Problem 1, p. 234 Goodman-Wallach.

Put $G = \text{SL}(2, \mathbb{C})$ and $F^{w} = \{f(x,y) = a_0 x^2 + 2a_1 xy + a_2 y^2 \mid a_0, a_1, a_2 \in \mathbb{C}\}$. The unique irreducible representation of $G$ on $F^{w}$ is given in the proof of Proposition 2.3.5, p. 86, namely

$$(g.f)(x,y) = f(ax+by, bx+dy)$$

for $g = (a \ b) \in G$ and $f \in F^{w}$. We identify $P(F^{w})$ with $\mathbb{C}[a_0, a_1, a_2]$.

(a) Consider $D \in P(F^{w})$, $D(f) = q^2 - a_0 a_2$. We'll show that $D$ is $G$-invariant, i.e., $D(g.f) = D(f)$ for all $g \in G$. For $f(x,y) = a_0 x^2 + 2a_1 xy + a_2 y^2$, we can write

$$f(x,y) = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a_0 & a_1 \\ a_1 & a_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Denote $f(g) = \begin{pmatrix} a_0 & a_1 \\ a_1 & a_2 \end{pmatrix}$. For $g = (c \ d) \in G$, we have

$$(g.f)(x,y) = f(ax+by, bx+dy) = (ax+by) + (bx+dy) \begin{pmatrix} a_0 & a_1 \\ a_1 & a_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Thus, $x(g.f) = g(x,y)g^t$. We have $D(f) = q^2 - a_0 a_2 = -\det(x(g))$.

Therefore, $D(g.f) = -\det(x(g)) = -\det(x(g)) = -\det(g) \cdot \det(x(g)) \cdot \det(g^t)$

$$= -\det(g) \cdot \det(x(g)) \cdot \det(g^t)$$

$$= -\det(g) \cdot \det(x(g)) \cdot \det(g^t).$$
\(- \det (\rho(g)) = D(q).\)

(b) Denote by \( \mathcal{P}^k(F^0) \) the space of polynomials that are homogeneous with order \( k \) on \( F^0 \). First, we show that the representation \( \mathcal{P}^k(F^0) \) has the decomposition

\[
\mathcal{P}^k(F^0) = F^{k+2} \oplus F^{k+4} \oplus F^{k+8} \oplus \ldots
\]

To do so, we will compute and compare the characters of both sides. The space \( \mathcal{P}^k(F^0) \) is isomorphic to \( S^k(F^0) \) as representations of \( G \) according to the proof of Theorem 4.1.20, page 189. Thus,

\[
\chi_{\mathcal{P}^k(F^0)}(d(g)) = \left[ \begin{array}{c} \frac{k+2}{2} \\ k \end{array} \right]_q \quad \forall q \in C \{10\},
\]

where \( d(g) = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix} \in G \). We have

\[
\left[ \frac{k+2}{2} \right]_q = \frac{[k+2]_q}{[2]_q! [k]_q!} = \frac{[k+1]_q [k+2]_q}{[2]_q} = \frac{q^{k+1} - q^{-k-1}}{q - q^{-1}} \frac{q^{k+2} - q^{-k-2}}{q - q^{-1}} = \frac{q^2 - 1}{q - q^{-1}}
\]

Therefore,

\[
\chi_{\mathcal{P}^k(F^0)}(d(g)) = \frac{q^{k+1} - q^{-k-1}}{q^2 - 1} \frac{q^{k+2} - q^{-k-2}}{q - q^{-1}}
\]

\[= \frac{(q^{k+1} - q^{-k-1})(q^{k+2} - q^{-k-2})}{(q^2 - 1)(q - q^{-1})} \quad (1).
\]

By Proposition 4.1.17,

\[
\chi_{\mathcal{P}^k(F^0)}(d(g)) = \chi_{F^{k+2}} + \chi_{F^{k+4}} + \chi_{F^{k+8}} + \ldots
\]

The formula at the bottom of page 188 reads

\[
\chi_{F^m}(d(q)) = q^m + q^{m-2} + \ldots + q^{-m+2} + q^{-m}
\]
\[
\frac{(q^2)^{m+1} - 1}{q^2 - 1} q^{-m} = \frac{q^{m+2} - q^{-m}}{q^2 - 1} = \frac{q^{m+1} - q^{-m-1}}{q - q^{-1}}
\]

\[
= \frac{(q^{m+1} - q^{-m-1})(q^2 - q^{-2})}{(q^2 - q^{-2})(q - q^{-1})} = \frac{(q^{m+3} + q^{-m-3}) - (q^{m-1} + q^{-m+1})}{(q^2 - q^{-2})(q - q^{-1})}
\]

Thus,

\[
\text{ch } F^{(2k)}(d(q)) + \text{ch } F^{(2k-2)}(d(q)) + \text{ch } F^{(2k-4)}(d(q)) + \ldots
\]

\[
= \frac{(q^{2k+3} + q^{-2k-3}) - (q + q^{-1})}{(q^2 - q^{-2})(q - q^{-1})} = \frac{(q^{k+1} - q^{-k-1})(q^{k+1} - q^{-k-1})}{(q^2 - q^{-2})(q - q^{-1})}
\]

From (1) and (2) we have

\[
\text{ch } P^k(F^{(2)}) = \text{ch } F^{(2k)}(d(q)) + \text{ch } F^{(2k-2)}(d(q)) + \text{ch } F^{(2k-4)}(d(q)) + \ldots
\]

Therefore, \( P^k(F^{(2)}) \sim F^{(2k)} \oplus F^{(2k-2)} \oplus F^{(2k-4)} \oplus \ldots \) as representations of \( G \).

Next, we'll show that \( \dim P^k(F^{(2)}) = \begin{cases} 0 & \text{if } k \text{ is odd}, \\ 1 & \text{if } k \text{ is even}. \end{cases} \)

because \( P^k(F^{(2)}) \sim F^{(2k)} \oplus F^{(2k-2)} \oplus F^{(2k-4)} \oplus \ldots \), we have a decomposition

\[
P^k(F^{(2)}) = A^{(2k)} \oplus A^{(2k-2)} \oplus A^{(2k-4)} \oplus \ldots
\]

where \( A^{(m)} \sim F^{(m)} \) as representations of \( G \). For each \( v \in P^k(F^{(2)}) \), we write \( v = v^{(2k)} + v^{(2k-2)} + v^{(2k-4)} + \ldots \) with \( v^{(m)} \in A^{(m)} \). We notice that

\[
g \cdot v = v \quad \forall g \in G
\]

\[
\Rightarrow \quad g \cdot v^{(2k)} + g \cdot v^{(2k-2)} + g \cdot v^{(2k-4)} + \ldots = v^{(2k)} + v^{(2k-2)} + v^{(2k-4)} + \ldots
\]

\[
\Rightarrow \quad \begin{cases} g \cdot v^{(2k)} = v^{(2k)}, \\ g \cdot v^{(2k-2)} = v^{(2k-2)}, \\ g \cdot v^{(2k-4)} = v^{(2k-4)}, \quad \forall g \in G \end{cases}
\]
\[ \begin{aligned}
&\begin{cases}
\varphi_{(2,1)} \in (A^{(2,1)})^G,
\varphi_{(2,2)} \in (A^{(2,2)})^G,
\end{cases}
\end{aligned}
\]

Therefore, \( \varphi^G(\varphi_{(2,1)}) \cong (\varphi_{(2,1)})^G \oplus (\varphi_{(2,2)})^G \oplus \cdots \)

Thus, \( \dim \varphi^G(\varphi_{(1)}) = \dim (\varphi_{(1)})^G + \dim (\varphi_{(2)})^G + \cdots \)

Hence, it suffices to show that
\[ \dim (\varphi_{(m)})^G = \begin{cases} 1 & \text{if } m = 0, \\
0 & \text{if } m \geq 1. \end{cases} \]

For \( m = 0 \), \( \varphi_{(0)} \cong \mathbb{C} \). We know that the trivial representation of \( G = \text{SL}(2, \mathbb{C}) \) on \( \mathbb{C} \) is irreducible. Thus, \( (\varphi_{(0)})^G = \varphi_{(0)} \cong \mathbb{C} \). Then \( \dim (\varphi_{(0)})^G = 1 \).

Consider \( m \geq 1 \). Then \( \varphi_{(m)} \) can be taken as the space of binary forms of degree \( m \), i.e.,
\[ \varphi_{(m)} = \{ f(x,y) = a_0 x^m + a_1 x^{m-1} y + \cdots + a_{m-1} x y^{m-1} + a_m y^m \mid a_0, a_1, \ldots, a_m \in \mathbb{C} \}. \]

The irreducible representation of \( G \) on \( \varphi_{(1)} \) is given in the proof of Proposition 2.3.5, page 86, Goodman-Wallach, namely,
\[ g \cdot f(x,y) = f(ax + cy, bx + dy), \text{ for } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{C}) \text{ and } f \in \varphi_{(m)}. \]

Take \( f \in \varphi_{(m)}^G \). Then \( f(ax + cy + bx + dy) \equiv f(x,y) \forall a, c, d \in \mathbb{C}, \ ad - bc = 1. \)

Write \( f(x,y) = \sum_{j=0}^m a_j y^{k-j} \).

Take \( b = c = 0 \) and \( d = a^{-1} \). We have \( f(ax, a^{-1} y) \equiv f(x,y) \forall a \in \mathbb{C}(0) \).

Thus, \( \sum_{j=0}^m a_j a^{j-m} x^j y^{m-j} = \sum_{j=0}^m a_j x^j y^{m-j} \).

This happens only if \( a_j = a_j y^{-t_j} \not\equiv 0 \forall 0 \leq j \leq m. \)
Thus, $g_j (a^{3-m}-1) = 0$ for $0 \leq j \leq m$ and $\forall a \in \mathbb{C} \setminus \{0\}$.

Thus, $g_j = 0$ if $j \neq \frac{m}{2}$. If $m$ is odd then $f \equiv 0$, i.e. $(F^{(m)})^G = \{0\}$.

If $m=2k$ then $f_{m+1}(y) = a_k x^k y^k$ ($k > 1$). Now choose

$$
\begin{pmatrix}
    a & b \\
    c & d
\end{pmatrix} = \begin{pmatrix}
    1 & 1 \\
    0 & 1
\end{pmatrix} \in SL(2, \mathbb{C})
$$

Then $f(ax+cy, bx+dy) = f(x, y) = a_k x^k (x+y)^k$. We must have

$$
a_k x^k y^k = a_k x^k (x+y)^k
$$

This is possible only if $a_k = 0$. Therefore $f_{m+1}(y) = 0$ and thus, $(F^{(m)})^G = \{0\}$.

Hence, $\dim (F^{(m)})^G = 0$ for all $m > 1$.

(c) We show that $\mathcal{B}(F^{(m)})^G = C[1]$.

We have the decomposition of $P(F^{(m)})$ into subrepresentations $P(F^{(m)}) = \bigoplus_{k=0}^{\infty} P^k(F^{(m)})$. Thus,

$$
\mathcal{B}(F^{(m)})^G = \bigoplus_{k=0}^{\infty} \mathcal{B}(P^k(F^{(m)}))^G.
$$

Because $\dim \mathcal{B}(P^k(F^{(m)}))^G = 0$ if $k$ is odd, we have

$$
\mathcal{B}(F^{(m)})^G = \bigoplus_{m=0}^{\infty} \mathcal{B}(P^{2m}(F^{(m)}))^G.
$$

For each $m > 0$, we put $\Psi_m \in \mathcal{B}(P^{2m}(F^{(m)}))$, $\Psi_m(f) = (D\phi)^m = (a^2 - \cos^2 \phi)^m$

for all $f \in F^{(m)}$. Then $\Psi_m(g \cdot f) = (D(g \cdot \phi))^m = (D\phi)^m = \phi_m(f)$ for all $g \in G$. Thus, $\Psi_m \in \mathcal{B}(P^{2m}(F^{(m)}))^G$. Because $\dim \mathcal{B}(P^{2m}(F^{(m)}))^G = 1$, we have

$$
\mathcal{B}(P^{2m}(F^{(m)}))^G = \langle \Psi_m \rangle = \langle D^m \rangle.
$$

From (c), we get $\mathcal{B}(F^{(m)})^G = \bigoplus_{m=0}^{\infty} \langle D^m \rangle = C[1]$. 

Problem 1, page 254, Goodman - Wallach.

Put $V = \mathbb{C}^n$, $G = GL(n, \mathbb{C})$. Let $G$ act on $V$ by usual multiplication, i.e. $g v = gv$ for all $g \in G, v \in \mathbb{C}^n$. Let $G$ act on $M_n$ by conjugation, i.e. $g X = g X g^{-1}$ for all $g \in G, X \in M_n$. Consider a linear map $T : V \otimes V^* \to M_n$ defined by $T(v \otimes v^*) = A \in M_n$ where $v^*(w) v = Aw + w \in V$.

We admit that $T$ is a linear isomorphism. Now we'll show that $T$ intertwines with the action of $G$ on $V \otimes V^*$ and $M_n$. Consider $g \in G, v \in V, v^* \in V^*$.

We have $g \cdot (v \otimes v^*) = (g \cdot v) \otimes (g \cdot v^*)$. For all $w \in V$,

$$T(g \cdot (v \otimes v^*)) = A_1,$$

where $A_1 w = (g \cdot v^*) (w) (g \cdot v) + w \in V$.

By the definition of the action of $G$ on $V^*$, we have

$A_1 w = v^* (g^{-1} w) (g \cdot v) = g \cdot (v^* (g^{-1} w) v) = g \cdot (A (g^{-1} w)) = (g A g^{-1}) w,$

where $A = T(v \otimes v^*)$. We have

$$g \cdot T(v \otimes v^*) = g A = g A g^{-1} \overset{(1)}{=} A_1 = T(g \cdot (v \otimes v^*)).$$

Thus, $T$ is an intertwining map, and $V \otimes V^*$ and $M_n$ are equivalent representation of $G$.

(a) The map $T^k : (V \otimes V^*)^k \to M_n^k$, $T^k(v_1 \otimes v_1^* \otimes \cdots \otimes v_k \otimes v_k^*) = T(v_1 \otimes v_1^*) \otimes \cdots \otimes T(v_k \otimes v_k^*)$ is also a linear isomorphism. For $g \in G$, we have

$$g \cdot (v_1 \otimes v_1^* \otimes \cdots \otimes v_k \otimes v_k^*) = g v_1 g v_1^* \otimes \cdots \otimes g v_k g v_k^*.$$

Thus, $T^k(g \cdot (v_1 \otimes v_1^* \otimes \cdots \otimes v_k \otimes v_k^*)) = T^k(g v_1 g v_1^* \otimes \cdots \otimes g v_k g v_k^*)$. 
\[= T(g_{v_{1}} \otimes g_{v_{1}}) \otimes \ldots \otimes T(g_{v_{k}} \otimes g_{v_{k}}) \]
\[= T(g_{v_{1}}(v_{1} \otimes v_{1}^{*})) \otimes \ldots \otimes T(g_{v_{k}}(v_{k} \otimes v_{k}^{*})) \]
\[= g_{v_{1}} T(v_{1} \otimes v_{1}^{*}) \otimes \ldots \otimes g_{v_{k}} T(v_{k} \otimes v_{k}^{*}) \quad \text{(because } T \text{ is intertwiner)} \]
\[= g_{v_{1}} T(v_{1} \otimes v_{1}^{*}) \otimes \ldots \otimes T(v_{k} \otimes v_{k}^{*}) \]
\[= g_{v_{1}} T^{\otimes k}(v_{1} \otimes v_{1}^{*} \ldots v_{k} \otimes v_{k}^{*}). \]

Therefore, \( T^{\otimes k} \) is intertwiner. Hence, \((V \otimes V^{*})^{\otimes k}\) and \(M_{\phi}^{\otimes k}\) are equivalent representations of \(G\).

(1) Consider a cyclic permutation \(C \in S_{k}\) of \(\{1, 2, \ldots, k\}\). Write
\[C = (m_{1} \rightarrow m_{2} \rightarrow \ldots \rightarrow m_{k} \rightarrow m_{1}).\]

Define \(\lambda_{C} \in [(V \otimes V^{*})^{\otimes k}]^{\otimes k}\) as follows,
\[\lambda_{C}(v_{1} \otimes v_{1}^{*} \otimes \ldots \otimes v_{k} \otimes v_{k}^{*}) = \prod_{j=1}^{k} \langle v_{m_{j}}^{*}, v_{m_{j+1}} \rangle.\]

Put \(X_{j} = T(v_{j} \otimes v_{j}^{*}) \neq 1\), \(j \leq k\). We want to show that
\[\lambda_{C}(v_{1} \otimes v_{1}^{*} \otimes \ldots \otimes v_{k} \otimes v_{k}^{*}) = tr(X_{m_{1}} X_{m_{2}} \ldots X_{m_{k}}).\]

This is equivalent to showing that
\[\prod_{j=1}^{k} \langle v_{m_{j}}^{*}, v_{m_{j+1}} \rangle = tr(X_{m_{1}} X_{m_{2}} \ldots X_{m_{k}}). \quad (2)\]

Because both sides of (2) are multilinear maps in \((v_{1}, v_{1}^{*}, \ldots, v_{k}, v_{k}^{*})\), it suffices to show (2) for \(v_{i}, v_{i}^{*} \in \{e_{1}^{*}, \ldots, e_{k}^{*}\}\) and \(v_{j}, v_{j}^{*} \in \{e_{1}, \ldots, e_{k}\}\). Write \(v_{j}^{*} = e_{j}^{*}\) and \(v_{j} = e_{j}\). Then
\[\text{LHS}(2) = \prod_{j=1}^{k} \langle e_{m_{j}}^{*}, e_{m_{j+1}} \rangle \]
\[= \prod_{j=1}^{k} \langle e_{s_{m_{j}}}^{*}, e_{r_{m_{j+1}}} \rangle. \]
Thus, \( \text{LHS}(2) = \begin{cases} 1 & \text{if } s_{n_j} = r_{m_{j+1}} \quad \forall j \leq k, \\ 0 & \text{otherwise.} \end{cases} \) \hfill (3)

We have \( X_{s_{m_j}} \circ T(v_{m_j} \otimes v_{m_j}^*) = T(e_{m_j} \otimes e_{m_j}^*) \). For every \( w \in V \),
\[
X_{s_{m_j}} w = e_{m_j}^*(w) e_{m_j} = w_{s_{m_j}} e_{m_j} = \left[ \begin{array}{c} y \\ \uparrow \\ s_{m_j} \end{array} \right] w.
\]

Thus, \( X_{s_{m_j}} = \left[ \begin{array}{c} 1 \\ \uparrow \\ s_{m_j} \end{array} \right] \).

Then \( X_{s_{m_1}} X_{s_{m_2}} \cdots X_{s_{m_k}} = \begin{cases} \left[ \begin{array}{c} 1 \\ \uparrow \end{array} \right] & \text{only one entry = 1, others are 0} \\ 0 & \text{otherwise.} \end{cases} \)

Thus, \( \text{RHS}(2) = \text{tr}(X_{s_{m_1}} X_{s_{m_2}} \cdots X_{s_{m_k}}) = \begin{cases} 1 & \text{if } s_{m_j} = r_{m_{j+1}} \quad \forall 1 \leq j \leq k, \\ 0 & \text{otherwise.} \end{cases} \) \hfill (4)

From (3) and (4), we have (2).

(c) Let \( s \in S_k \). Write \( s = c_1 c_2 \cdots c_r \) where \( c_1, c_2, \ldots, c_r \) are disjoint cycle permutations. Write \( c_i = (m_{1,i} \rightarrow m_{2,i} \rightarrow \cdots \rightarrow m_{k,i} \rightarrow m_{1,i}) \). Define \( \delta_s \in \text{Sym}(V^* \otimes V^*) \) as follows.
\[
\delta_s (v_1 \otimes v_1^* \otimes \cdots \otimes v_k \otimes v_k^*) = \prod_{i=1}^{r} \prod_{j=1}^{k} \langle v_{m_{j,i}}^*, v_{m_{j+1,i}} \rangle.
\] \hfill (5)

By Equation (2) in part (b), we have \( \frac{1}{k} \prod_{j=1}^{k} \langle v_{m_{j,i}}^*, v_{m_{j+1,i}} \rangle = \text{tr}(X_{m_{1,i}} X_{m_{2,i}} \cdots X_{m_{k,i}}) \).
Therefore, (5) becomes \( \lambda (\eta_{\sigma_1} \circ \cdots \circ \eta_{\sigma_r}) = \prod_{i=1}^{r} \text{tr}(X_{m_{i,i}} X_{m_{i,2}} \cdots X_{m_{i,n_i}}) \).

(3) Problem 2, page 255, Goodman-Wallach.

Put \( G = \text{GL}(n, \mathbb{C}) \). Let \( G \) act on \( M_n \) by conjugation, i.e. \( g \cdot X = g X g^{-1} \) for all \( g \in G \), \( X \in M_n \). For \( 1 \leq i \leq n \), we define \( u_i \in \mathfrak{p}(M_n) \) as \( u_i(X) = \text{tr}(X^i) \). We'll show that \( \mathfrak{p}(M_n)^G = \mathbb{C}[u_1, u_2, \ldots, u_n] \).

First, we'll show that \( \mathbb{C}[u_1, u_2, \ldots, u_n] \subseteq \mathfrak{p}(M_n)^G \). Because \( \mathfrak{p}(M_n)^G \) is an algebra over \( \mathbb{C} \), it suffices to show that \( u_i \in \mathfrak{p}(M_n)^G \) for every \( 1 \leq i \leq n \). For any \( g \in G \), we have

\[
    u_i(g \cdot X) = u_i(g X g^{-1}) = \text{tr}(g X g^{-1} g^i) = \text{tr}(g X^i g^{-1}) = \text{tr}(X^i) = u_i(X).
\]

Therefore, \( u_i \in \mathfrak{p}(M_n)^G \).

Next, we'll show that \( \mathfrak{p}(M_n)^G \subseteq \mathbb{C}[u_1, u_2, \ldots, u_n] \). We have the decomposition

\[
    \mathfrak{p}(M_n) = \bigoplus_{k=0}^{\infty} \mathfrak{p}^k(M_n),
\]

where \( \mathfrak{p}^k(M_n) \) is the set of all homogeneous polynomials on \( M_n \) of degree \( k \). Thus, \( \mathfrak{p}(M_n)^G = \bigoplus_{k=0}^{\infty} \mathfrak{p}^k(M_n)^G \). Hence, it suffices to show that for every \( k \geq 0 \), \( \mathfrak{p}^k(M_n)^G \subseteq \mathbb{C}[u_1, u_2, \ldots, u_n] \). Because \( \mathfrak{p}^0(M_n) = \mathbb{C} \), we only need to consider the case \( k \geq 1 \). We'll follow 3 steps.

**Step 1** Consider a map \( \phi : \mathfrak{p}^k(M_n) \to (\mathfrak{s}^k(M_n))^\ast 
\)

\[
    f \mapsto (X^k \mapsto f(X)).
\]
Recall that $S^k(M_n)$ is linearly spanned by the elements $X^{\otimes k}$, for $X \in M_n$ (see Lemma B2.3, page 651). We admit that $\phi$ is well-defined and is a linear isomorphism. We'll show that $\phi$ intertwines with the action of $G$ on $P^k(M_n)$ and $(S^k(M_n))^\ast$. Consider $g \in G$, $f \in P^k(M_n)$.

$$f \in P^k(M_n), \quad g \cdot \phi(X) = \phi(g^{-1}Xg).$$

Then $\phi(gf) = \left( Z \mapsto \phi(f)(Z) \right) = \left( Z^{\otimes k} \mapsto \phi(f)(g^{-1}Zg) \right)$. (1)

On the other hand,

$$g \cdot (\phi(f)) = \left( Z \mapsto \phi(f)(g^{-1}Z) \right)$$

$$= \left( Z^{\otimes k} \mapsto \phi(f)(g^{-1}Z^{\otimes k}) \right)$$

$$= \left( Z^{\otimes k} \mapsto \phi(f)(g^{-1}Z)^{\otimes k} \right)$$

$$= \left( Z^{\otimes k} \mapsto \phi(f)(g^{-1}Z) \right)$$

$$= \left( Z^{\otimes k} \mapsto \phi(f)(g^{-1}Zg) \right)$$  \hspace{2cm} (2)

By (1) and (2), $\phi(gf) = g \cdot (\phi(f))$. Thus, $\phi$ intertwines. Therefore, $P^k(M_n)$ and $(S^k(M_n))^\ast$ are equivalent representations of $G$.

**Step 2** We have a natural injection map $S^k(M_n) \to M_n^{\otimes k}$. Thus, we have a natural dual map, which is surjective, $(M_n^{\otimes k})^\ast \to (S^k(M_n))^\ast$. Hence,

$$[S^k(M_n)]^G = \left\{ f \big|_{S^k(M_n)} : f \in (M_n^{\otimes k})^\ast \right\}.$$
Step 3. Consider the map $B : V^* \otimes V^{*\otimes k} \rightarrow [(V \otimes V^*)^{\otimes k}]^*$, 

$$B(v_1 \otimes \ldots \otimes v_k \otimes w_1 \otimes \ldots \otimes w_k) = \left( w_1 \otimes \ldots \otimes w_k \rightarrow \prod_{i=1}^{k} \langle v_i^*, w_i \rangle \langle w_i^*, v_i \rangle \right).$$

We admit that $B$ is a linear isomorphism. We'll show that $B$ is intertwining.

For any $g \in G$, we have

$$B(g \cdot (v_1 \otimes \ldots \otimes v_k \otimes w_1 \otimes \ldots \otimes w_k)) = B(g \cdot (v_1 \otimes \ldots \otimes v_k \otimes g \cdot w_1 \otimes \ldots \otimes g \cdot w_k))$$

$$= (w_1 \otimes \ldots \otimes w_k \otimes \rightarrow \prod_{i=1}^{k} \langle g \cdot v_i^*, w_i \rangle \langle w_i^*, g \cdot v_i \rangle)$$

$$= (w_1 \otimes \ldots \otimes w_k \otimes \rightarrow \prod_{i=1}^{k} \langle v_i^*, g \cdot w_i \rangle \langle w_i^*, v_i \rangle)$$

(3)

$$g \cdot B(v_1 \otimes \ldots \otimes v_k \otimes w_1 \otimes \ldots \otimes w_k) = g \cdot (w_1 \otimes \ldots \otimes w_k \otimes \rightarrow \prod_{i=1}^{k} \langle v_i^*, w_i \rangle \langle w_i^*, v_i \rangle)$$

$$= (w_1 \otimes \ldots \otimes w_k \otimes \rightarrow \prod_{i=1}^{k} \langle v_i^*, g \cdot w_i \rangle \langle g \cdot w_i^*, v_i \rangle)$$

$$= (w_1 \otimes \ldots \otimes w_k \otimes \rightarrow \prod_{i=1}^{k} \langle v_i^*, g \cdot w_i \rangle \langle w_i^*, g \cdot v_i \rangle)$$

(4)

From (3) and (4), we conclude that $B$ is an intertwining map. Therefore, $V^* \otimes V^{*\otimes k}$ and $[(V \otimes V^*)^{\otimes k}]^*$ are equivalent representations of $G$.

Step 4. By Theorem 5.3.1, page 247, Goldman-Wallach,

$$(V^* \otimes V^{*\otimes k})_G = \text{span} \{ \xi_s : s \in G_k \}, \text{ where } G_k = \sum_{|I|=k} e_{i_1} \otimes \ldots \otimes e_{i_k},$$

$s \cdot \xi = s \cdot (i_1, \ldots, i_k) = (s_i i_1, \ldots, s_k i_k)$, $\xi I = e_{i_1} \otimes \ldots \otimes e_{i_k}$.

Now by Step 3, we get

$$\left\{ [ (V \otimes V^*)^{\otimes k} ]^*_G \right\}^* = \text{span} \{ B(\xi_s) : s \in G_k \}.$$
Step 5. We will compute $B(C_s)$.

\[ B(C_s) = B\left( \sum_{I \subseteq k} e_{s,I} \otimes e^* \right) = \sum_{I \subseteq k} B(e_{s,I} \otimes e^*). \]  

\[ B(e_{s,I} \otimes e^*) = B(e_{s(I)} \otimes e_{s(I)}^*) \]

\[ = \left( \nu_i \otimes \nu_i^* \otimes \nu_k \otimes \nu_k^* \mapsto \prod_{j=1}^d \langle e^*_{j}, \nu_j \rangle \prod_{j=1}^d \langle \nu_j, e_{s(j)} \rangle \right) \]

\[ = \left( \nu_i \otimes \nu_i^* \otimes \nu_k \otimes \nu_k^* \mapsto \prod_{j=1}^d \langle e_{s(j)}, \nu_j \rangle \prod_{j=1}^d \langle \nu_{s(j)}, e_j \rangle \right) \]

\[ = \left( \nu_i \otimes \nu_i^* \otimes \nu_k \otimes \nu_k^* \mapsto \prod_{j=1}^d (\nu_j^*) (\nu_{s(j)})_j \right). \]

Then (5) becomes

\[ B(C_s) = \sum_{I \subseteq k} \left( \nu_i \otimes \nu_i^* \otimes \nu_k \otimes \nu_k^* \mapsto \prod_{j=1}^d (\nu_{s(j)})_j (\nu_{s(j)})_j \right) \]

\[ = \left( \nu_i \otimes \nu_i^* \otimes \nu_k \otimes \nu_k^* \mapsto \sum_{I \subseteq k} \prod_{j=1}^d (\nu_{s(j)})_j (\nu_{s(j)})_j \right) \]

\[ = \left( \nu_i \otimes \nu_i^* \otimes \nu_k \otimes \nu_k^* \mapsto \langle \nu_{s(k)}, \nu_i \rangle \cdots \langle \nu_{s(k)}, \nu_k \rangle \right). \]  

Define $\lambda_s \in (V \otimes V^*)^{\otimes k}$ as follows.

\[ \lambda_s(e_{s(I)} \otimes e_{s(I)}^*) = \langle \nu_i^*, \nu_{s(I)} \rangle \cdots \langle \nu_k^*, \nu_{s(k)} \rangle. \]  

Then (6) implies $B(C_s) = \lambda_s$.

Step 6. With the result $B(C_s) = \lambda_s$, Step 4 gives us

\[ \{ (V \otimes V^*)^{\otimes k} \}^\mathbb{C} = \text{span} \{ \lambda_s^*: s \in B_k \} = \text{span} \{ \lambda_s : s \in B_k \}. \]
For \( s \in \mathfrak{S}_k \), we write \( s = \sigma_1 \sigma_2 \cdots \sigma_r \) where \( \sigma_1, \sigma_2, \ldots, \sigma_r \) are disjoint cycle decompositions. Write \( \sigma_i = (m_{1i} \rightarrow m_{2i} \rightarrow \cdots \rightarrow m_{k_i}, i \rightarrow m_{i}, i) \). Then

\[
\lambda_s (\gamma_1 \otimes \gamma_2 \otimes \cdots \otimes \gamma_k) \overset{(a)}{=} < \nu_1^*, \nu_2^*, \ldots, \nu_k^* > \otimes < \gamma_1, \gamma_2, \ldots, \gamma_k > \\
= \prod_{i=1}^r \prod_{j=1}^{k_i} \langle \nu_{m_{ji}, i}^*, \nu_{m_{ji}, i} \rangle \\
= \prod_{i=1}^r \text{tr} (X_{m_{1i}, i} X_{m_{2i}, i} \cdots X_{m_{k_i}, i}) \quad \text{(by Part (c), Prob 2)} \tag{9}
\]

where \( X_j = T (\gamma_j \otimes \gamma_j^*) \).

Step 7: By Part (a), Problem 2, we have an isomorphism of representations

\[
T^{\otimes k} : (V \otimes V^*)^k \rightarrow M_n^{\otimes k}.
\]

Thus, we have an isomorphism \( (T^{\otimes k})^* : (M_n^{\otimes k})^* \rightarrow [(V \otimes V^*)^k]^* \),

\[
f \mapsto f \circ T^{\otimes k}.
\]

A map \( f \in (M_n^{\otimes k})^* \) can be considered as a multi-linear map on \( M_n^k \).

Under \( T^{\otimes k} \), \( f (X_1 \otimes \cdots \otimes X_k) \mapsto (\gamma_1 \otimes \gamma_2 \otimes \cdots \otimes \gamma_k^*) \mapsto f (\gamma_1 \otimes \cdots \otimes \gamma_k) \)

where \( \gamma_j = T (\gamma_j \otimes \gamma_j^*) \).

We have

\[
f_s := [(T^{\otimes k})^*]^{-1} (\gamma_s) \overset{(a)}{=} \left( X_1 \otimes \cdots \otimes X_k \mapsto \prod_{i=1}^r \text{tr} (X_{m_{1i}, i} X_{m_{2i}, i} \cdots X_{m_{k_i}, i}) \right) \tag{9}
\]

because \( \left( (V \otimes V^*)^{\otimes k} \right)^* = \text{span} \{ \lambda_s : s \in \mathfrak{S}_k \} \), we have

\[
\left( (M_n \otimes k) \right)^* = \text{span} \{ f_s : s \in \mathfrak{S}_k \}.
\]
Step 8. According to Step 2, we have

\[ S_k^{*(M_n)} \cong \text{span} \left\{ s \mid S_k^{*(M_n)} : s \in G_k \right\} \]

Put \( g_k = f_k \mid S_k^{*(M_n)} \). We have

\[ g_k(X^k) = f_k(X^k) \left( \overset{r}{\underset{i=1}{\sum}} \text{tr}(X^k) = u_1(X)u_2(X) \ldots u_r(X) \right). \]

Step 9. According to Step 2, we have

\[ \phi^{-1}(g_k) = (X \mapsto u_1(X) \ldots u_r(X)) \in C[u_1, \ldots, u_n]. \]

Thus \( \phi^{-1}(M_n) \cong \text{span} \left\{ \phi^{-1}(g_k) : s \in G_k \right\} \subset C[u_1, \ldots, u_n]. \)

Problem #4, page 274, Goodman–Wallach.

Let \( \Omega \) be the following bilinear form on \( C^4 \),

\[ \Omega(x,y) = x^T J y \quad \forall x,y \in C^4, \]

where \( J_z = \begin{pmatrix} 0 & z \cr -z & 0 \end{pmatrix} \) and \( z = \begin{pmatrix} 0 & 1 \cr 1 & 0 \end{pmatrix} \). Put \( G = \text{Sp}(C^4, \Omega) \), which is a classical group of type \( E \).

Put \( V = C^4 \). Let \((\pi, V)\) be the representation of \( G \) in usual sense, i.e.

\[ \pi(g)v = g \cdot v \quad \forall g \in G, \forall v \in V. \]

This representation induces a representation \( \pi \odot \pi \) on \((\pi \odot \pi, V \otimes V)\) of \( G \). Put \( \tilde{V} = \Lambda^2(V) \subset V \otimes V \). Denote by \((\pi, \tilde{V})\) the subrepresentation of \((\pi \odot \pi, V \otimes V)\).

(a) We will find the weights of \( \tilde{V} \). We understand these weights as the weights of the representation \((\pi, \tilde{V})\) of \( G = \text{Sp}(C^4, \Omega) \), which are defined in
Theorem 3.1.16, page 138. Let \((e_1, e_2, e_3, e_4)\) be the standard basis of \(V = C^4\).

Then an ordered basis of the vector space \(V = \mathfrak{h}(V)\) is

\[
(e_1 e_2, e_1 e_3, e_1 e_4, e_2 e_3, e_2 e_4, e_3 e_4, e_4 e_1, e_4 e_2, e_4 e_3).
\]

By a result on page 73, Goodman-Wallach, a Cartan subalgebra of \(\mathfrak{g}\) is

\[
\mathfrak{h} = \{ \text{diag}(a_1, a_2, -a_1, -a_2) : a_1, a_2 \in C \}.
\]

Thus, a basis of the dual vector space \(\mathfrak{h}^*\) is \(\{e_1, e_2\}\), where

\[
e_1(\text{diag}(a_1, a_2, -a_1, -a_2)) = a_1,
\]

\[
e_2(\text{diag}(a_1, a_2, -a_1, -a_2)) = a_2.
\]

For each \(\lambda \in \mathfrak{h}^*\), put

\[
V(\lambda) = \{ w \in V : \lambda(h)w = \langle \lambda, h \rangle w \quad \forall h \in \mathfrak{h} \}.
\]

We write

\[
w = c_1 e_1 e_2 + c_3 e_3 e_4 + c_4 e_4 e_1 + c_5 e_2 e_3 + c_6 e_3 e_2 + c_7 e_4 e_3 + c_8 e_1 e_4,
\]

\[
h = \text{diag}(a_1, a_2, -a_1, -a_2),
\]

\[
\lambda = \beta_1 e_1 + \beta_2 e_2.
\]

Then \(\langle \lambda, h \rangle = \beta_1 \langle e_1, h \rangle + \beta_2 \langle e_2, h \rangle = \beta_1 a_1 + \beta_2 a_2\). Thus,

\[
\langle \lambda, h \rangle w = \sum_{1 \leq i < j \leq 4} (\beta_1 a_1 + \beta_2 a_2)c_{ij} e_i e_j \tag{1}
\]

we have

\[
d\langle \lambda, h \rangle w = \sum_{1 \leq i < j \leq 4} c_{ij} \left( \frac{d}{dt} \right)_{t=0} \exp(\lambda(t)) \langle e_i e_j, h \rangle
\]

\[
= \sum_{1 \leq i < j \leq 4} c_{ij} \left. \frac{d}{dt} \right|_{t=0} (h e_i e_j, h) =
\]
\[
\begin{aligned}
\sum_{1 \leq i, j \leq 4} c_{ij} \left[ \frac{d}{dt} \left. A_k e_i \right|_{t=0} \right] e_j + e_i \wedge \left( \frac{d}{dt} \left. A_k e_j \right|_{t=0} \right) \\
\text{We have } \frac{d}{dt} \left. A_k \right|_{t=0} = \frac{d}{dt} \exp(\theta t) = \theta. \text{ Thus, } (*) \text{ implies }
\end{aligned}
\]

\[
\begin{aligned}
dp(h)w &= \sum_{1 \leq i, j \leq 4} c_{ij} \left[ (he_i \wedge e_j + e_i \wedge (he_j)) \right]. \\
\text{We have:}
he_1 &= \begin{pmatrix} a_1 & a_2 \\ -a_2 & -a_1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a_1 \\ 0 \end{pmatrix} = a_1 e_1. \\
he_2 &= a_2 e_2, \\
he_3 &= -a_2 e_3, \\
he_4 &= -a_1 e_4.
\end{aligned}
\]

Similarly, 
\[
\begin{aligned}
\text{Then } (**) \text{ can be written as:}
\end{aligned}
\]

\[
\begin{aligned}
dp(h)w &= c_{12} (a_1 e_i \wedge e_j + a_2 e_i \wedge e_j) + c_{13} (a_1 e_i \wedge e_3 + a_2 e_i \wedge e_3) + \\
&\quad + c_{14} (a_1 e_i \wedge e_4 + a_2 e_i \wedge e_4) + c_{23} (a_2 e_i \wedge e_3 - a_1 e_i \wedge e_3) + \\
&\quad + c_{24} (a_2 e_i \wedge e_4 - a_1 e_i \wedge e_4) + c_{34} (-a_2 e_i \wedge e_4 - a_1 e_i \wedge e_4) \\
&= c_{12}'(a_1 + a_2) e_i \wedge e_j + c_{23}'(a_2 - a_1) e_i \wedge e_3 + c_{34}'(a_2 - a_1) e_i \wedge e_4 + c_{42}'(a_1 - a_2) e_i \wedge e_4.
\end{aligned}
\]

We now compare (1) and (3). The equation \( \langle A, h \rangle w = dp(h)w \) is equivalent to

\[
\begin{cases}
(p_1 q_1 + p_2 q_2) c_{12} = c_{12}'(a_1 + a_2), \\
(p_1 q_1 + p_2 q_2) c_{13} = c_{23}'(a_2 - a_1), \\
(p_1 q_1 + p_2 q_2) c_{14} = 0, \\
(p_1 q_1 + p_2 q_2) c_{23} = 0, \\
(p_1 q_1 + p_2 q_2) c_{24} = c_{24}'(a_2 - a_1), \\
(p_1 q_1 + p_2 q_2) c_{34} = c_{34}'(a_2 - a_1).
\end{cases}
\]
By (8) and (7), \( c_{14} = c_{23} = 0 \). If \((\beta_1, \beta_2) \notin \{(\pm 1, \pm 1)\}\) then the system (4)-(9) only has trivial solution \( c_{ij} \equiv 0 \).

- \( \beta_1 = \beta_2 = 1 \): then \( \lambda = \varepsilon_1 + \varepsilon_2 \). The system (4)-(9) has solutions

\[
\begin{cases}
  c_{14} \in \mathcal{C}, \\
  c_{23} = c_{45} = c_{14} = 0.
\end{cases}
\]

Thus, \( \tilde{\nu} (\varepsilon_1 + \varepsilon_2) = \text{span} \{ \varepsilon_1, \varepsilon_2 \} \).

- \( \beta_1 = \beta_2 = -1 \): then \( \lambda = -\varepsilon_1 - \varepsilon_2 \). The system (4)-(9) has solutions

\[
\begin{cases}
  c_{14} \in \mathcal{C}, \\
  c_{23} = c_{12} = c_{45} = 0.
\end{cases}
\]

Thus, \( \tilde{\nu} (-\varepsilon_1 - \varepsilon_2) = \text{span} \{ -\varepsilon_1, -\varepsilon_2 \} \).

- \( \beta_1 = 1, \beta_2 = -1 \): then \( \lambda = \varepsilon_1 - \varepsilon_2 \). The system (4)-(9) has solutions

\[
\begin{cases}
  c_{14} \in \mathcal{C}, \\
  c_{12} = c_{45} = c_{14} = 0.
\end{cases}
\]

Thus, \( \tilde{\nu} (\varepsilon_1 - \varepsilon_2) = \text{span} \{ \varepsilon_1, \varepsilon_2 \} \).

- \( \beta_1 = -1, \beta_2 = 1 \): then \( \lambda = -\varepsilon_1 + \varepsilon_2 \). The system (4)-(9) has solutions

\[
\begin{cases}
  c_{14} \in \mathcal{C}, \\
  c_{12} = c_{45} = c_{14} = 0.
\end{cases}
\]

Thus, \( \tilde{\nu} (-\varepsilon_1 + \varepsilon_2) = \text{span} \{ \varepsilon_1, \varepsilon_2 \} \).

We conclude that the set of weights of \((\rho, \tilde{\nu})\) is \( \mathcal{X}(\tilde{\nu}) = \{ \pm \varepsilon_1, \pm \varepsilon_2 \} \).

Next, we'll show that \( \mathcal{X}(\tilde{\nu}) \) is invariant under the action of Weyl group.
Let $W$ be the Weyl group of $G = Sp(4, \mathbb{R})$. By Lemma 3.1.6, p.133, $W$ is generated by the reflections $\{s_\alpha : \alpha \in \Delta\}$. Recall that
\[ \Delta = \{ \alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = 2\varepsilon_2 \} \]
is the set of simple roots of $G = Sp(4, \mathbb{R})$ (see page 102, Goodman-Wallach). By Equation (3.6), page 132, the reflection $s_\alpha : \varphi^* \to \varphi^*$ is given by
\[ s_\alpha(\varphi) = \varphi - \frac{2(\varphi, \alpha)}{\langle \alpha, \alpha \rangle} \alpha. \]
Thus,
\[ s_{\alpha_1}(\beta) = \beta - \frac{2(\beta, \alpha_1)}{\langle \alpha_1, \alpha_1 \rangle} \alpha_1 = \beta - \frac{2(\beta, \varepsilon_1 - \varepsilon_2)}{2} (\varepsilon_1 - \varepsilon_2) \]
\[ = \beta - (\beta, \varepsilon_1 - \varepsilon_2) (\varepsilon_1 - \varepsilon_2) \quad (10), \]
\[ s_{\alpha_2}(\beta) = \beta - \frac{2(\beta, \alpha_2)}{\langle \alpha_2, \alpha_2 \rangle} \alpha_2 = \beta - \frac{2(\beta, 2\varepsilon_2)}{4} (2\varepsilon_2) = \beta - 2(\beta, \varepsilon_2) \varepsilon_2 \quad (11). \]

Because the group $W$ is generated by $\{s_{\alpha_1}, s_{\alpha_2}\}$, to show that $X(\overline{V})$ is invariant under the action of $W$, it suffices to show that $X(\overline{V})$ is invariant under $s_{\alpha_1}$ and $s_{\alpha_2}$. Moreover, because $s_\alpha(-\beta) = -s_\alpha(\beta)$, it suffices to show that
\[ s_{\alpha_1}(\varepsilon_1 + \varepsilon_2), s_{\alpha_1}(\varepsilon_1 - \varepsilon_2), s_{\alpha_2}(\varepsilon_1 + \varepsilon_2), s_{\alpha_2}(\varepsilon_1 - \varepsilon_2) \in X(\overline{V}). \]

Thanks to (10) and (11), we have
\[ s_{\alpha_1}(\varepsilon_1 + \varepsilon_2) = (\varepsilon_1 + \varepsilon_2) - (\varepsilon_1 + \varepsilon_2, \varepsilon_1 - \varepsilon_2) (\varepsilon_1 - \varepsilon_2) = \varepsilon_1 + \varepsilon_2 \in X(\overline{V}). \]
\[ s_{\alpha_1}(\varepsilon_1 - \varepsilon_2) = (\varepsilon_1 - \varepsilon_2) - (\varepsilon_1 - \varepsilon_2, \varepsilon_1 - \varepsilon_2) (\varepsilon_1 - \varepsilon_2) = -\varepsilon_1 + \varepsilon_2 \in X(\overline{V}). \]
\[ S_{\alpha}(e_1 + e_2) = (e_1 + e_2)^{-2(e_1 + e_2, e_2)} e_2 = e_1 + e_2 - 2e_2 = e_1 - e_2 \in \mathfrak{X}(\tilde{V}). \]
\[ e_\alpha(e_1 - e_2) = (e_1 - e_2)^{-2(e_1 - e_2, e_2)} e_2 = e_1 - e_2 + 2e_2 = e_1 + e_2 \in \mathfrak{X}(\tilde{V}). \]

(5) Problem 1, Section 7.1.4, page 339, Goodman-Wallach.

We will verify the Weyl Denominator Formula for a classical group \( G \) of types \( A, E \),
\[ G \{ A, E \} : \quad D_G = \sum_{s \in W} \text{sgn}(s) e^{fs}. \]

By the definition of \( D_G \), we are supposed to show that
\[ e^{\sum_{y \in \Omega^+} (1 - e^{-y})} = \sum_{s \in W} \text{sgn}(s) e^{fs}, \quad (1) \]

where \( s = \frac{1}{2} \sum_{e \in \Omega^+} \alpha \) and \( W \) is the Weyl group of \( G \). A survey of \( \Omega^+ \) is given on pages 100-102, Goodman-Wallach.

Type \( A_e \) (\( G = SL(l+1, C) \)): \( \Omega^+ = \{ e_i - e_j : 1 \leq i < j \leq l+1 \} \),

Type \( B_2 \) (\( G = SO(2l+1, C) \)): \( \Omega^+ = \{ e_i - e_j, e_i + e_j : 1 \leq i < j \leq l \} \cup \{ e_i : 1 \leq i \leq l \} \),

Type \( C_e \) (\( G = Sp(l, C) \)): \( \Omega^+ = \{ e_i - e_j, e_i + e_j : 1 \leq i < j \leq l \} \cup \{ 2e_i : 1 \leq i \leq l \} \),

Type \( D_e \) (\( G = SO(2l, C), l > 3 \)): \( \Omega^+ = \{ e_i - e_j, e_i + e_j : 1 \leq i < j \leq l \} \).

We will verify (1) in each type of \( G \).

Type \( A_e \) : \( G = SL(l+1, C) \)
\[ s = \frac{1}{2} \sum_{e \in \Omega^+} \alpha = \frac{1}{2} \sum_{1 \leq i < j \leq l+1} (e_i - e_j) = \frac{l}{2} e_1 + \frac{l-2}{2} e_2 + \cdots + \frac{-2}{2} e_l + \frac{-1}{2} e_{l+1}. \quad (2) \]

Recall that \( \gamma = \{ \text{diag}(a_1, \ldots, a_{l+1}) : \sum_{i=1}^{l+1} a_i = 0 \} \) and \( e_i(\text{diag}(a_1, \ldots, a_{l+1})) = a_i. \)
Thus, $e_1 + e_2 + \ldots + e_{l+1} = 0$. Then (2) becomes

$$S = \frac{\ell}{2} e_1 + \frac{\ell - 2}{2} e_2 + \ldots + \frac{-1}{2} e_{l+1} + \frac{1}{2} (\varepsilon_1 + \varepsilon_e) = \varepsilon_1 + \ell \varepsilon e + \ldots + \varepsilon_e. \quad (3)$$

Thus, as a character on $H = \Pi h = \text{diag} (x_1, \ldots, x_{\ell+1}) | x_1 \cdots x_{\ell+1} = 1 |

$$e^S = \frac{\ell}{2} x_1^{\ell-1} \ldots x_e. \quad (4)$$

For $x = e_1 - y$, we have $e^{-x} = e^{x_1 - y} = x_1^{-1} x$. Thus,

$$\text{LHS}(1) = x_1^{\ell-1} \ldots x_e \prod_{1 \leq i < j \leq \ell+1} (1 - x_i x_j^{-1}) = \prod_{1 \leq i < j \leq \ell+1} (x_i - x_j). \quad (5)$$

Now we will consider $\text{RHS}(1)$. For $x = \lambda_1 e_1 + \ldots + \lambda_{\ell+1} e_{\ell+1}$ with $\lambda_i \in \mathbb{Z}$,

we have $x = \lambda_1 e_1 + \ldots + \lambda_{\ell+1} e_{\ell+1} = (\lambda_1 - \sigma_{\ell+1}) e_1 + \ldots + (\lambda_{\ell+1} - \sigma_1) e_1$.

Thus,

$$e^x = x_1^{\lambda_1 - \sigma_1} \ldots x_e^{\lambda_{\ell+1} - \sigma_{\ell+1}} = x_1^{\lambda_1} x_2^{\lambda_2} \ldots x_{\ell+1}^{\lambda_{\ell+1}} = x_1 \cdots x_{\ell+1}. \quad (6)$$

In the proof of Lemma 3.16, page 138, the Weyl group $W$ acts on $g^*$ by permuting $e_1, e_2, \ldots, e_{\ell+1}$. By (3) we get

$$S \cdot e^x = \frac{e_1}{2} \left( x_1^{\ell-1} \ldots x_e^{-1} \right) + \ldots + \left( \frac{e_{\ell+1}}{2} \right) \left( x_1 x_2 \ldots x_{\ell+1} \right). \quad (7)$$

Then by (6) we get $e^S = \frac{\ell}{2} x_1^{\ell-1} \ldots x_e^{-1}$. Thus,

$$\text{RHS}(1) = \sum_{\sigma \in S_{\ell+1}} \text{sgn} (\sigma) x_1^{\ell-1} x_2^{\ell-1} \ldots x_{\ell+1} = \det \left( \begin{array}{cccc} x_1 & x_1 & \cdots & 1 \\ x_2 & x_2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ x_{\ell+1} & x_{\ell+1} & \cdots & 1 \end{array} \right) = \prod_{1 \leq i < j \leq \ell+1} (x_i - x_j).$$

by (5) and (7), we obtain the identity (1).

Type $B_n$: $G = SO(2\ell+1, \mathbb{C})$
\[ s = \frac{1}{2} \sum_{x \in \mathcal{E}_\ell} x = \frac{1}{2} \sum_{1 \leq i < j \leq \ell} \left[(\varepsilon_i \cdot y) + (\varepsilon_j \cdot y)\right] + \frac{1}{2} \sum_{i=1}^{\ell} \varepsilon_i = \sum_{1 \leq i < j \leq \ell} \varepsilon_i + \frac{1}{2} \sum_{i=1}^{\ell} \varepsilon_i \]
\[ = \left(1 - \frac{1}{2}\right) \varepsilon_1 + \left(1 - \frac{2}{2}\right) \varepsilon_2 + \cdots + \frac{1}{2} \varepsilon_\ell. \quad (8) \]

Because the coefficients of \( \varepsilon_1, \ldots, \varepsilon_\ell \) are not integers, we don't know how to define \( \varepsilon \). The textbook (near the bottom of page 330) suggested that we need some knowledge about the spin group \( \text{Spin}(2l+1, \mathbb{C}) \) to make sense of \( \varepsilon \).

However, we will continue to work formally on this type.

\[ e^s = x_{1}^{-\frac{1}{2}} x_{2}^{-\frac{1}{2}} \cdots x_{\ell}^{-\frac{1}{2}} = x_{1}^{\ell-1} x_{2}^{\ell-2} \cdots x_{\ell} \left(x_{1} \cdots x_{\ell}\right)^{\frac{1}{2}}. \quad (9) \]

For \( x = \varepsilon_i - \varepsilon_j \), \( e^{-s} = e^{-\varepsilon_i + \varepsilon_j} = x_{i}^{1} x_{j}^{-1}. \)

For \( x = \varepsilon_i + \varepsilon_j \), \( e^{-s} = e^{-\varepsilon_i + \varepsilon_j} = x_{i}^{-1} x_{j}^{1}. \)

For \( x = \varepsilon_i \), \( e^{-s} = e^{-\varepsilon_i} = x_{i}^{-1}. \)

Thus, \( \text{LHS}(1) = x_{1}^{-1} x_{2}^{-2} \cdots x_{\ell} \left(x_{1} \cdots x_{\ell}\right)^{\frac{1}{2}} \left(\prod_{1 \leq i < j \leq \ell} \left(1 - x_{i}^{-1} x_{j} \right) \left(1 - x_{i} x_{j}^{-1}\right)\right) \left(\prod_{k=1}^{\ell} x_{k} x_{k}^{-1}\right) \)
\[ = \left(\prod_{1 \leq i < j \leq \ell} x_{i} \left(1 - x_{i}^{-1} x_{j} \right) \left(1 - x_{i}^{-1} x_{j}\right)\right) \left(\prod_{k=1}^{\ell} x_{k} x_{k}^{-1}\right) \]
\[ = \prod_{1 \leq i < j \leq \ell} \left(x_{i} + x_{j}^{-1} - x_{j} - x_{i}^{-1}\right) \left(\prod_{k=1}^{\ell} x_{k} \right) \left(x_{k} \right) \left(x_{k}^{-1}\right). \quad (10) \]

On page 128, Goodman-Wellach, an action of \( W \) on \( T(H) \) is defined by \( s \cdot e^s = e^s. \)

By Lemma 3.13, page 131, Goodman-Wellach, the Weyl group \( W \) acts on the coordinate functions in \( T(H) \) by \( x_{i} \mapsto (x_{i}(\sigma))^{\frac{1}{2}} \) (i = 1, 2, ..., l) for every \( \sigma \in \mathcal{E}_\ell \) and choice \( \pm 1 \) of exponents. Then,
\section*{22}

\[ s(e^g) = e(s(x_1^{-1} x_2^{-\frac{1}{2}} ... x_l^{-\frac{1}{2}})) = \left( e(x_1) e(x_2)^{-\frac{1}{2}} ... e(x_l)^{-\frac{1}{2}} \right)^{\frac{1}{2}} \]

\[ = \left( e(x_1) e(x_2)^{-\frac{1}{2}} \right) e(e(x_1)) \pm (e(x_2)^{-\frac{1}{2}} ... e(x_l)^{-\frac{1}{2}}) \pm l \quad (11) \]

When viewed as an element in \( GL(n^e) \), we have \( s(e^g) = \pm e \) (the choice of \( \pm 1 \) is according to that of \( e(x_i)^{\pm 1} \)). By the definition of \( \text{sgn}(s) \) on the top of page 331,

\[ \text{sgn}(s) = \det((e_i e_2 ... e_l) \mapsto (\pm e(x_1), ... , \pm e(x_l))) = \text{sgn}(e) (\pm 1)^\# \text{ of minus signs} \quad (12) \]

by (11) and (12), we have

\[ \text{RHS}(I) = \sum_{\delta \in \mathbb{Z}_e} \text{sgn}(s)(-1)^\# \text{ of } e^{-\frac{1}{2}} \pm (e^{-\frac{1}{2}})(e(x_1)) \pm (e^{-\frac{1}{2}} ... (e(x_l)) \pm l \quad (13) \]

When (10) is expanded, it will be a sum of \( 4 \cdot 2^l = 4 \cdot 2^{l-1} \cdot 2^l = 2^{l+1} \) terms, each of which is of the form \( \pm x_1 x_2 ... x_l \), where each \( x_i \) is an integer added by \( \frac{1}{2} \). We will show that after the sum of these \( 2^{l+1} \) terms is reduced, only the terms with \( 1x_1, 1x_2, ... 1x_l \) pairwise distinct remain. To do so, let us consider two typical terms: \( x_1 x_2 ... x_l \) and \( x_1 x_2 ... x_l \). Write

\[ \prod_{1 \leq i < j \leq l} \left( x_i + x_j - y - y^j \right) \prod_{k=1}^l (x_k - y - y^k) = \sum_{r} A_r x_1^r x_2^r + B \], \quad (14) \]

where \( A_r \) is independent of \( x_1 \) and \( x_2 \), and \( B \) is a sum of terms \( \pm x_1 x_2 ... x_l \) with \( x_1 \neq x_2 \). Switching \( x_1 \) and \( x_2 \), we get

\[ - \prod_{1 \leq i < j \leq l} \left( x_i + x_j - y - y^j \right) \prod_{k=1}^l (x_k - y - y^k) = \sum_{r} A_r x_1^r x_2^r + B' \], \quad (15) \]
where $B'$ is a sum of terms $\pm x_1 x_2 \ldots x_L$ with $x_i \neq x_j$ (but the choices of $\pm 1$ may be different from those in $B$). Adding (14) and (15), we get

$$\sum_{r} A_r (x_i x_j)^r = \frac{-B + B'}{2}.$$ \hspace{1cm} (16)

LHS(16) is a polynomial of $x_1 x_2$ while RHS(16) is not unless it is identically zero. Thus $\sum_{r} A_r (x_i x_j)^r \equiv 0$. Therefore, each $A_r \equiv 0$. Similarly, we write

$$\prod_{1 \leq i < j \leq l} (x_i + x_j - x_i x_j) \prod_{d=1}^{L} (x_i x_j - x_k x_l) = \sum_{r} C_r x_i^r x_j^r + D,$$ \hspace{1cm} (17)

where $C_r$ is independent of $x_1$ and $x_2$, and $D$ is a sum of terms $\pm x_1 x_2 \ldots x_L$ with $x_i \neq x_j$. Switching $x_i$ and $x_j^{-1}$, we get

$$-\prod_{1 \leq i < j \leq l} (x_i + x_j^{-1} - x_i x_j^{-1}) \prod_{d=1}^{L} (x_i x_j^{-1} - x_k x_l) = \sum_{r} C_r x_i^r x_j^{-r} + D',$$ \hspace{1cm} (18)

where $D'$ is a sum of terms $\pm x_1 x_2 \ldots x_L$ with $x_i \neq x_j$ (but the choices of $\pm 1$ may be different from those in $D$). Adding (17) and (18), we get

$$\sum_{r} C_r x_i^r x_j^{-r} = \frac{D + D'}{2}.$$ \hspace{1cm} (19)

LHS(19) is a polynomial of $x_i x_j$ while RHS(19) is not unless it is identically zero. Thus, $\sum_{r} C_r (x_i x_j)^r \equiv 0$. Therefore, each $C_r \equiv 0$.

We have showed that after (10) is expanded and reduced, only the terms $\pm x_1 x_2 \ldots x_L$ with $|r_1|, |r_2|, \ldots, |r_L|$ pairwise distinct remain. Put $a_i = x_i + x_i^{-1}$. Then

$$\prod_{1 \leq i < j \leq l} (x_i + x_j^{-1} - x_i x_j^{-1}) = \prod_{1 \leq i < j \leq l} (a_i - a_j) = \sum_{\sigma} \text{sign}(\sigma) a_{\sigma(1)}^{a_{\sigma(1)} - 2} a_{\sigma(2)}^{a_{\sigma(2)} - 2} \ldots a_{\sigma(L)}^{a_{\sigma(L)} - 2}.$$
Thus, (10) becomes
\[
\text{LHS}(1) = \sum_{\mathcal{E}} \text{sgn}(\epsilon) \left( x_{c(1)} + x_{c(1)}^{-1} \right)^{\frac{\ell_1}{2}} \left( x_{c(2)} + x_{c(2)}^{-1} \right)^{\frac{\ell_2}{2}} \cdots \left( x_{c(\ell-1)} + x_{c(\ell-1)}^{-1} \right)^{\frac{\ell_{\ell-1}}{2}} \prod_{k=1}^{\ell} (x_{k}^{\frac{1}{2}} - x_{k}^{-\frac{1}{2}})
\]
\[
= \sum_{\mathcal{E}} \text{sgn}(\epsilon) \left( x_{c(1)} \right)^{\frac{1}{2}} \left( x_{c(1)}^{-\frac{1}{2}} \right)^{\frac{1}{2}} \cdots \left( x_{c(\ell-1)} \right)^{\frac{1}{2}} \left( x_{c(\ell-1)}^{-\frac{1}{2}} \right)^{\frac{1}{2}} \prod_{k=1}^{\ell} (x_{k}^{\frac{1}{2}} - x_{k}^{-\frac{1}{2}}) \quad (20)
\]
(Where $\epsilon$'s are numbers depending on each summand)
\[
= \sum_{\mathcal{E}} \text{sgn}(\epsilon) \left( x_{c(1)} \right)^{\frac{1}{2}} \left( x_{c(1)}^{-\frac{1}{2}} \right)^{\frac{1}{2}} \cdots \left( x_{c(\ell-1)} \right)^{\frac{1}{2}} \left( x_{c(\ell-1)}^{-\frac{1}{2}} \right)^{\frac{1}{2}} \prod_{k=1}^{\ell} (x_{k}^{\frac{1}{2}} - x_{k}^{-\frac{1}{2}}). \quad (21)
\]

Note that $e_1, e_2, \ldots, e_{l-1}$ are integers satisfying $|e_k| \leq l - k$. As we showed earlier, in the reduced form of (21), the values $|e_1 \pm \frac{1}{2}, e_2 \pm \frac{1}{2}, \ldots, e_{l-1} \pm \frac{1}{2}|$ are pairwise distinct. Suppose by contradiction that there exists $k \leq l - 1$ such that $|e_k| \leq l - k - 1$. Then $|e_1, e_{k+1}, \ldots, e_{l-1}| \leq l - k - 1$. Then
\[
|e_1 \pm \frac{1}{2}, e_{k+1} \pm \frac{1}{2}, \ldots, e_{l-1} \pm \frac{1}{2}| \leq l - k - \frac{1}{2}.
\]
Thus,
\[
\left|\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, -\frac{1}{2}\right| \subset \left\{ \frac{1}{2}, \frac{1}{2}, \ldots, l - k - \frac{1}{2} \right\}.
\]
\[
\begin{align*}
\text{l-k distinct numbers} & \\
\text{l-k-1 elements} & \\
\end{align*}
\]
This is a contradiction. Thus, $|e_k| = l - k$ for all $1 \leq k \leq l - 1$. Thus, $e_k = \pm(l - k)$.

Consequently, the factors $\alpha$ in (20) and (21) are $\pm 1$. We have
\[
|e_k \pm \frac{1}{2}| \leq |e_k + \frac{1}{2}| = l - k + \frac{1}{2}.
\]
Thus,
\[
|e_1 \pm \frac{1}{2}, e_2 \pm \frac{1}{2}, \ldots, e_{l-1} \pm \frac{1}{2}| \subset \left\{ \frac{1}{2}, \frac{3}{2}, \ldots, l - \frac{1}{2} \right\}.
\]
Hence
\[
|e_k \pm \frac{1}{2}| = l - k + \frac{1}{2} \quad \forall 1 \leq k \leq l - 1.
\]
Hence, if we chose $\varepsilon_k = \ell - k$ in (21) for some $k$ then we must have

$$\varepsilon_k + \frac{1}{2} = \varepsilon_k + \frac{1}{2} = \ell - k + \frac{1}{2},$$

i.e. the factor $x_k^{1/2}$ in (20) is chosen to multiply with $(x_{\alpha(1)})^k (x_{\alpha(2)})^2 \cdots (x_{\alpha(k)})^{k+1}$.

If $\varepsilon_k = -(\ell - k)$ for some $k$ then we must have

$$\varepsilon_k - \frac{1}{2} = \varepsilon_k - \frac{1}{2} = -\ell + k + \frac{1}{2},$$

i.e. the factor $(-x_k^{-1/2})$ in (20) is chosen to multiply with $(x_{\alpha(1)})^k (x_{\alpha(2)})^2 \cdots (x_{\alpha(k)})^{k+1}$.

Therefore,

$$\alpha = \begin{cases} 
-1 & \text{if the number of "-" chosen for } \pm (\ell - k + \frac{1}{2}), \ 1 \leq k \leq l, \ \text{is odd} \\
+1 & \text{otherwise.} 
\end{cases}$$

Then (21) can be written as

$$\text{LHS}(1) = \sum_{\varepsilon \in \mathcal{E}} \text{sgn}(\varepsilon) (-1)^{\# \text{"-"}} (x_{\alpha(1)})^{\ell - \frac{1}{2}} (x_{\alpha(2)})^{k - \frac{3}{2}} \cdots (x_{\alpha(k)})^{k + 1}.$$

Comparing with (13), we get the identity (21).

**Type $G_2$: $G = Sp(\ell, \mathbb{C})$**

$$p = \frac{1}{2} \sum_{\varepsilon \in \mathcal{E}^{\pm}} \alpha = \frac{1}{2} \sum_{1 \leq i < j \leq \ell} [(\varepsilon_i - \varepsilon_j) + (\varepsilon_i + \varepsilon_j)] + \frac{1}{2} \sum_{i=1}^{\ell} \varepsilon_i = \sum_{1 \leq i < j \leq \ell} \varepsilon_i + \frac{\ell}{2} \varepsilon_i$$

$$\ell \varepsilon_i + (\ell - 1) \varepsilon_i + \cdots + \varepsilon_i. \quad (22)$$

Thus, $\varepsilon = \varepsilon_1 \varepsilon_2 \cdots \varepsilon_\ell$. \quad (23)

For $\alpha = \varepsilon_i - \varepsilon_j$, $e^{\alpha} = e^{\varepsilon_i - \varepsilon_j} = x_i^{-1} y_j$.

For $\alpha = \varepsilon_i + \varepsilon_j$, $e^{\alpha} = e^{\varepsilon_i + \varepsilon_j} = x_i^{-1} y_j$.

For $\alpha = 2\varepsilon_k$, $e^{\alpha} = e^{2\varepsilon_k} = x_k^{-2}$. 
Thus, \[ \text{LHS}(4) = \prod_{i < j} x_i (1 - x_i^{-1} x_j) (1 - x_i^{-1} y_j) \prod_{k=1} \left( 1 - x_k^{-2} \right) \]
\[ = \prod_{1 \leq i < j \leq \ell} x_i (1 - x_i^{-1} y_j) (1 - x_i^{-1} y_j) \prod_{k=1} x_k (1 - x_k^{-2}) \]
\[ = \prod_{1 \leq i < j \leq \ell} (x_i + x_i^{-1} y_j - x_j - y_j) \prod_{k=1} (x_k - x_k^{-1}). \tag{24} \]

by Lemma 3.12, page 134, Goodman- Wallach, the Weyl group \( W \) acts on the coordinate functions in \( \mathfrak{X}(H) \) by \( x_i \rightarrow (x_{\sigma(i)})^{\pm 1} \) \((1 \leq i \leq \ell)\) for every \( \sigma \in S_\ell \) and choice \( \pm 1 \) of exponents. Then
\[ s \cdot \left( x_1 x_2 \cdots x_\ell \right) = (s x_1) (s x_2) \cdots (s x_\ell) \]
\[ = (x_{\sigma(1)})^{\ell} (x_{\sigma(2)})^{\ell-1} \cdots (x_{\sigma(\ell)})^{\pm 1}. \tag{25} \]

From now, the arguments will follow very closely to those in Type B. When viewed as an element in \( GL(\mathfrak{g}^\mathbb{C}) \), \( s \) satisfies \( s(\xi) = \pm \xi_{\sigma(i)} \) (the choice of \( \pm 1 \) is according to that of \( (x_{\sigma(i)})^{\pm 1} \)). By the definition of \( \text{sgn}(s) \) on top of page 331,
\[ \text{sgn}(s) = \det((\xi_1, \ldots, \xi_\ell) \rightarrow (\pm \xi_{\sigma(1)}), \ldots, \pm \xi_{\sigma(\ell)}) = \text{sgn}(s) (-1)^{\# \text{ of minus signs}}. \tag{26} \]

by (25) and (26), we have
\[ \text{RHS}(4) = \sum_{\sigma \in S_\ell} \text{sgn}(s)(-1)^{\# \text{ of minus signs}} (x_{\sigma(1)})^{\ell} (x_{\sigma(2)})^{\ell-1} \cdots (x_{\sigma(\ell)})^{\pm 1}. \tag{27} \]

When (24) is expanded, it will be a sum of \( 4^{\ell} \frac{\ell!}{2!} = 2^\ell \) terms, each of which is of the form \( \pm x_1^{r_1} x_2^{r_2} \cdots x_\ell^{r_\ell} \), where each \( r_i \) is an integer.
Show that after the sum of these $2^k$ terms is reduced, only the terms with $|x_1, x_2, \ldots, x_k|$ pairwise distinct remain. The proof is exactly the same as in type $B_k$, so we will not rewrite it here. Among these remaining terms, we'll show that $|x_1| \geq 1$ for all $1 \leq k \leq k$. Suppose by contradiction that there exists some $1 \leq k \leq k$ such that $x_k = 0$. We can assume $k = 1$. Write
\[
\prod_{1 \leq i < j \leq k} (x_i + x_j - y_i - y_j) \prod_{k=1}^{k} (x_k - x_k) = A + B, \tag{28}
\]
where $A$ is independent of $x_1$, and $B$ is a sum of terms $\pm x_i^2 \ldots x_k^2$ with $i \neq 0$.

Switching $x_1$ and $x_1^\prime$, we get
\[
-\prod_{1 \leq i < j \leq k} (x_i + x_j - y_i - y_j) \prod_{k=1}^{k} (x_k - x_k^\prime) = A + B', \tag{29}
\]
where $B'$ is a sum of terms $\pm x_i^2 \ldots x_k^2$ with $i \neq 0$ (but the choices of $\pm 1$ may be different from those in $B$). Adding (28) and (29), we get
\[
A = -\frac{B + B'}{2}. \tag{30}
\]

LHS (30) is independent of $x_1$ while RHS (30) is not unless $i$ is identically zero. Thus $A \geq 0$. This is a contradiction.

We have showed that after (24) is expanded and reduced, only the terms $\pm x_1^2 \ldots x_k^2$ with $|x_1, x_2, \ldots, x_k| \geq 1$ and pairwise distinct remain. By the virtue of (20) and (21) we have:
\[ \text{LHS}(l) = \sum_{c \in C_l} \alpha \text{sign}(c) (x_{cu})^{c_1} (x_{cuu})^{c_2} \cdots (x_{cu(l-1)})^{c_{l-1}} \prod_{k=1}^{l} (y_k - y_k^{-1}) \] (31)

\[ = \sum_{c \in C_l} \alpha \text{sign}(c) (x_{cu})^{c_1 \pm 1} (x_{cuu})^{c_2 \pm 1} \cdots (x_{cu(l-1)})^{c_{l-1} \pm 1} \cdot x_{cu(l)} \cdot x_{cu}^{-1} \] (32)

Note that \(|k| \leq l-k\). Suppose by contradiction that there is \(l-k \leq k \leq l-1\), such that \(|k| \leq l-k-1\). Then \(|k_1|, |k_{k+1}|, \ldots, |k_{l-1}| \leq l-k-1\). Then

\[ |k_1 \pm 1|, |k_2 \pm 1|, \ldots, |k_{l-1} \pm 1|, l \leq l-k-1. \]

\[ l-k \text{ integers in } \{1, 2, \ldots, l\} \]

This is a contradiction. Thus, \(|k| = l-k\) for all \(1 \leq k \leq l-1\). Thus, \(c_k = \pm (l-k)\).

Consequently, the factors \(\alpha\) in (31) and (32) are \(\pm 1\). We have

\[ |k_1 \pm 1| \leq |k_1| + 1 = l-k+1 \]

Thus, \(|k_1 \pm 1|, |k_2 \pm 1|, \ldots, |k_{l-1} \pm 1|, l \leq l-k+1\). Hence, \(|k_1 \pm 1| = l-k+1\).

Hence, if we choose \(c_k = l-k\) in (31) for some \(k\) then we must have

\[ c_k \pm 1 = l-k+1, \]

i.e. the factor \(\alpha\) in (31) is chosen to multiply with \((x_{cu})^{c_1} (x_{cuu})^{c_2} \cdots (x_{cu(l-1)})^{c_{l-1}}\).

If \(c_k = -(l-k)\) for some \(k\) then we must have

\[ c_k \pm 1 = -(l-k)-1, \]

i.e. the factor \((-z_k^{-1})\) in (31) is chosen to multiply with \((x_{cu})^{c_1} (x_{cuu})^{c_2} \cdots (x_{cu(l-1)})^{c_{l-1}}\).

Therefore,

\[ \alpha = \begin{cases} -1 & \text{if the number of \text{"-" chosen for } \pm (l-k+1), 1 \leq k \leq l, \text{ is odd,} } \\ +1 & \text{otherwise.} \end{cases} \]

Then (32) can be written as
LHS(1) = \sum_{c \in \mathfrak{g}} \text{sgn}(c) \cdot (1)^{\# \mathfrak{g}^+} \cdot (x(c))^{\pm 1} \cdot (x(c))^{\pm 1}.

Comparing with (27), we get the identity (A).

Type D_6: G = SO(2l, C), l > 3

\( g = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha = \frac{1}{2} \sum_{1 \leq i < j \leq l} [(\xi_i - \xi_j) + (\xi_i + \xi_j)] = \sum_{1 \leq i < j \leq l} \xi_i = (\xi_1 \xi_2 + (\xi_2 \xi_3 + \ldots + \xi_{l-1} \xi_l).

Thus, \( e^g = \chi_1 \chi_2 \ldots \chi_{l-1}. \) (33)

For \( \lambda = \xi_i - \xi_j \), \( e^{-\lambda} = e^{\xi_j - \xi_i} = \chi_i \chi_j. \)

For \( \lambda = \xi_i + \xi_j \), \( e^{-\lambda} = e^{-\xi_i - \xi_j} = \chi_i^{-1} \chi_j^{-1}. \)

Thus, LHS(1) = \( \chi_1 \chi_2 \ldots \chi_{l-1} \prod_{1 \leq i < j \leq l} (1 - \chi_i^{-1} \chi_j)(1 - \chi_i \chi_j^{-1}) \)

= \( \prod_{1 \leq i < j \leq l} \chi_i(1 - \chi_i^{-1} \chi_j)(1 - \chi_i \chi_j^{-1}) \)

= \( \prod_{1 \leq i < j \leq l} (\chi_i + \chi_i^{-1} - \chi_j - \chi_j^{-1}) \). \) (34)

by Lemma 3.14, page 132, Gozmann-Wallach, the Weyl group W acts on the coordinate functions in \( \mathcal{F}(H) \) by \( x_i \mapsto (x_{\alpha(c)})^{\pm 1} \) (1 \leq i \leq l) for every \( c \in \mathfrak{g}_e \) and choice \( \pm 1 \) of exponent with an even number of negative exponents. Then

s \cdot e^g (33) s \cdot (x_1^{l-1} x_2^{l-2} \ldots x_{l-1}) = (s \cdot x_1)^{l-1} (s \cdot x_2)^{l-2} \ldots (s \cdot x_{l-1})

= (x_{\alpha(c)})^{(l-1)} (x_{\alpha(c)})^{(l-2)} \ldots (x_{\alpha(c)})^{(l-1)}. \) (35)

When viewed as an element in \( \mathfrak{h}^* \), s satisfies \( s(\xi_i) = \pm \xi_i \) (the choice of sign is according to that of \( (x_{\alpha(c)})^{\pm 1} \)). By the definition of \( \text{sgn}(s) \) on the
top of page 331,
\[ \text{sgn}(s) = \det \left( \left( \varepsilon_1, \ldots, \varepsilon_k \right) \mapsto (\pm \varepsilon_{2(1)}, \ldots, \pm \varepsilon_{2(k)}) \right) = \text{sgn}(\sigma) \# \text{ of } ^{-} = \text{sgn}(\sigma). \] 

By (35) and (36), we have
\[ \text{RHS}(s) = \sum_{\sigma \in \mathcal{S}_k} \text{sgn}(\sigma) (x_{2(1)})^{\pm(\ell-1)} (x_{2(2)})^{\pm(\ell-2)} \cdots (x_{2(k)})^{\pm 1}. \] 

When the product at (34) is expanded, (34) can be written as
\[ \text{LHS}(s) = \prod_{1 \leq i < j \leq \ell} (x_i + x_i^{-1} - x_j - x_j^{-1}) \]
\[ = \prod_{\tau \in \mathcal{E}_i} (x_{\tau(i)} + x_{\tau(i)}^{-1})^{l-1} (x_{\tau(2)} + x_{\tau(2)}^{-1})^{l-2} \cdots (x_{\tau(\ell-1)} + x_{\tau(\ell-1)}^{-1}) \]
\[ = \sum \alpha (x_{\tau(1)})^{k_1} (x_{\tau(2)})^{k_2} \cdots (x_{\tau(\ell-1)})^{k_{\ell-1}}, \] 

where \( \alpha \) is a number depending on each summand and \( \ell \in \mathbb{Z}, \ 1 \leq \ell \leq k \) for all \( 1 \leq k \leq \ell - 1 \). We'll show that in the reduced form, the sum at (39) contains only terms \( (x_{\tau(1)})^{k_1} (x_{\tau(2)})^{k_2} \cdots (x_{\tau(\ell-1)})^{k_{\ell-1}} \) with \( k_1, \ldots, k_{\ell-1} \) pairwise distinct. The proof is exactly the same as in Type \( B_\ell \), so we will not rewrite it here. Suppose by contradiction that among those remaining terms, there are terms with \( x_k = 0 \) for some \( 1 \leq k \leq \ell - 1 \). Such a term is independent of at least two of the factors \( x_1, x_2, \ldots, x_\ell \). Write
\[ \prod_{1 \leq i < j \leq \ell} (x_i + x_i^{-1} - x_j - x_j^{-1}) = A + B, \] 

where \( A \) is independent of \( x_1 \) and \( x_\ell \), and \( B \) is a sum of terms.
\[ \pm x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n} \text{ with } (a_1, a_2) \neq (0,0). \text{ Switching } x_1 \text{ and } x_2, \text{ we get} \]
\[ \prod_{1 \leq i < j \leq n} (x_i + x_j - y_j) = A + B', \quad (41) \]
where \( B' \) is another sum of terms \( \pm x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n} \) with \( (a_1, a_2) \neq (0,0) \). Adding (40) and (41) together, we get
\[ A = - \frac{B + B'}{2}. \quad (42) \]

LHS (42) is independent of \( x_1 \) and \( x_2 \), while RHS (42) is not unless it is identically zero. Thus, \( A = 0 \). We have showed that after (39) is reduced, only the terms with \( 1 \leq \mid a_1 \mid, 1 \leq \mid a_2 \mid, \ldots, 1 \leq \mid a_{l-1} \mid \) pairwise distinct and \( 1 \leq k \leq l \) for all \( 1 \leq k \leq l - 1 \) remain. By the same argument as in Type C_e, we get \( 1 \leq k = l - k \) for all \( 1 \leq k \leq l - 1 \). Thus, the coefficients \( x_i \)'s in (39) are equal to 1. Thus, (39) can be written as
\[ \text{LHS(1)} = \sum_{\tau \in S_l} (x_{c_1})^{\pm 1} (x_{c_2})^{\pm 1} \ldots (x_{c_{l-1}})^{\pm 1} \]

Comparing with (37), we get the identity (1).

6) Problem 1, Section 9.1.4, Goodman-Wallach, p. 396.

We'll show that \( T^{[k-2,2]} \cong G^{[k-2,2]} \oplus G^{[k-1,0]} \oplus G^{[k]}. \) By Proposition 5.1.3, so called the reciprocity law for multiplicities of the irreducible representations of \( G_n \) in \( T^m \), we have
\[ T^{[k-2,2]} \cong \bigoplus_{\lambda \in \text{Par}(k,n)} \dim F_{\lambda n}(C^{k-2,2}) G^\lambda. \]
We need to show that
\[ \dim F_n^\lambda (\lambda \cdot 2, 2) = \begin{cases} 
1 & \text{if } \lambda \in \{[k], [k-1, 1], [k-2, 2]\} \\
0 & \text{otherwise}
\end{cases} \]

By Corollary 3.7.1.1, so-called the Gelfand-Tsetlin Basis, \( \dim F_n^\lambda (\lambda \cdot 2, 2) \) is equal to the number of \( n \)-fold branching patterns of shape \( \lambda \) and weight \([k-2, 2]\). Write such an \( n \)-fold branching pattern as \( \gamma = \{\mu^{(n)}, \mu^{(n-1)}, \ldots, \mu^{(0)}\} \).

Because \( \gamma \) has weight \([k-2, 2]\), its Young diagram consists of \( k \) boxes, \( k-2 \) of which are filled with \( 1 \) and two are filled with \( 2 \). By the rule of writing a Young diagram for an \( n \)-fold branching pattern in page 366, Goodman-Wallach, there are only 3 possibilities for \( \gamma \):

\[ \gamma = \{[k-2, 2], [k-2]\} \quad \gamma = \{[k-1, 1], [k-2]\} \quad \gamma = \{[k], [k-2]\} \]

Recall that \( \lambda \) is the shape of \( \gamma \). For case (a), \( \lambda = [k-2, 2] \). For case (b), \( \lambda = [k-1, 1] \). For case (c), \( \lambda = [k] \). Therefore,

\[ \dim F_n^\lambda (\lambda \cdot 2, 2) = \begin{cases} 
1 & \text{if } \lambda \in \{[k-2, 2], [k-1, 1], [k]\} \\
0 & \text{otherwise}
\end{cases} \]

Problem 4, Section 4.1.5, p.490, Goodman-Wallach.

\[ \nu' = \sum \nu(x) = \binom{n}{1} \cdot \nu(\lambda) : \lambda \in \mathbb{C} \quad \text{and} \quad \gamma = \sum \binom{1}{0} \cdot \gamma(\lambda) : \lambda \in \mathbb{Z}^2 \]
We will show that $P$ is Zariski dense in $N$.

By Lemma A.1.7, p. 615, Goodman-Wallach, the Zariski topology on $N$ consists of finitely many finite unions of principal open sets

$$N^f = \{ u(\xi) \in N : f(u(\xi)) \neq 0 \} \quad \forall f \in \mathbb{F}[N] \setminus \{0\}.$$ 

For each $f \in \mathbb{F}[GL(2, \mathbb{C})]$, $f((x_{11}, x_{12}, x_{21}, x_{22}) \cdot (y_{12} - x_{12} x_{21})^{-1})$ is a polynomial of variables $x_{11}, x_{12}, x_{21}, x_{22}, y_{12}$. Thus, $f(u(\xi))$ is simply a polynomial of $\xi$.

We can, therefore, identify $\mathbb{F}[N]$ with $\mathbb{C}[\xi]$. Then

$$N^f = \{ u(\xi) \in N : f(\xi) \neq 0 \} \quad \forall f \in \mathbb{C}[\xi].$$

Take $u(a) \in N$ and let $V$ be a Zariski open nbhd of $u(a)$ in $N$. We'll show that $V \cap P \neq \emptyset$. Because $V$ is a finite union of principal open sets, we can assume that $V$ is a principal open set, i.e. $V = N^f$ for some $f \in \mathbb{C}[\xi]$. Because $f(a) \neq 0$, $f$ is not the zero polynomial. Thus, $f$ has finitely many roots in $\mathbb{C}$. Thus, there exists $b \in \mathbb{C}$ such that $f(b) \neq 0$.

Then $u(b) \in N^f = V$. Hence, $u(b) \in V \cap P$.

2 Problem 1, Section 12.1.4, Goodman-Wallach, page 550.

Let $G$ be a reductive group acting linearly on a vector space $V$. In other words, $V$ is a representation of the group $G$. This representation induces a representation of $G$ on $\mathbb{C}(V)$, the space of polynomial functions on $V$, via
\((g, f)(v) := f(g^{-1}v) \quad \forall g \in G, f \in \mathcal{P}(V), v \in V.\)

Let \(G^x\) act linearly on \(V\) by scalar multiplication. This action induces a representation of \(G^x\) on \(\mathcal{P}(V)\) via

\[
(g, f)(v) := f(g^{-1}v) \quad \forall g \in G^x, f \in \mathcal{P}(V), v \in V.
\]

The actions of \(G\) and of \(G^x\) on \(\mathcal{P}(V)\) commute because

\[
(g, (g, f))(v) = (g, f)(g^{-1}v) = f(g^{-1}g^{-1}v) = (g, f)(g^{-1}v) = (g, (g, f))(v)
\]

\(\forall g \in G^x, g \in G, f \in \mathcal{P}(V).\)

Then we can define an action of \(G \times G^x\) on \(\mathcal{P}(V)\) via

\[(g, \lambda, f)(v) := (g, (\lambda, f))(v) = f(g^{-1}g^{-1}v) \quad \forall g \in G^x, \lambda \in G, f \in \mathcal{P}(V).\]

For each integer \(k \geq 0\), denote by \(\mathcal{P}^k(V)\) the space of homogeneous polynomials of degree \(k\). We have a decomposition of vector spaces

\[
\mathcal{P}(V) = \bigoplus_{k \geq 0} \mathcal{P}^k(V).
\]

Denote by \(\bar{G}\) the set of all equivalent classes of finite-dimensional irreducible \(G\)-modules. The primary decomposition of \(\mathcal{P}^k(V)\) as a \(G\)-module is

\[
\mathcal{P}^k(V) = \bigoplus_{\omega \in \bar{G}} \mathcal{P}^k(V)(\omega),
\]

where

\[
\mathcal{P}^k(V)(\omega) = \bigoplus_{U \subseteq \mathcal{P}^k(V)} U \quad \text{ s.t. } [U] = \omega.
\]

Hence, we get a decomposition of vector spaces

\[
\mathcal{P}(V) = \bigoplus_{k \geq 0} \bigoplus_{\omega \in \bar{G}} \mathcal{P}^k(V)(\omega).
\]
We want to show that (2), after eliminating trivial summands on the right hand side, is the primary decomposition of $\mathcal{P}(V)$ as a $G \times C^x$-module. First, we show that each $\mathcal{P}^k(V)_{(w)}$ is a $G \times C^x$-module. For each $(g, \lambda) \in G \times C^x$ and $f \in \mathcal{P}^k(V)_{(w)}$ we have

$$(g, \lambda) \cdot f = (v \mapsto f(g^{-1} \cdot v)) = (v \mapsto f(g^{-1} \cdot \lambda v)) = \lambda^k (v \mapsto f(g^{-1} \cdot v)) = \lambda^k (g, \lambda) \cdot f.$$ Since $\mathcal{P}^k(V)_{(w)}$ is a $G$-module, $g \cdot f \in \mathcal{P}^k(V)_{(w)}$ and thus $\lambda^k (g, \lambda) \cdot f \in \mathcal{P}^k(V)_{(w)}$. Hence, $(g, \lambda) \cdot f \in \mathcal{P}^k(V)_{(w)}$. We have showed that $\mathcal{P}^k(V)_{(w)}$ is a $G \times C^x$-module. Thus, (2) is a decomposition of $G \times C^x$-modules.

Denote by $\widehat{G \times C^x}$ the set of all equivalent classes of finite dimensional irreducible $G \times C^x$-modules. Then we have the primary decomposition

$$\mathcal{P}(V) = \bigoplus_{(g, \lambda) \in \widehat{G \times C^x}} \mathcal{P}(V)_{(g, \lambda)} \quad (3)$$

For each $g \in G \times C^x$, we show that there exist $k_0 > 0$ and $w_0 \in G$ such that

$$\mathcal{P}(V)_{(g)} \subseteq \mathcal{P}^{k_0}(V)_{(w_0)} \quad (4)$$

If this can be done then

$$\bigoplus_{k \geq 0} \mathcal{P}(V)_{(w)} \overset{(4)}{=} \mathcal{P}(V) \overset{(3)}{=} \bigoplus_{(g, \lambda) \in \widehat{G \times C^x}} \mathcal{P}(V)_{(g, \lambda)} \subseteq \bigoplus_{(g, \lambda) \in \widehat{G \times C^x}} \mathcal{P}^{k_0}(V)_{(w_0)},$$

then the equality must hold; thus (1) is the primary decomposition of $\mathcal{P}(V)$ as a $G \times C^x$-module.
By the definition of isotypic components, (4) is equivalent to
\[ \sum_{W \in \mathcal{O}(V)} W \subseteq \mathcal{P}^k(V)_{(\omega)} \quad (5) \]

For each $W \in \mathcal{O}(V)$, $W \not\in \mathcal{O}$, we have
\[ W = W \cap \mathcal{P}(V) \stackrel{\mathcal{O}}{\rightarrow} \oplus \mathcal{O} \theta \bigoplus_{W \in \mathcal{O}} \left( W \cap \mathcal{P}^0(V)_{(\omega)} \right) \]

Since $W$ is an irreducible $G \times C^\times$-module, there exist an integer $k_{ij}w > 0$ and $\omega_{ij,w} \in \hat{G}$ such that $W = W \cap \mathcal{P}^{k_{ij}}(V)_{(\omega_{ij,w})}$. Thus, $W \subseteq \mathcal{P}^{k_{ij}w}(V)_{(\omega_{ij,w})}$.

For $W_1, W_2 \in \mathcal{O}$ and $W_1 \subseteq \mathcal{P}^{i_1}(V)_{(\omega_{i_1})}$, $W_2 \subseteq \mathcal{P}^{i_2}(V)_{(\omega_{i_2})}$, we show that $i_1 = i_2$ and $\omega_{i_1} = \omega_{i_2}$.

Because $W_i \simeq W_2$ as $G \times C^\times$-modules, there exists a linear isomorphism $\varphi: W_1 \rightarrow W_2$ which intertwines with the actions of $G \times C^\times$ on $W_1$ and $W_2$.

For each $\lambda \in C^\times$ and $f \in W_1$, we have $\varphi(\lambda f) = \lambda \varphi(f)$. On the other hand,
\[ \varphi(\lambda f) = (v \mapsto \varphi(\lambda f)) = (v \mapsto \lambda^{-i_2} \varphi(f)) = \lambda^{-i_2} \varphi(f) \]
\[ \varphi(f) = (v \mapsto \varphi(f)(\omega_{i_2}v)) = (v \mapsto \lambda^{-i_2} \varphi(f)(v)) = \lambda^{-i_2} \varphi(f) \]

By (6) and (7), $\lambda^{-i_2} \varphi(f) = \lambda^{-i_2} \varphi(f)$ for all $\lambda \in C^\times$ and $f \in W_1$. Thus,
\[ i_1 = i_2 \quad (\neq i). \]

Hence, $W_1, W_2 \subseteq \mathcal{P}^0(V)$.

By identifying $G$ with $G \times \{1\} \subseteq G \times C^\times$, we have $W_1 \simeq W_2 \cong$
$G$-modules via the intertwining map $\phi$. Now suppose by contradiction that $W_i$ is a reducible $G$-module. Write

$$W_i = U_i \oplus U_2,$$

where $U_i, U_2 \neq \{0\}$ are $G$-modules. For $\lambda \in C^x, g \in G, f \in U_2$,

$$(g, \lambda) \cdot f = (v \mapsto f(g^{-1}g^{-1}v)) = (v \mapsto \lambda^i \cdot f(g^{-1}v)) = \lambda^i (v \mapsto f(g^{-1}v)) = \lambda^i (g \cdot f).$$

Since $U_i$ is a $G$-module, $g \cdot f \in U_i$ and thus $\lambda^i \cdot f \in U_i$. Hence, $(g, \lambda) \cdot f \in U_i$. Thus, $U_i$ is a $G \times C^x$-module. Similarly, $U_2$ is also a $G \times C^x$-module. Then (8) contradicts the fact that $W_i$ is an irreducible $G \times C^x$-module.

Therefore, $W_i$ is an irreducible $G$-module. Similarly, $W_2$ is also an irreducible $G$-module. Moreover, $W_1$ and $W_2$ are isomorphic irreducible $G$-submodules of $\Phi(V)$. Moreover, $W_i$ and $W_2$ are finite-dimensional. Thus, they are contained in the same isotypic component of $\Phi(V)$ as a $G$-module. This means $\omega_i = \omega_2$. Therefore,

$$\begin{cases}
\omega_{i, W} = \omega_0 \\
\omega_{0, W} = \omega_0
\end{cases} \quad \forall W \in \Phi(V), W \in \Sigma.$$

We obtain (5).