline{w}_{k+1}\}$ be an orthonormal. Thus $\text{Im } S$ is generated by finitely many elements $S(v_1')$, $\ldots$, $S(v_k')$. Let $\{v_1, v_2, \ldots, v_{k+1}\}$ be an orthonormal set generating $v_1', \ldots, v_k'$. Let $\{v'_1, v'_2, v'_3, \ldots\}$ be an orthonormal basis of $V$. Then $S(v'_{j+1}) = S(v'_{j+2}) = \cdots = 0$.

Let $\{v'_1, \ldots, v'_k\}$ be an orthonormal basis of $\ker S$. Then $\{v'_1, \ldots, v'_j, v'_{j+1}, \ldots\}$ is an orthonormal basis of $V$. We have

$$\sum_{n=1}^{\infty} |S(v'_n)|^2 = \sum_{n=1}^{\infty} |S(v'_n)|^2 < \infty$$

thus $S \in B_2(V, W)$.

Next, we'll show that $B_{fn}(V, W)$ is dense in $B_2(V, W)$. For each $T \in B_2(V, W)$, we have

$$Tv = \sum_{n=1}^{\infty} \langle Tv, w_n \rangle w_n$$
For each $m \in \mathbb{N}$, we define the finite-rank operator
\[
T_m v = \sum_{k=1}^{m} \langle v_k, w_k \rangle w_k \quad \forall v \in V
\]
Then
\[
(1 - T_m)(v) = \sum_{k=m+1}^{\infty} \langle v_k, w_k \rangle w_k
\]
and
\[
| (1 - T_m)(v) |^2 = \sum_{k=m+1}^{\infty} | \langle v_k, w_k \rangle |^2 = \sum_{k=m+1}^{\infty} \langle v_k, T^* w_k \rangle
\]
Then
\[
| (1 - T_m) v_n |^2 = \sum_{k=m+1}^{\infty} \langle v_n, T^* w_k \rangle
\]
and
\[
\|1 - T_m\|_{HS}^2 = \sum_{n=1}^{\infty} \| (1 - T_m) v_n \|_2^2 = \sum_{n=1}^{\infty} \sum_{k=m+1}^{\infty} | \langle v_n, T^* w_k \rangle |^2
\]

\[
= \sum_{k=m+1}^{\infty} | T^* w_k |^2
\]

Since
\[
\sum_{k=1}^{\infty} | T^* w_k | = \sum_{k=1}^{\infty} | T w_k | < \infty,
\]
we have
\[
\sum_{k=m+1}^{\infty} | T^* w_k | \to 0
\]
as $m \to \infty$. Thus,
\[
\|1 - T_m\|_{HS} \to 0 \quad \text{as} \quad m \to \infty.
\]

There exists a linear isomorphic isometry between $B_2(V, W)$ and $\ell^2(W)$. Consequently, $B_2(V, W)$ is a Hilbert space isomorphic to $\ell^2(W)$. 

For $\ell^2(W)$. 

(3) \( \ell_2(V, W) \) is a Hilbert space with inner product
\[
\langle T, S \rangle = \sum_{i=1}^{\infty} \langle S^* T v_i, v_i \rangle = \sum_{i=1}^{\infty} \langle T v_i, S v_i \rangle
\]

Proof. First we show that the sum \( \sum_{i=1}^{\infty} \langle T v_i, S v_i \rangle \) does not depend on the choice of orthonormal basis \((v_i)\).

\[
T v_i = \sum_{n=1}^{\infty} \langle T v_i, w_n \rangle w_n
\]

\[
S v_i = \sum_{n=1}^{\infty} \langle S v_i, w_n \rangle w_n
\]

Then
\[
\langle T v_i, S v_i \rangle = \sum_{n=1}^{\infty} \langle T v_i, w_n \rangle \langle w_n, S v_i \rangle
\]

\[
= \sum_{n=1}^{\infty} \langle v_i, T^* w_n \rangle \langle S^* w_n, v_i \rangle
\]

We have
\[
|a_{in}| \leq \frac{1}{2} \left( |\langle v_i, T^* w_n \rangle|^2 + |\langle S^* w_n, v_i \rangle|^2 \right)
\]

Thus
\[
\sum_{in} |a_{in}| \leq \frac{1}{2} \left\{ \sum_{n} \sum_{i} |\langle v_i, T^* w_n \rangle|^2 + \sum_{n} \sum_{i} |\langle S^* w_n, v_i \rangle|^2 \right\}
\]

\[
= \frac{1}{2} \left( \sum_{n} |T^* w_n|^2 + \sum_{n} |S^* w_n|^2 \right) < \infty
\]

By Fubini's Theorem,
\[
\sum_{i=1}^{\infty} \langle T v_i, S v_i \rangle = \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} a_{in} = \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} a_{in}
\]
\[
= \sum_{n=1}^{N_0} \sum_{k=1}^{b_0} \left[ \sum_{l=1}^{b_0} \left< \left< S_{w_n}, v_l \right> w_{n_l}, T^* w_n \right> \right]
\]
\[
= \sum_{n=1}^{N_0} \left[ \sum_{l=1}^{b_0} \left< S_{w_n}, T w_{n_l} \right> \right], \text{ which is independent of } (v_l)_{l=b_0}.
\]

Next, we have to check the following 3 properties:

* Linear with respect to the first argument

* Conjugate symmetric

* Positive definite

Obviously, \( \langle \cdot, \cdot \rangle \) is linear in the first argument:

\[
\left< S, T \right> = \sum_{n=1}^{N_0} \left< S w_n, T w_n \right> = \sum_{n=1}^{N_0} \left< T w_n, S w_n \right> = \sum_{n=1}^{N_0} \left< T w_n, S w_n \right> = \left< \overline{T}, S \right>.
\]

Thus, \( \langle \cdot, \cdot \rangle \) is conjugate symmetric.

\[
\left< T, T \right> = \sum_{n=1}^{N_0} \left< T w_n, T w_n \right> = \sum_{n=1}^{N_0} | T w_n |^2 = \| T \|_{HS}^2
\]

Thus, \( \langle \cdot, \cdot \rangle \) is positively definite. Up to now, we verified that \( \langle \cdot, \cdot \rangle \) is an inner product on \( B_2(U, W) \) which generates the norm \( \| \cdot \|_{HS} \).

Next, we have to show that this norm is complete.
Let \((T_m)\) be a Cauchy sequence in \(B_2(V,W)\). Then
\[
\sum_{n=1}^{\infty} |T_m v_n|^2 < \infty
\]
Thus, \(\{T_m v_n\}_n \in c^0(W)\). Put \(u_m = \{T_m v_n\}_n\). Then
\[
\|u_m - u_k\|^2 = \sum_{n=1}^{\infty} |T_m v_n - T_k v_n|^2 = \|T_m - T_k\|_{HS}^2,
\]
or
\[
\|u_m - u_k\|^2 = \|T_m - T_k\|_{HS}^2.
\]
That means \(\{u_m\}\) is a Cauchy sequence in \(c^0(W)\). Since \(c^0(W)\) is complete, there exists \(u \in c^0(W)\) such that \(u_m \to u\). Denote \(a_n^m\). We write \(u = \{a_n^m\}_{m,n}^\infty\).

Define a mapping \(T \in B(V,W)\) such that \(T v_n = a_n^m v_m\). We have
\[
\sum_{n=1}^{\infty} |T_m v_n - T v_n|^2 = \sum_{n=1}^{\infty} |T_m v_n - a_n^m v_m|^2 = \|u_m - u\|^2_{c^0} \to 0
\]
Thus \(T_m \to T\) in \(B_2(V,W)\).

1. Tensor product of two vector spaces

1. Definition

Let \(V\) and \(W\) be two modules on ring \(K\). Their tensor product of \(V\) and \(W\) is a pair \((L, \phi)\) consisting of a vector space \(L\), a free module \(L\) and a bilinear mapping \(\phi\) from \(V \times W\) to \(L\) such that for each bilinear map \(l\) from \(V \times W\) to a vector space.
There exists uniquely a linear map \( \Phi \) from \( L \) to \( Y \) such that \( h = \Phi \circ \phi \), i.e. the following diagram is commutative:

\[
\begin{array}{ccc}
V \times W & \xrightarrow{\phi} & L \\
\downarrow \Phi & & \downarrow \Phi \\
Y & \xrightarrow{h} & Y
\end{array}
\]

(2) Suppose that \((L_1, \phi_1)\) and \((L_2, \phi_2)\) are two tensor products of vector spaces \( V \) and \( W \) if and only if there exists a linear isomorphism between \( L_1 \) and \( L_2 \).

Proof (2) Let \((L_i, \phi_i)\) be a tensor product of \( V \) and \( W \). Then a pair \((L_2, \phi_2)\) is also a tensor product of \( V \) and \( W \) if and only if there exists a linear isomorphism between \( L_1 \) and \( L_2 \).

Proof: The backward part.

Let \( \psi : L_1 \to L_2 \) be an isomorphism. For each vector space \( Y \) and bilinear map \( h : V \times W \to Y \), we'll show that there exists a unique linear map \( \tilde{h} : L_2 \to Y \) such that \( h = \tilde{h} \circ \phi \).

\[
\begin{array}{ccc}
V \times W & \xrightarrow{\phi_1} & L_1 \\
\downarrow h & & \downarrow \psi \\
Y & \xrightarrow{\tilde{h}} & L_2
\end{array}
\]
Since \((L_1, \psi_1)\) is a tensor product, there exists a linear map \(h_1 : L_1 \to \gamma\) such that \(h_1 = h_1 \psi_1\). Put \(\tilde{h}_1 = h_1 \psi_1^\dagger\). Then \(\tilde{h}_1 : L_2 \to \gamma\) and \(h_1 = \tilde{h}_1 \psi_1\). If there is another \(\tilde{h}_1 : L_2 \to \gamma\) such that \(h_1 = \tilde{h}_1 \psi_1\), then \(\tilde{h}_1 = \tilde{h}_1 \psi_1\) is a linear map from \(L_1\) to \(\gamma\) such that \(h_1 = \tilde{h}_1 \psi_1\). Since \((L_1, \psi_1)\) is a tensor product, \(\tilde{h}_1 = h_1\). Thus, \(\tilde{h}_1 = h_1 \psi_1^\dagger = h_1 \psi_1^\dagger = h_1\). Thus, \(\tilde{h}_1\) is unique. That means \((L_2, \psi_1\psi_1)\) is also a tensor product.

The forward part:

\[
\begin{array}{ccc}
V \times W & \xrightarrow{\psi_1} & L_1 \\
\downarrow \phi_2 & & \downarrow h_1 \\
L_2 & \xrightarrow{h_2} & L_1
\end{array}
\]

Since \((L_1, \psi_1)\) is a tensor product, there exists uniquely a linear map \(h_1 : L_1 \to L_2\) such that \(\psi_1 \equiv h_1 \phi_2\).

Thus, \(h_1 = h_2 \phi_2 = h_2 \psi_1 \psi_1\), i.e., \(h_2 \psi_1 \equiv \text{id} \text{ on } \text{Im} \psi_1\).

To show that \(h_2 \psi_1 \equiv \text{id} \text{ on } L_2\), we have to show that \(L_2\) can be linearly spanned from \(\text{Im} \psi_1\).
\[ S = \langle \text{Im} \phi \rangle. \] Suppose that \( S \nsubseteq L_1. \) Since \( L_1 \) is a semisimple module, \( S \) is its direct summand. There exists a vector space \( T \) such that \( S \oplus T = L_1. \) Let \( \{v_3, v_2\} \) be a basis of \( T. \)

Let \( k_1 : L_1 \to S \) be such that
\[ k_1(x) = x \quad \forall x \in S \]
and
\[ k_1(v_i) = 0 \quad \forall i \in \{i_3, i_2\} \]

Then \( \hat{k}_1 = \chi = k_1 \).

Let \( k_2 : L_1 \to S \) be such that
\[ k_2(x) = x \quad \forall x \in S \]
\[ k_2(v_i) = 0 \quad \forall i \in \{i_3, i_2\} \]
\[ k_2(v_0) = v_0 \neq 0 \]

Then \( \hat{k}_2 = k_2 \hat{\phi}. \) Since \((L_1, \phi)\) is a tensor product, \( \hat{k}_1 \) and \( \hat{k}_2 \) must be the same. This is a contradiction. \( \Box \)

In short, \( k_2 \hat{\phi} = \text{id}_{L_1}. \) Thus, \( k_2 = \phi_1^{-1} \) is the linear isomorphism between \( L_1 \) and \( L_2. \)

Point (2) guarantees that the property mentioned in Point (1) is a universal property, i.e., it contains all attributes of tensor product. Accordingly, tensor products are unique up to a linear isomorphism.
(3) Construction of tensor product

Let \( F = K^{V \times W} \) be a direct sum, i.e., each element of \( F \) is a map from \( V \times W \) to \( K \) that is zero for all but finitely many elements in \( V \times W \). Then \( F \) is also a module with addition

\[(f + g)(x) := f(x) + g(x) \quad \forall x \in V \times W,\]

and scalar multiplication

\[(\lambda f)(x) := \lambda f(x) \quad \forall \lambda \in K, \quad \forall x \in V \times W.\]

For each \((v, w) \in V \times W\) we denote \( I_{(v, w)} \) the mapping from \( V \times W \) to \( K \) such that

\[I_{(v, w)}(u) = \begin{cases} 1 & \text{if } u = (v, w) \\ 0 & \text{otherwise} \end{cases}\]

Then \( F \) is a free module with basis \( \{ I_{(v, w)} : v \in V, w \in W \} \).

Let \( R \) be a submodule of \( F \) spanned by the set

\[\{ I_{(v_1 + v_2, w)} - a I_{(v_1, w)} - b I_{(v_2, w)} : a, b \in K, \ v_1, v_2 \in V, \ w \in W \}\]

\[\cup \{ I_{(v, w_1 + w_2)} - a I_{(v, w_1)} - b I_{(v, w_2)} : a, b \in K, \ v \in V, \ w_1, w_2 \in W \}\]

On \( F \), we define an equivalence relation

\[x \sim y \iff x - y \in R\]

and denote \( F/R \) the set of all equivalence classes.
We define the map \( \phi : V \times W \to F/R \):
\[
(\mathbf{v}, \mathbf{w}) \mapsto \mathbf{1}(\mathbf{v}, \mathbf{w}) + R
\]
Then we claim \((F/R, \phi)\) is a tensor product between \(V\) and \(W\).

Proof. First, we show that \(\phi\) is bilinear:
\[
\phi(\mathbf{v} + \mathbf{v'}, \mathbf{w}) = \mathbf{1}(\mathbf{v} + \mathbf{v'}, \mathbf{w}) + R = \mathbf{1}(\mathbf{v}, \mathbf{w}) + \mathbf{1}(\mathbf{v'}, \mathbf{w}) + \frac{\mathbf{1}(\mathbf{v} + \mathbf{v'}, \mathbf{w}) - \mathbf{1}(\mathbf{v}, \mathbf{w}) - \mathbf{1}(\mathbf{v'}, \mathbf{w})}{R} + R
\]
\[
= \mathbf{1}(\mathbf{v}, \mathbf{w}) + \mathbf{1}(\mathbf{v'}, \mathbf{w}) + R
\]
\[
= \phi(\mathbf{v}, \mathbf{w}) + \phi(\mathbf{v'}, \mathbf{w})
\]
\[
\phi(a \mathbf{v}, \mathbf{w}) = \mathbf{1}(a \mathbf{v}, \mathbf{w}) + R = a \mathbf{1}(\mathbf{v}, \mathbf{w}) + \mathbf{1}(a \mathbf{v}, \mathbf{w}) - a \mathbf{1}(\mathbf{v}, \mathbf{w}) + R
\]
\[
= a \mathbf{1}(\mathbf{v}, \mathbf{w}) + R = a \phi(\mathbf{v}, \mathbf{w})
\]

Similarly, \(\phi\) is linear with respect to \(W\).

Let \(Y\) be a vector space. Suppose \(h : V \times W \to Y\) is a bilinear map. We'll show that there exists uniquely a linear map \(\tilde{h} : F/R \to Y\) such that \(h = \tilde{h} \phi\).
The uniqueness:

If \( \overline{h} : F/R \to \gamma \) is a linear map such that \( h = \overline{h} \phi \) then

\[
h(v, w) = \overline{h} \phi (v, w) \quad \forall (v, w) \in V \times W
\]

Then

\[
\overline{h} (\ell (v, w) + R) = h(v, w)
\]

Since the set \( \{ \ell (v, w) + R : (v, w) \in V \times W \} \) generates \( F/R \), \( \overline{h} \) is determined uniquely over \( F/R \).

The existence:

Since \( \{ \ell (v, w) : v \in V, w \in W \} \) is a basis of \( F \), there exists a linear map \( h_1 : F \to \gamma \) such that \( h_1(\ell (v, w)) = h(v, w) \forall (v, w) \in V \times W \).

Let suppose \( u_1, u_2 \in F \) such that satisfy \( u_1 - u_2 \in R \). Then

\[
u_1 - u_2 = \sum_{i=1}^{m} \alpha_i \left[ \frac{1}{b_i} (a_i^1 f_i^1 + b_i^2 f_i^2, g_i) - \alpha_i \frac{1}{b_i} (a_i, g_i) - b_i \frac{1}{b_i} (a_i, g_i) \right]
\]

\[
+ \sum_{j=1}^{n} \beta_j \left[ b_j \left( c_j f_j^1 + d_j f_j^2, g_j \right) - g_j \left( c_j f_j, g_j \right) - d_j \left( c_j f_j, g_j \right) \right]
\]

By definition, \( h_1 \) is linear. Thus,

\[
h_1(u_1 - u_2) = \sum_{i=1}^{m} \alpha_i \left[ h_1 \left( \frac{1}{b_i} (a_i^1 f_i^1 + b_i^2 f_i^2, g_i) \right) - \alpha_i \frac{1}{b_i} h_1 (a_i, g_i) - b_i \frac{1}{b_i} h_1 (a_i, g_i) \right]
\]

\[
+ \sum_{j=1}^{n} \beta_j \left[ b_j \left( c_j f_j^1 + d_j f_j^2, g_j \right) - g_j h_1 (c_j f_j, g_j) - d_j h_1 (c_j f_j, g_j) \right]
\]
\[ \sum_{i=1}^{\infty} \lambda_i \left[ h(a_1 f_i + b_i g_i) - a_i h(f_i, \xi_i) - b_i h(g_i, \xi_i) \right] \\
+ \sum_{j=1}^{\infty} \beta_j \left[ \left( f_j | x \xi_j + g_j | x \xi_j \right) - c_j h(f_j, \xi_j) - d_j h(g_j, \xi_j) \right] \]

= 0 \quad \text{because } h \text{ is bilinear.}

Thus, \( h_i(w) = h_x(w) \). By this reason, we can define a map 
\[ \tilde{h}_i : F/R \to Y \]
\[ \tilde{h}_i(f + R) = h_i(f) \]

For each \((v, w) \in V \times W\),
\[ \tilde{h}_i(v, w) = \tilde{h}_i(i(v, w) + R) = h_i(i(v, w)) = h_i(v, w) \]

Thus, \( \tilde{h}_i = h_i \).

4. Let \( \{ v_i \}_{i \in I} \) be a basis of \( V \), \( \{ w_j \}_{j \in J} \) a basis of \( W \). Then
\[ S = \{ i(v, w) + R : i \in I, j \in J \} \]

is a basis of \( F/R \).

Proof: First we show that \( S \) can linearly generate \( F/R \). We know the set \( \{ i(v, w) + R : v \in U, w \in W \} \) can generate \( F/R \). Thus, it is sufficient to show that for each \( i(v, w) + R \), \( i(v, w) + R \) is a linear combination of elements of \( S \). We can write 
\[ v = \sum_{i=1}^{\infty} \alpha_i v_i, \quad w = \sum_{j=1}^{\infty} \beta_j w_j \]
Then
\[ I_{i,j} + R = I_{i,j} (z_{i,j}, z_{i,j}) + R = \sum_{i,j} \alpha_i \beta_j \delta_i (z_{i,j}) + R \]

Next, we'll show that \( S \) is linearly independent. Suppose that \( c_j \in K, \forall i = 1, \ldots, n; j = 1, \ldots, m \) be such that
\[ \sum_{i,j} c_j \delta_i (z_{i,j}) + R = 0 \quad \text{or} \quad \sum_{i,j} c_j \delta_i (z_{i,j}) \in R. \]

For each bilinear map \( h_i : V \times W \rightarrow Y \), we define as in point 3) the map linear map \( h_i : F \rightarrow Y \) such that \( h_i (1_{i,j}) = h_i (z_{i,j}) \).

By point 3), if \( u_1 = u_2 \in R \) then \( h_i (u_1) = h_i (u_2) \). Note that
\[ \sum_{i,j} c_j \delta_i (z_{i,j}) = 0 \in R. \]

Thus,
\[ 0 = h_i \left( \sum_{i,j} c_j \delta_i (z_{i,j}) \right) = \sum_{i,j} c_j h_i (\delta_i (z_{i,j})) = \sum_{i,j} c_j h_i (1_{i,j}). \]

Therefore,
\[ \sum_{i,j} c_j h_i (1_{i,j}) = 0 \quad (\ast) \]

\( \forall \) bilinear \( h_i \) from \( V \times W \) to \( Y \).

For each pair of indices \( (i, j) \), we define the bilinear map
\[ h_{i,j} : V \times W \rightarrow K \]
\[ (v, w) \mapsto \alpha_i \beta_j \]

where \( v = \sum v_i, w = \sum w_j \).
defines \( h_{ij} \) as \( (\psi, \varphi) \) if \( i = j \) and \( 0 \) otherwise.

Applying (iv) for \( h = h_{ij} \), we get \( h_{ij} \psi = 0 \). \( \square \)

Hereafter, we denote \( V \otimes W \) the space \( F/R \) together with the bilinear map \( \phi \). That means, \( V \otimes W \) is the tensor product of \( V \) and \( W \). Also, we define \( v \otimes w := \lambda \psi \varphi + R \).

### II. Hilbert-space tensor product

Let \( V \) and \( W \) be two separable Hilbert spaces. Let \( (v_i)_{i \in \mathbb{N}} \) and \( (w_j)_{j \in \mathbb{N}} \) be respectively an orthonormal basis of \( V \) and \( W \).

\[
V_0 = \langle v_1, v_2, \ldots \rangle,
\]
\[
W_0 = \langle w_1, w_2, \ldots \rangle,
\]

i.e. each element of \( V_0 \) is a finite linear combination of \( \{v_i, v_2, \ldots \} \).

Note that \( V_0 \subset V \) and \( W_0 \subset W \) if \( V \) and \( W \) are infinite dimensional. In part II, we defined the algebraic tensor product of \( V \) and \( W \). In this case, \( V \) and \( W \) have noncountable bases. However, orthogonal bases \( (v_i) \) and \( (w_j) \) are very important in these spaces. We should introduce another tensor product between \( V \) and \( W \) that involves these orthogonal bases. Such a kind of tensor product...
is Hilbert-space tensor product.

1. $V_0 \otimes W_0$ is independent of the choice of orthonormal bases of $V$ and $W$.

Proof: Let $(v_i)$ and $(w_j)$ be respectively orthonormal bases of $V$ and $W$. We denote

$$V_0' = \langle \{v_1, v_2, \ldots \} \rangle$$
$$W_0' = \langle \{w_1, w_2, \ldots \} \rangle$$

By Point 4, Part II, $\{v_i \otimes w_j / i, j \in \mathbb{N}\}$ is a basis of $V_0 \otimes W_0$, and $\{v_i' \otimes w_j' / i, j \in \mathbb{N}\}$ is a basis of $V_0' \otimes W_0'$. We can introduce a linear isomorphism between $V_0 \otimes W_0$ and $V_0' \otimes W_0'$

$$\psi : V_0 \otimes W_0 \rightarrow V_0' \otimes W_0'$$

$$\sum_{ij} v_i \otimes w_j \rightarrow \sum_{ij} v_i' \otimes w_j'$$

Thus, $V_0' \otimes W_0'$ is simply a tensor product of $V_0$ and $W_0$.

2. $V_0 \otimes W_0$ can be equipped with the following inner product

$$\langle \sum_{ij} v_i \otimes w_j, \sum_{kl} v_k \otimes w_l \rangle = \sum_{ij} \langle v_i, v_k \rangle \langle w_j, w_l \rangle$$

Proof: Because $(x_i)$ and $(y_j)$ vanish at all but finitely many entries, the map $\langle \cdot, \cdot \rangle$ is well-defined. Moreover, by its definition,
\( \langle \cdot, \cdot \rangle \) is linear in the first argument. We have
\[
\left\langle \sum_{k} f_{k} \psi_{k}, \sum_{j} 2 \alpha_{j} \psi_{j} \right\rangle = \sum_{k} f_{k} \bar{\alpha}_{k} = \sum_{k} f_{k} \bar{f}_{k}
\]
Thus, \( \langle \cdot, \cdot \rangle \) is conjugate symmetric. We have
\[
\left\langle \psi_{j}, \sum_{k} 2 \psi_{k} \right\rangle = \sum_{k} \alpha_{k} \bar{\alpha}_{j} = \sum_{k} |\alpha_{k}|^2 ≥ 0
\]
The equality holds if and only if \( \alpha_{j} = 0 \leftrightarrow \psi_{j} \), i.e. \( \sum_{k} \alpha_{k} \psi_{k} = 0 \)
Thus, \( \langle \cdot, \cdot \rangle \) is an inner product and induces a norm of \( V \otimes W \)
\[
\| \sum_{j} \alpha_{j} \psi_{j} \| = \left( \sum_{j} |\alpha_{j}|^2 \right)^{1/2} = \| \sum_{j} |\alpha_{j}|^2 e_{j} \psi_{j} \|
\]
\( 3 \) By the previous point, \( \langle V \otimes W, \cdot, \cdot \rangle \) is a norm space.
Definition: The A completion of \( \langle V \otimes W, \cdot, \cdot \rangle \) is called Hilbert-space tensor product of \( V \) and \( W \), and denoted \( V \otimes W \).
\( 4 \) By this definition, Hilbert-space tensor product of \( V \) and \( W \) is unique up to a linear isometric isomorphism.
\( 4 \) In this point, we'll construct a specific Hilbert-space tensor product of two separable Hilbert spaces \( V \) and \( W \).
Put $G = C^{V \times W}$ - the set all maps from $V \times W$ to $C$.

We define the following subsets of $G$:

\[ R_0 = \left\{ \frac{1}{4} \left( a_1 f_1 + b_1 f_2, c_1 g_1 + d_1 g_2 \right) - \frac{1}{4} \left( a_1 f_3 + b_1 f_4, c_1 g_3 + d_1 g_4 \right) \middle| a,b,c,d \in C; f_1, f_2 \in V_0; g_1, g_2 \in W_0 \right\} \]

\[ R = \left\{ \frac{1}{4} \left( 2x_i v_i, 2y_i w_i \right) - \sum_{ij} \alpha_{ij} f_{(i,j)} \middle| (i,j) \in \mathbb{Z}^2 \right\} \]

Remember that an element $v \in V$ corresponds one to one to a sequence $(x_i) \in l^2(\mathbb{N})$ by the relation

\[ v = \sum_{i=1}^{\infty} x_i v_i \]

There is one thing worth noticing in the definition of $R$. The map

\[ f = \sum_{ij} \alpha_{ij} f_{(i,j)} \]

is simply a map from $V \times W$ to $C$ such that

\[ f(x) = \begin{cases} \alpha_{ij} & x = (v_i, w_j) \\ 0 & \text{otherwise} \end{cases} \]

by definition, $R_0$ and $R$ are vector spaces and $R_0 \subseteq R$. Then we have two equivalence relations on $G$

\[ u \sim u' \text{ if and only if } u - u' \in R_0. \]

\[ u \sim u' \text{ if and only if } u - u' \in R. \]
As we know, a special subset of the $R_0$-equivalence classes is the algebraic tensor product of $V_0$ and $W_0$

$$V_0 \otimes W_0 = \tilde{F}_0 = \left\{ \sum_{ij} \gamma_{ij} A_{(i,j)} + R_0 / (\alpha_{ij}) \text{vanishes a all but finitely many entries} \right\}$$

We define a special set of the $R$-equivalence classes

$$\tilde{F} = \left\{ \sum_{ij} \gamma_{ij} A_{(i,j)} + R / (\alpha_{ij}) \in C(N \times W) \right\}$$

Then $\tilde{F} = V \otimes W$.

Proof: Put $\psi : \tilde{F}_0 \rightarrow \tilde{F}$

$$\sum_{ij} \gamma_{ij} A_{(i,j)} + R_0 \mapsto \sum_{ij} \gamma_{ij} A_{(i,j)} + R$$

Then $\psi$ is well-defined and linear. To show that $\psi$ is injective, we only need to show that $\ker \psi = 0$. Suppose that

$$\sum_{ij} \gamma_{ij} A_{(i,j)} + R = 0,$$

i.e.

$$\sum_{ij} \gamma_{ij} A_{(i,j)} \in R$$

Thus,

$$\sum_{ij} \gamma_{ij} A_{(i,j)} = \sum_{k=1}^{N} \sum_{ij} \gamma_{ij} A_{(i,j)} - \sum_{ij} \sum_{k=1}^{N} \alpha_{ij} \beta_{ij} A_{(i,j)}$$

(1)
\[ \sum_{k=1}^{N} \xi_k \mathbf{1}(\sum_{l}^{k} \psi_l, \sum_{j}^{k} \eta_j) - \sum_{ij} \lambda_{ij} \mathbf{1}(\epsilon_i, \omega_j) \]

where \( \lambda_{ij} = \sum_{k=1}^{N} \xi_k \psi_i \eta_j \).

We can assume that \( \xi_k \neq 0 \) and \( \mathbf{1}(\sum_{l}^{k} \psi_l, \sum_{j}^{k} \eta_j) \)'s are distinct.

Then \( (\sum_{l}^{k} \psi_l, \sum_{j}^{k} \eta_j) = (\nu_k, \omega_k) \) \( \forall k = 1, \ldots, N \) and therefore the sum with \( \epsilon_i \) over \( i \) and \( j \) on the right hand side of (***) must be finite. Thus,

\[ \sum_{ij} \lambda_{ij} \mathbf{1}(\nu_i, \omega_j) \leq R_0, \]

i.e.

\[ \sum_{ij} \lambda_{ij} \mathbf{1}(\nu_i, \omega_j) + R_0 = 0. \]

Hence, \( \psi \) is linear and injective. Then \( \psi(\mathbb{F}_0) \) is a linear isomorphism to \( \tilde{\mathbb{F}}_0 \) and thus a tensor product of \( \mathbb{V}_0 \) and \( \mathbb{W}_0 \).

\[ \psi(\mathbb{F}_0) = \mathbb{V}_0 \otimes \mathbb{W}_0 \]

(Here the corresponding bilinear map is implicitly understood). Since \( \mathbb{F}_0 \) is endowed with the inner product mentioned in Point (2), we can define an inner product on \( \psi(\mathbb{F}_0) \) as follow

\[ \langle \psi(x), \psi(y) \rangle_{\psi(\mathbb{F}_0)} := \langle x, y \rangle_{\mathbb{F}_0} \]
Then the inner product induces a norm on \( \mathcal{V}(\mathcal{F}_0) \), and 
\( (\mathcal{V}(\mathcal{F}_0), \| \cdot \|) \) is linearly isometrically isomorphic to \((\mathcal{F}_0, \| \cdot \|)\). Thus, the completion space of \((\mathcal{V}(\mathcal{F}_0), \| \cdot \|)\) is a Hilbert-space tensor product of \( V \) and \( W \). The problem now becomes to show that this completion is actually \( \mathcal{F} \).

We know that each element \( v \in \mathcal{F} \) has the form

\[
  f = \sum \gamma_j \delta(v, w_j) + R, \quad \text{where} \quad (\gamma_j) \in \ell^1(\mathbb{N} \times \mathbb{N})
\]

First, we'll show that this representation is unique. Suppose that

\[
  f = \sum \gamma'_j \delta(v, w_j) + R, \quad \text{where} \quad (\gamma'_j) \in \ell^1(\mathbb{N} \times \mathbb{N}).
\]

Put \( \delta_j = \gamma_j - \gamma'_j \). Then \( (\delta_j) \in \ell^1(\mathbb{N} \times \mathbb{N}) \) and

\[
  \sum \delta_j \delta(v, w_j) \in \mathbb{R}
\]

Thus,

\[
  \sum \gamma_j \delta(v, w_j) = \sum_{k=1}^{N} \gamma_k \left[ \sum_{j} \delta_j (2 \alpha_{k,w_j} \geq \beta_{k,w_j}) - \frac{2}{y} \sum \delta_j \delta(v, w_j) \right]
\]

\[
  = \sum_{k=1}^{N} \gamma_k \delta(2 \alpha_{k,w_j} \geq \beta_{k,w_j}) - \frac{2}{y} \sum \delta_j \delta(v, w_j)
\]

where

\[
  \delta_j = \sum_{k=1}^{N} \alpha_{k,w_j} \beta_{k,w_j}
\]

Then \( (\sum \alpha_{k,v_i} \delta \beta_{k,w_j}) = (v_i, w_j) \), and the sum over \( ij \) at \( \sum \) is actually a finite sum.
Thus, \( \sum \alpha_{ij} d(w_i, w_j) \in R_0 \). Hence \( \alpha_{ij} = 0 \), \( \forall i, j \in N \). We introduce an inner product on \( \tilde{F} \):

\[
\left< \sum \alpha_{ij} d(w_i, w_j) + R, \sum \beta_{kl} d(w_k, w_l) + R \right>_\tilde{F} = \sum \alpha_{ij} \beta_{ij}
\]

\( \forall (\alpha_{ij}), (\beta_{kl}) \in \ell^2(N \times N) \)

Notice that the sum at the right hand side always converges. It is easy to see that \( \left< , \right>_\tilde{F} \) is linear in the first argument, conjugate symmetric and positively definite. Thus, \( \left< , \right>_\tilde{F} \) is indeed an inner product on \( \tilde{F} \). In fact, \( \left< , \right>_\tilde{F} \) is an extension of \( \left< , \right>_{\ell^2(F_0)} \) on \( \tilde{F} \). It induces a norm on \( \tilde{F} \). Define

\[
\gamma: \ell^2(N \times N) \rightarrow \tilde{F}
\]

\[
(\alpha_{ij}) \mapsto \sum \alpha_{ij} d(w_i, w_j) + R
\]

Then \( \gamma \) is well-defined, surjective, injective, linear and norm-preserving.

\[
|| \gamma(\alpha_{ij}) || = || \sum \alpha_{ij} d(w_i, w_j) + R || = || (\alpha_{ij})_F ||
\]

Since \( \ell^2(N \times N) \) is complete, \( (\tilde{F}, ||.||) \) is also complete. The only task left is to show that \( \Phi(F_0) \) is dense in \( \tilde{F} \). Let

\[
f = \sum \delta_{ij} d(w_i, w_j) + R \in \tilde{F}
\]

For each \( \forall i \in N \), we define
\[ f_m = \sum_{i,j=1}^{m} \gamma_{ij} A_{ij}(e_i, w_j) + R \in \Psi(F) \]

Then
\[ f - f_m = \sum_{i,j>m} \gamma_{ij} A_{ij}(e_i, w_j) + R \]
\[ \|f - f_m\|^2 = \sum_{i,j>m} |\gamma_{ij}|^2 \]

because \( \sum_{i,j=1}^{\infty} |\gamma_{ij}|^2 \) converges, \( \lim_{m \to \infty} \sum_{i,j>m} |\gamma_{ij}|^2 = 0 \). Thus

\[ \|f - f_m\| \to 0 \quad \text{or} \quad f_m \to f. \]

Therefore, \( \Psi(F) \) is dense in \((F, \|\cdot\|)\).

In conclusion, \((\tilde{F}, \langle \cdot, \cdot \rangle_F)\) is the Hilbert–space tensor product of \( V \) and \( W \).

(5) With the notation \( v \otimes w := A_{ij}(e_i, w_j) + R \quad \forall v \in V, w \in W \), we have \( \langle v \otimes w, v' \otimes w' \rangle_F = \langle v, v' \rangle \langle w, w' \rangle \).

**Proof.** We can write

\[ v = \sum_{i=1}^{\infty} \alpha_i e_i, \quad w = \sum_{j=1}^{\infty} \beta_j w_j \]

\[ v' = \sum_{k=1}^{\infty} \alpha'_k e_k, \quad w' = \sum_{l=1}^{\infty} \beta'_l w_l \]

where \((\alpha_i), (\beta_j), (\alpha'_k), (\beta'_l) \in l^2(N \times N)\). Then

\[ v \otimes w = \sum_{i,j=1}^{\infty} \gamma_{ij}(e_i, w_j) + R = \sum_{i,j} \gamma_{ij}(e_i, w_j) + R \]

for \( \gamma_{ij} = \alpha_i \beta_j \).
\[
\sum \lambda_{ij} f_i (v, w) + R
\]

Similarly,

\[
\langle v \hat{\otimes} w, v \hat{\otimes} w \rangle = \langle \sum \lambda_{ij} f_i (v, w) + R, \sum \lambda'_{ij} f_i (v, w) + R \rangle
\]

By definition,

\[
\frac{Z \sum \lambda_{ij} \bar{f_i} \bar{f_j}}{\sum \lambda_{ij} \bar{f_i} \bar{f_j}} = \sum \frac{(\bar{v_i}) (\bar{w_j})}{\sum \lambda_{ij} \bar{f_i} \bar{f_j}}
\]

We have

\[
\sum \lambda_{ij} \bar{v_i} \bar{w_j} = \sum \frac{(\bar{v_i}) (\bar{w_j})}{\sum \lambda_{ij} \bar{f_i} \bar{f_j}} < \infty
\]

Thus,

\[
\sum \lambda_{ij} \bar{v_i} \bar{w_j} = \left( \sum \frac{(\bar{v_i})}{\sum \lambda_{ij} \bar{f_i}} \right) \left( \sum \frac{(\bar{w_j})}{\sum \lambda_{ij} \bar{f_j}} \right)
\]

\[
= \left( \sum \bar{v_i} \right) \left( \sum \bar{w_j} \right) = \left\langle \sum \bar{v_i}, \sum \bar{w_j} \right\rangle = \left\langle \sum \beta_i \bar{v_i}, \sum \beta_j \bar{w_j} \right\rangle
\]

\[
= \left\langle v, w \right\rangle \left\langle w, w \right\rangle
\]

Therefore, \( \langle v \hat{\otimes} w, v \hat{\otimes} w \rangle = \langle v, w \rangle \langle w, w \rangle \).

\[\text{IV} \quad \text{Some examples of Hilbert space tensor product} \]

\[\text{1. } V \hat{\otimes} W = B_{2} (V, W) \]

\[\text{Proof. Let } (v_i)_{i \in V} \text{ be an orthonormal basis of } V \]

\[(w_j)_{j \in W} \text{ be an orthonormal basis of } W \]

As in previous points, we pat
\[ V_0 = \langle \{v_1, v_2, \ldots \} \rangle \]
\[ W_0 = \langle \{w_1, w_2, \ldots \} \rangle \]

First, we'll show that \( V_0 \otimes W_0 = \text{Bim}(V_0, W_0) \). For each \( i \in \mathbb{N} \), we denote \( v_i^* \) the linear map from \( V \) to \( \mathbb{C} \) such that
\[ v_i^*(v_j) = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{otherwise} \end{cases} \]

More explicitly, \[ v_i^*(v) = a_i, \] where \( v = \sum_{j=1}^{\infty} a_j v_j \).

Then \( v_i^* \in V^* \). We define the map
\[ \phi: V_0 \otimes W_0 \rightarrow \text{Bim}(V_0, W_0) \]
\[ \phi(\sum \lambda_j v_i \otimes w_j) = \sum \lambda_j v_i^* w_j \]

Then \( \phi \) is well-defined and linear. If \( \phi(\sum \lambda_j v_i \otimes w_j) = 0 \) then
\[ \sum \lambda_j v_i^* w_j = 0, \text{ i.e. } \sum \lambda_j v_i^* w_j = 0 \quad \forall v \in V_0 \]

For each \( k \in \mathbb{N} \), we substitute \( v \) by \( v_k \) and obtain
\[ 0 = \sum_j \lambda_j v_i^* v_k w_j = \sum_j \lambda_j v_k w_j \]

Thus \( \lambda_j = 0 \forall j \). Thus \( \lambda_j = 0 \forall v_j \). Hence, \( \phi \) is injective.

For each \( f \in \text{Bim}(V_0, W_0) \), we have
\[ f(v) = \sum_j f_j(v) w_j \quad (\text{finite sum}) \]
It is easy to see that $f$ is linear. Since $v \in V$, it can be expressed as the finite sum $v = \sum \xi_i v_i = \sum \xi_i^*(v) v_i$.

Then

$$f(v) = \sum_j f_j(v) w_j = \sum_j f_j \left( \sum_j \xi_i^*(v) v_i \right) w_j = \sum_j \xi_i^*(v) f_j(v_i) w_j$$

Put $a_j = f_j(v_i)$. Then $f(v) = \sum_j a_j \xi_i^*(v) w_j$, or

$$f = \sum_j a_j \xi_i^* w_j = \phi \left( \sum_j a_j v_j^* w_j \right)$$

Thus, $\phi$ is surjective. That means $\phi$ is a linear isomorphism. Hence, $b_{\text{lin}}(V_0, W_0)$ is a tensor product of $V_0$ and $W_0$. The inner product on $b_{\text{lin}}(V_0, W_0)$ induced by $\phi$ is

$$\left\langle \sum a_j \xi_i^* w_j, \sum f_k \xi_l^* w_k \right\rangle = \sum a_j f_k \xi_i^* \xi_l^*$$

Let $f = \sum a_j \xi_i^* w_j$. Then

$$\langle f, f \rangle = \sum a_j a_l = \sum |a_j|^2$$

we have

$$f(v) = \sum_j a_j \xi_i^* (v_k) w_j = \sum_j a_j \xi_i^* v_k \xi_l^* w_j = \sum_j a_j \xi_i^* w_j$$

Thus

$$|f(v)|^2 = \sum_j |a_j|^2$$

and

$$\sum_k |f(v_k)|^2 = \sum_j |a_j|^2$$

Hence

$$\langle f, f \rangle = \sum_k |f(v_k)|^2.$$
The norm on $B_{pn}(V_0, W_0)$ induced by this inner product is therefore:

$$
\|f\| = \left( \sum_{k} |f(v_k)|^2 \right)^{1/2},
$$

i.e. the Hilbert–Schmidt norm. For each $f \in B_{pn}(V_0, W_0)$ and $x \in V$, there exists a sequence $(x_n)$ in $V_0$ such that $x_n \to x$. We define $f(x) = \lim_{n \to \infty} f(x_n)$. Then $(x_n)$ is a Cauchy sequence in $V$.

$$
|f(x_n) - f(x_m)| = |f(x_n - x_m)| \leq \|f\| \|x_n - x_m\|
$$

Thus, $\{f(x_n)\}$ is a Cauchy sequence in $W$. Since $W$ is complete, the sequence converges. Moreover, the limit is independent of the choice of sequence $(x_n)$. Thus, we can define

$$
\tilde{f}(x) = \lim_{n \to \infty} f(x_n)
$$

Then $\tilde{f}$ is a finite-rank operator from $V$ to $W_0$. Define

$$
\Psi : B_{pn}(V_0, W_0) \to B_2(V_1, W)
$$

$$
\begin{align*}
\Psi(f) & \mapsto \tilde{f} \\
\Psi & \Psi
\end{align*}
$$

Then $\Psi$ is well-defined and linear. If $\tilde{f} = 0$ then $\tilde{f}(v_i) = 0$ for all $v_i$, i.e. $f = 0$. Thus $\Psi$ is injective. Hence,

$$
G = \Psi(B_{pn}(V_0, W_0)) = V_0 \otimes W_0.
$$

We can define an inner product on $G$

$$
\langle \Psi(f), \Psi(g) \rangle_G = \langle f, g \rangle_{B_{pn}(V_0, W_0)}.
$$
This product is respect the restriction of Hilbert–Schmidt inner product on $G$

$$<f, g>_{B_2(V,W)} = \sum_\lambda \langle f_\lambda, g_\lambda \rangle_W$$

To show that $V \otimes W = B_2(V,W)$, we have to show that $B_2(V,W)$ is the completion of $(G, \| \cdot \|_{HS})$. By Point 3, Part 4, $B_2(V,W)$ is a Banach space. The task left is to show that $(G, \| \cdot \|_{HS})$ is dense in $B_2(V,W)$. For each $T \in B_2(V,W)$,

$$T_0 = \sum_{n=1}^{\infty} \langle T_0, w_n \rangle w_n$$

For each $m \in \mathbb{N}$, we define

$$T_m v = \sum_{n=1}^{m} \langle T_0, w_n \rangle w_n$$

Then $T_m \in \mathcal{F}(\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)) = G$ and

$$\| T - T_m \|_{HS}^2 = \sum_{k=1}^{\infty} \left| T_k - T_m w_n \right|^2 = \sum_{k=1}^{\infty} \left| \sum_{n=m+1}^{\infty} \langle T_k, w_n \rangle w_n \right|^2$$

$$= \sum_{k=1}^{\infty} \sum_{n=m+1}^{\infty} \left| \langle T_k, w_n \rangle \right|^2$$

$$= \sum_{n=m+1}^{\infty} \sum_{k=1}^{\infty} \left| \langle w_k, T^* w_n \rangle \right|^2 = \sum_{n=m+1}^{\infty} \| T^* w_n \|^2$$
Since \( \sum_{n=1}^{\infty} |T_{n}w_{n}|^{2} = \sum_{n=1}^{\infty} |T_{n}v_{n}|^{2} = \|T\|_{L^{2}}^{2} < \infty \),

\[ \lim_{m \to \infty} \sum_{n=m+1}^{\infty} |T_{n}w_{n}|^{2} = 0. \]

Thus, \( \|T - T_{m}\|_{L^{2}} \to 0 \), and \( T_{m} \to T \). Therefore, \((G, \|\|_{L^{2}})\) is dense in \( B_{2}(V, W) \) and

\[ B_{2}(V, W) = V \hat{\otimes} W. \]

(2) \[ L^{2}(X, \mu) \otimes L^{2}(Y, \nu) = L^{2}(X \times Y, \mu \times \nu) \]

Proof: Let \((f_{i})\) be an orthonormal basis of \( L^{2}(X, \mu) \),

\[ (f_{i}) \quad \rightarrow \quad L^{2}(Y, \nu). \quad (i \in W, j \in W) \]

Put \( \psi : L^{2}(X, \mu) \otimes L^{2}(Y) \to L^{2}(X \times Y, \mu \times \nu) \) be a linear map such that \( \psi(f_{i} \otimes g_{j}) = \delta_{i}^{j} \) where \( \delta_{i}^{j} \) is the Kronecker delta. To make sure that \( \psi \) is well-defined, we show that \( \delta_{i}^{j} \in L^{2}(X \times Y) \). Let \( X \times Y \) be the product \( \sigma \)-algebra on \( X \times Y \). Then \( f_{i} \) and \( g_{j} \) are also \( X \times Y \)-measurable. Thus \( \delta_{i}^{j} \) is \( X \times Y \)-measurable. Moreover,

\[ \int_{X \times Y} |\delta_{i}^{j}|^{2} \, d(x, y) = \int_{X} |f_{i}(x)|^{2} \, d\mu(x) \int_{Y} |g_{j}(y)|^{2} \, d\nu(y) = \|f_{i}\|_{L^{2}(X, \mu)}^{2} \|g_{j}\|_{L^{2}(Y, \nu)}^{2} = 1 < \infty \]

Thus, \( \delta_{i}^{j} \in L^{2}(X \times Y) \), and \( \psi \) is well-defined.
Next, we show that $\Psi$ is injective. Suppose that $x_j \in \mathcal{C}$ and $\sum x_j d_{ij} = 0$.

Then

$$\sum x_j f_i(x) g_j(y) = 0 \quad \text{for a.e. } (x,y) \in X \times Y$$

Then

$$0 = \sum_{ij} x_{ij} \int f_i(x) g_j(y) \, dx \, dy = \sum_{ijk} a_{ij} \int f_i(x) \overline{f_k(x)} g_j(y) \overline{g_l(y)} \, dx \, dy$$

Then

$$0 = \int_{X \times Y} \sum_{ijk} a_{ij} \int f_i(x) \overline{f_k(x)} g_j(y) \overline{g_l(y)} \, dx \, dy$$

$$= \sum_{ijk} a_{ij} \int_{X \times Y} f_i(x) \overline{f_k(x)} g_j(y) \overline{g_l(y)} \, dx \, dy$$

$$= \sum_{ijk} a_{ij} \overline{a_{kl}} \left< f_i, f_k \right>_L^X \left< g_j, g_l \right>_L^Y$$

$$= \sum_{ij} a_{ij} \overline{a_{ij}} = \sum_{ij} |a_{ij}|^2$$

Thus, $a_{ij} = 0$ for $i \neq j$, and $\Psi$ is injective. Thus, $\Psi(L^2(X) \otimes L^2(Y))$ is linearly isomorphic to $L^2(X) \otimes L^2(Y)$. Hence $\text{Im} \Psi = L^2(X) \otimes L^2(Y)$. 

The inner product on $L^p(X) \otimes L^q(Y)$ is

$$\langle \sum \delta_y \otimes f, \sum \delta_x \otimes g \rangle = \sum \delta_y \bar{g}_y$$

The inner product on $G = \Psi(L^p(X) \otimes L^q(Y))$ induced by $\Psi$ is

$$\langle \sum \delta_y \otimes f, \sum \delta_x \otimes g \rangle = \sum \delta_y \bar{g}_y$$

This is simply the restriction of the inner product on $L^p(X \times Y)$ onto $G$.

Indeed,

$$\langle \sum \delta_y \otimes f, \sum \delta_x \otimes g \rangle_{L^p(X \times Y)} = \sum \delta_y \bar{g}_y \int_{X \times Y} \langle f, g \rangle$$

$$= \sum \delta_y \bar{g}_y \int_{X \times Y} \int_X f(x) \overline{g(x)} f(x) \overline{g(x)} \, dx$$

$$= \sum \delta_y \bar{g}_y \int_X \langle f(x), \overline{g(x)} f(x) \overline{g(x)} \rangle$$

$$= \sum \delta_y \bar{g}_y$$

$$= \langle \sum \delta_y \otimes f, \sum \delta_x \otimes g \rangle_G$$

We know that $(L^p(X \times Y), \| \cdot \|_{L^p(X \times Y)})$ is a complete space. Thus, to show that $L^p(X \times Y) = L^p(X) \otimes L^p(Y)$, we only need to show that $(G, \| \cdot \|)$ is dense in $(L^p(X \times Y), \| \cdot \|)$. 


Each function \( h \in L^c(X \times Y) \) can be written as

\[ h = h^+ - h^- \]

where \( h^+ = \max \{0, h\} \), \( h^- = \max \{0, -h\} \) and \( h^+, h^- \in \ell^c(X \times Y) \).

There exist sequences of simple functions \( (s_n), (t_n) \in \ell^c(X \times Y) \) such that \( s_n \uparrow h^+ \) and \( t_n \uparrow h^- \). Thus \( u_n = s_n - t_n \in \ell^c(X \times Y) \) is a simple function, and

\[ \| h - u_n \|_{L^c} = \| (h^+ - h^-) - (s_n - t_n) \|_{L^c} \leq \| h^+ - s_n \|_{L^c} + \| h^- - t_n \|_{L^c} \to 0 \to 0 \]

Hence \( u_n \to h \) in \( \ell^c(X \times Y) \).

That means the set of simple functions in \( \ell^c(X \times Y) \) is dense in \( \ell^c(X \times Y) \); thus we only need to show that \( G \) is dense in this set. Moreover, each simple function in \( \ell^c(X \times Y) \) is a linear combination of characteristic functions in \( \ell^c(X \times Y) \). Hence we the task is left as follow.

Let \( A \) be \( \mathcal{S} \times \mathbb{T} \)-measurable and \( (\mu \circ \chi_A) < \infty \). Find a sequence \( (G_n) \) in \( G \) that converges to \( \chi_A \).

We have 3 following lemmas that will be proved in the end.

**Lemma 1:** Let \( D = A_1 \times B_1 \) be a measurable rectangle in \( X \times Y \). Then \( X_D \) is the limit of a sequence in \( G \); if \( \mu(A), \nu(B) < \infty \).
We called $A \in \mathcal{C} \times \mathcal{Y}$ an elementary set if $A = R_1 \cup \ldots \cup R_n$ where each $R_i$ is a measurable rectangle and $R_i \cap R_j = \emptyset$ for $i \neq j$. The class of all elementary sets is denoted by $E$.

Lemma 2: If $P, Q \in E$ then $P \cup Q, P \cap Q, P \setminus Q \in E$.

Lemma 3: Let $Q \in E$. Then $X_Q$ is a limit of a sequence in $G$.

By Lemma 3, we only have to show that the set $\{X_Q : Q \in E\}$ is dense in $\{X_A : A \in \mathcal{C} \times \mathcal{Y}\}$. We have

$$\|X_Q - X_A\|_p^p = \int_{\mathcal{Y}} |X_Q(x) - X_A(x)|^p = (\mu(\mathcal{D})) (Q \Delta A)$$

where $Q \Delta A = (Q \setminus A) \cup (A \setminus Q)$ is the symmetric difference of $Q$ and $A$. For simplicity, we put $\lambda = \mu(\mathcal{D})$. Put

$$N = \{A \in \mathcal{C} \times \mathcal{Y} : \forall \varepsilon > 0, \exists Q \in E \text{ such that } \lambda(Q \Delta A) < \varepsilon\}$$

The task left is to show that $N = \mathcal{C} \times \mathcal{Y}$. Because $E \subseteq N \subseteq \mathcal{C} \times \mathcal{Y}$, and $\mathcal{C} \times \mathcal{Y}$ is the smallest $\sigma$-algebra on $\mathcal{C} \times \mathcal{Y}$ containing all elementary sets, we only need to show that $N$ is a $\sigma$-algebra.

We have $\phi, X \times Y \in E \subseteq N$. 
Let \( A \in \mathbb{N} \). We show that \( X \times Y \setminus A \in \mathbb{N} \). For each \( \varepsilon > 0 \), there exists \( Q_\varepsilon \in E \) such that \( \lambda(Q_\varepsilon \setminus A) < \varepsilon \). By Lemma 2, \( Q'_\varepsilon = (X \times Y) \setminus Q_\varepsilon \in E \). Put \( A' = X \times Y \setminus A \).

\[
Q'_\varepsilon \setminus A' = (Q'_\varepsilon \setminus A') \cup (A' \setminus Q'_\varepsilon) = (A \setminus Q_\varepsilon) \cup (Q_\varepsilon \setminus A) = Q_\varepsilon \setminus A
\]

Thus \( \lambda(Q'_\varepsilon \setminus A') = \lambda(Q_\varepsilon \setminus A) < \varepsilon \). Hence \( A' \in \mathbb{N} \).

Let \( (A_n) \) be a sequence in \( \mathbb{N} \) and \( A = \bigcup_{n=1}^{\infty} A_n \). We'll show that \( A \in \mathbb{N} \). For each \( \varepsilon > 0 \) and \( n \in \mathbb{N} \), there exists \( Q_{\varepsilon,n} \in E \) such that \( \lambda(Q_{\varepsilon,n} \setminus A_n) < \frac{\varepsilon}{4} \cdot \frac{1}{2^n} \).

For each \( m \in \mathbb{N} \), we put \( p_{m}^{\varepsilon} = \bigcup_{n=1}^{m} Q_{\varepsilon,n} \). By Lemma 2, \( p_{m}^{\varepsilon} \in E \).

We have

\[
\lambda(A \setminus \bigcup_{m=1}^{\infty} p_{m}^{\varepsilon}) = \lambda\left[ \bigcup_{n=1}^{\infty} (A_n \setminus p_{m}^{\varepsilon}) \right]
\leq \sum_{n=1}^{\infty} \lambda(A_n \setminus p_{m}^{\varepsilon}) \leq \sum_{n=1}^{\infty} \lambda(A_n \setminus A_{n}^{\varepsilon})
\leq \sum_{n=1}^{\infty} \lambda(Q_{\varepsilon,n} \setminus A_n) = \sum_{n=1}^{\infty} \frac{\varepsilon}{4} \cdot \frac{1}{2^n} = \frac{\varepsilon}{4}
\]

Since \( \lambda(A \setminus \bigcup_{m=1}^{\infty} p_{m}^{\varepsilon}) = \lim_{m \to \infty} \lambda(A \setminus p_{m}^{\varepsilon}) \), we get
Moreover,
\[
\lambda (B_m \setminus A) = \lambda \left( \bigcup_{n=1}^{m} B_n \setminus A \right) \leq \sum_{n=1}^{m} \lambda (B_n \setminus A) \leq \sum_{n=1}^{\infty} \lambda (B_n \setminus A_n) \\
\leq \sum_{n=1}^{\infty} \lambda (A_n \setminus A_n) \leq \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n 2^n} = \frac{\varepsilon}{4}
\]

Together with (1), we have
\[
\lim_{m \to \infty} \lambda (A \Delta B_m) = \operatorname{liminf} \left[ \lambda (A \setminus B_m) + \lambda (B_m \setminus A) \right] \leq \frac{\varepsilon}{2}
\]

Thus, there exists $m_0 \in \mathbb{N}$ such that \( \lambda (A \setminus B_{m_0}) < \varepsilon \). Therefore, \( A \in \mathcal{N} \). That completes the proof.

*Proof of Lemma 1*

Let \( A \in \mathcal{S} \) and \( B \in \mathcal{S} \) such that \( \mu(A), \nu(B) < \infty \). Then \( X_A \in \mathcal{L}(X) \) and \( X_B \in \mathcal{L}(Y) \). For each \( \varepsilon > 0 \), there exist a finite sum \( \sum_{i} d_i f_i \in \mathcal{L}(X) \) such that
\[
\| \sum_{i} d_i f_i - X_A \|_{\mathcal{L}(X)} < \varepsilon
\]
and a finite sum \( \sum_{j} e_j g_j \in \mathcal{L}(Y) \) such that
\[
\| \sum_{j} e_j g_j - X_B \|_{\mathcal{L}(Y)} < \varepsilon
\]
Put \( h \in L^\infty(X \times Y) \) given by

\[
h(x,y) = \left( \sum_i x_i f_i(x) \right) \left( \sum_j y_j g_j(y) \right)
\]

Then \( h \in G \). We have

\[
\| h - \chi_{A \times B} \|_{L^\infty(X \times Y)} = \| \left( \sum_i x_i f_i \right) \left( \sum_j y_j g_j \right) - \chi_A \chi_B \|_{L^\infty(X \times Y)}
\]

Put \( f = \sum x_i f_i \) and \( g = \sum y_j g_j \). We have

\[
\| h - \chi_{A \times B} \|_{L^\infty(X \times Y)} = \| f(x)g(y) - \chi_A(x)\chi_B(y) \|_{L^\infty(X \times Y)}
\]

\[
= \| \left( f(x) - \chi_A(x) \right) g(y) + \chi_A(x) \left( g(y) - \chi_B(y) \right) \|_{L^\infty(X \times Y)}
\]

\[
\leq \| f(x) - \chi_A(x) \|_{L^\infty(X)} \| g(y) \|_{L^\infty(Y)} + \| \chi_A(x) \|_{L^\infty(X)} \| g(y) - \chi_B(y) \|_{L^\infty(Y)}
\]

\[
= \| f - \chi_A \|_{L^\infty(X)} \| g \|_{L^\infty(Y)} + \| \chi_A \|_{L^\infty(X)} \| g - \chi_B \|_{L^\infty(Y)}
\]

\[
\leq \varepsilon \| g \|_{L^\infty(Y)} + \mu(A)^{\frac{m^2}{2}} \varepsilon
\]

\[
\leq \varepsilon \left( \| \chi_B \|_{L^\infty(Y)} + \varepsilon \right) + \mu(A)^{\frac{m^2}{2}} \varepsilon
\]

\[
= \varepsilon \left( \| \chi_B \|_{L^\infty(Y)} + \varepsilon \right) + \mu(A)^{\frac{m^2}{2}} \varepsilon
\]

Thus \( \chi_{A \times B} \) is a limit of a sequence in \( G \).

* Proof of Lemma 2
First, we see that
\[(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D),\]
i.e. the intersection of two measurable rectangles is also a measurable rectangle. Let \(P, Q \in \mathcal{E}\)
\[P = UR_i \quad Q = UR_j\]
Then \(P \cap Q = (UR_i) \cap (UR_j) = U(R_i \cap R_j') \in \mathcal{E}\)
measurable rectangle
Consequently, every finite intersection of elements in \(\mathcal{E}\) belongs to \(\mathcal{E}\).
Let \(A \times B \in \mathcal{E}\) be a measurable rectangle. Then
\[(x, y) \setminus (A \times B) = \left[ A \setminus (y \setminus B) \right] \cup \left[ (x \setminus A) \times (y \setminus B) \right] \cup \left[ (x \setminus A) \times B \right] \]
\[R_1' \quad R_2 \quad R_3\]
\(R_1', R_2', R_3\) are measurable rectangles and pairwise disjoint. Thus, \((x, y) \setminus (A \times B) \in \mathcal{E}\)
For each \(P = UR_i \in \mathcal{E}\), we have
\[(x, y) \setminus P = (x, y) \setminus UR_i = \bigcap_{E \in \mathcal{E}} [(x, y) \setminus R_i] \]
This is a finite intersection of elements in \(\mathcal{E}\). Thus, \((x, y) \setminus P \in \mathcal{E}\).
For each \(P, Q \in \mathcal{E}\), we have
\[P \setminus Q = P \cap \left[ (x, y) \setminus Q \right] \in \mathcal{E}\]
To show that $P \cup A \in \mathcal{E}$, we only need to show that $(X \times Y) \setminus (P \cup A) \in \mathcal{E}$. We have

$$(X \times Y) \setminus (P \cup A) = \left(\frac{(X \times Y) \setminus P}{\mathcal{E}} \cap \frac{(X \times Y) \setminus A}{\mathcal{E}}\right) \in \mathcal{E}$$

\textbf{Proof of Lemma 3}

Let $Q \in \mathcal{E}$ and $\lambda(Q) < \infty$. We can write $Q = \cup R_i$ where $R_i$ is a measurable rectangle and $R_i \cap R_j = \emptyset$ for $i \neq j$. Thus,

$$\lambda(Q) = \sum_{i=1}^{n} \lambda(R_i),$$

and hence $\lambda(R_i) < \infty \forall i$. By Lemma 1, for each $\varepsilon > 0$, there exists $f_i^{\varepsilon} \in G$ such that

$$\|f_i^{\varepsilon} - X_{R_i}\|_{L^\infty(X \times Y)} < \frac{\varepsilon}{n}.$$ 

Put

$$f^{\varepsilon} = \sum_{i} f_i^{\varepsilon} \in G.$$ 

Then

$$\|f^{\varepsilon} - X_{\lambda}\|_{L^\infty(X \times Y)} = \|\sum_{i=1}^{n} f_i^{\varepsilon} - \sum_{i=1}^{n} X_{R_i}\| \leq \sum_{i=1}^{n} \|f_i^{\varepsilon} - X_{R_i}\| \leq \frac{n}{n} \varepsilon = \varepsilon.$$ 

Thus

$$\|f^{\varepsilon} - X_{\lambda}\|_{L^\infty(X \times Y)} < \varepsilon$$

Therefore, $X_{\lambda}$ is a limit of a sequence in $(G, \|\cdot\|_{\infty})$. 