
Let $X$ and $Y$ be two random variables from a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to measurable space $(\mathbb{X}, \mathcal{G})$. Assume that the set $A = \{w: X(w) \neq Y(w)\}$ is an event having probability 0. We show that $X$ and $Y$ have the same distribution.

The distribution of $X$ is defined as a measure $Q_X$ on $(\mathbb{X}, \mathcal{G})$,

$$Q_X(B) = \mathbb{P}(X^{-1}(B)) \quad \forall B \in \mathcal{G}.$$

The distribution of $Y$ is defined as a measure $Q_Y$ on $(\mathbb{X}, \mathcal{G})$,

$$Q_Y(B) = \mathbb{P}(Y^{-1}(B)) \quad \forall B \in \mathcal{G}.$$

Take $B \in \mathcal{G}$. We want to show that $Q_X(B) = Q_Y(B)$.

$$Q_X(B) - Q_Y(B) = \mathbb{P}\left(\frac{X^{-1}(B_1)}{B_1}\right) - \mathbb{P}\left(\frac{Y^{-1}(B_2)}{B_2}\right)$$

$$= \mathbb{P}(B_1 \setminus B_2) - \mathbb{P}(B_2 \setminus B_1).$$

For each $w \in B_1 \setminus B_2$, $Y(w) \notin B$ and $X(w) \in B$ and hence $w \in A$. This implies $B_1 \setminus B_2 \subseteq A$. Similarly, $B_2 \setminus B_1 \subseteq A$.

Because $\mathbb{P}(A) = 0$, $\mathbb{P}(B_1 \setminus B_2) = \mathbb{P}(B_2 \setminus B_1) = 0$. Therefore,

$$Q_X(B) = Q_Y(B).$$

② Problem 6, Fristedt-Gray, page 13.
Let $(\Omega, F), (\Psi, G), (\Theta, H)$ be measurable spaces and $X: \Omega \rightarrow \Psi$, $Y: \Psi \rightarrow \Theta$ be measurable functions. We show that $Z = Y \circ X: \Omega \rightarrow \Theta$ is also a measurable function.

Take $B \in H$. We need to show $Z^{-1}(B) \in F$.

$$Z^{-1}(B) = \{\omega \in \Omega : Z(\omega) = Y(X(\omega)) \in B\}$$

$$= \{\omega \in \Omega : X(\omega) \in Y^{-1}(B)\}$$

$$= X^{-1}(Y^{-1}(B)).$$

Because $Y$ is measurable, $Y^{-1}(B) \in G$. Because $X$ is measurable, $X^{-1}(Y^{-1}(B)) \in F$.

Thus, $Z^{-1}(B) \in F$.


Let $\mu$ be a translation-invariant measure on $(\mathbb{R}, B)$, where $B$ is the Borel $\sigma$-field, satisfying $\mu([0,1)) = 1$. We show that $\mu$ is the Lebesgue measure on $B$.

That is to show $\mu((a,b)) = b-a$ for all $a,b \in \mathbb{R}$, $a < b$.

Define a function $f: [0,\infty) \rightarrow [0,\infty]$, $f(r) = \mu([0,r))$. We want to show that $f(r) = r$ for all $r > 0$. Because $\mu([0,1)) = 1$, $f(0) = 0$ and $f(1) = 1$. For $r, s > 0$,

$$f(r+s) = \mu([0, r+s)) = \mu([0, r)) \cup [r, r+s)$$

$$= \mu([0, r)) + \mu([r, r+s))$$

$$= \mu([0, s))$$ since $\mu$ is translation-invariant

$$= f(r) + f(s).$$

Thus, $f$ is finitely additive. For each $n \in \mathbb{N}$,
\[ f(\frac{1}{n}) = f\left(\frac{1}{n} + \frac{1}{n} + \ldots + \frac{1}{n}\right) = f\left(\frac{1}{n}\right) + f\left(\frac{1}{n}\right) + \ldots + f\left(\frac{1}{n}\right) = n f\left(\frac{1}{n}\right). \]

Thus, \( f\left(\frac{1}{n}\right) = \frac{f(1)}{n} = \frac{1}{n} \).

For \( m,n \in \mathbb{N} \),

\[ f\left(\frac{m}{n}\right) = f\left(\frac{1}{n} + \frac{1}{n} + \ldots + \frac{1}{n}\right) = f\left(\frac{1}{n}\right) + f\left(\frac{1}{n}\right) + \ldots + f\left(\frac{1}{n}\right) = m f\left(\frac{1}{n}\right) = \frac{m}{n}. \]

Thus, \( f(r) = r \) for all \( r \in \mathbb{Q}, r > 0 \). Because \( \mathbb{Q} \cap (0,0) \) is dense in \( (0,\infty) \), each \( r \in (0,\infty) \) is the limit of a decreasing sequence of rational numbers \( \{r_n\} \).

Then \( (0,r) = \bigcap_{n=1}^{\infty} [0,r_n) \). Because of the continuity of measure \( \mu \), \( \mu((0,r)) = \lim_{n \to \infty} \mu((0,r_n)) \). In other words,

\[ f(r) = \lim_{n \to \infty} f(r_n) = \lim_{n \to \infty} r_n = r. \]

Therefore, \( f(r) = r \) for all \( r \in (0,\infty) \), which means

\[ \mu([0,r)) = r \quad \forall r > 0 \quad (1) \]

Take \( a,b \in \mathbb{R}, a < b \). We want to show \( \mu((a,b)) = b-a \). Because \( [a,b) = [a+b-a+\ldots+0, b-a) \) and that \( \mu \) is translation-invariant, \( \mu([a,b)) = \mu([0, b-a)) \Rightarrow b-a \).

4) Additional problem.

Let \( \mu \) be a measure on a measurable space \((\Omega, F)\). Define

\[ \overline{F} = \{ B \in \mathcal{B} \cup \mathcal{N} : \exists E \in F : \mu(E) = 0 \text{ and } N \subset C \}. \]

We show the following statements.

(i) \( \overline{F} \) is a \sigma-field.
(ii) Define \( \bar{\mu} : \mathbb{F} \to [0,1] \), \( \bar{\mu} (\text{BUN}) = \mu (B) \). Then \( \bar{\mu} \) is a measure on \( (\mathbb{R}, \mathcal{B}) \) that coincides with \( \mu \) on \( \mathbb{F} \).

**Proof of (ii)**

Because \( \mathbb{F} = \mathbb{F} \cup \phi \), \( \mathbb{F} \in \mathbb{F} \), \( \mu (\phi) = 0 \), we get \( \mathbb{F} \in \mathbb{F} \). Take \( A \in \mathbb{F} \). Write \( A = \text{BUN} \) where \( \text{B} \in \mathbb{F} \) and \( N \subseteq C \) for some \( C \in \mathbb{F} \), \( \mu (C) = 0 \). Then

\[
\mathbb{F} \setminus A = \mathbb{F} \setminus \text{BUN} = (\mathbb{F} \setminus \text{B}) \cap (\mathbb{F} \setminus N)
\]

\[
= (\mathbb{F} \setminus \text{B}) \cap \left[ (\mathbb{F} \setminus C) \cup (C \setminus N) \right]\]

\[
= \left[ (\mathbb{F} \setminus \text{B}) \cap (\mathbb{F} \setminus C) \right] \cup \left[ (\mathbb{F} \setminus \text{B}) \cap (C \setminus N) \right]
\]

\[
= \text{B} \setminus N \cup N
\]

Because \( \text{B}, C \in \mathbb{F} \), \( \text{B} \setminus N \in \mathbb{F} \). Because \( N \subseteq C \) and \( \mathbb{F} \setminus A = \text{B} \setminus N \), we conclude that \( \mathbb{F} \setminus A \in \mathbb{F} \).

Let \((A_n)\) be a sequence in \( \mathbb{F} \). We want to show that \( A = \bigcup_{n=1}^{\infty} A_n \in \mathbb{F} \).

Write \( A_n = \text{B}_n \cup N_n \), where \( \text{B}_n \in \mathbb{F} \) and \( N_n \subseteq C_n \) for some \( C_n \in \mathbb{F} \), \( \mu (C_n) = 0 \).

Then

\[
A = \bigcup_{n=1}^{\infty} (\text{B}_n \cup N_n) = \left( \bigcup_{n=1}^{\infty} \text{B}_n \right) \cup \left( \bigcup_{n=1}^{\infty} N_n \right)
\]

Since \( \mathbb{F} \) is a field, \( \text{B} \cup \text{N} \subseteq \mathbb{F} \). Then \( N \subseteq C \) because \( N_n \subseteq C_n \subseteq C \) for each \( n \in \mathbb{N} \). By Problem 2 of Homework #2,

\[
\mu (C) = \sum_{n=1}^{\infty} \mu (C_n) = 0.
\]

Thus \( \mu (C) = 0 \). Because \( A = \text{BUN} \), \( \text{B} \in \mathbb{F} \), \( N \subseteq C \) and \( \mu (C) = 0 \), we conclude that \( A \in \mathbb{F} \).
Proof of (ii)

First, we show that $\overline{\mu}$ is well-defined. Suppose that a set $A \subset \mathcal{C}$ is written as $A = B \cup N = B' \cup N'$ where $B, B' \in \mathcal{F}$, $N \in \mathcal{C}$, $N \in \mathcal{C}'$ for some $C, C' \in \mathcal{F}$, $\mu(C) = \mu(C') = 0$. We want to show that $\mu(B) = \mu(B')$.

Because $B$ and $B'$ play the same role, it suffices to show that $\mu(B) \leq \mu(B')$. This is true if $\mu(B') = \infty$. Consider the case $\mu(B') < \infty$.

We have $B \setminus B' \subset (B \cup N) \setminus B' = (B' \cup N') \setminus B' \subset N' \subset C'$.

Thus, $\mu(B \setminus B') \leq \mu(C') = 0$. Hence, $\mu(B) - \mu(B') \leq 0$.

Next, we show that $\overline{\mu}$ coincides with $\mu$ on $\mathcal{F}$. For each $B \in \mathcal{F}$, $B = B \cup \emptyset$.

Thus, $B \in \mathcal{F}$ and $\overline{\mu}(B) = \overline{\mu}(B \cup \emptyset) = \mu(B)$. Thus, $F \subseteq \mathcal{F}$ and $\overline{\mu}|_F = \mu$.

Now we show that $\overline{\mu}$ is a measure on $\mathcal{F}$. Since $\emptyset \in \mathcal{F}$, $\overline{\mu}(\emptyset) = \mu(\emptyset) = 0$.

Let $(A_n)$ be a sequence of disjoint sets in $\mathcal{F}$. Write $A_n = B_n \cup N_n$ where $B_n \in \mathcal{F}$ and $N_n \in \mathcal{C}$ for some $C_n \in \mathcal{F}$, $\mu(C_n) = 0$. We pointed out in Part (i) that the set $A = \bigcup_{n=1}^{\infty} A_n$ is of the form $A = B \cup N$ where $B = \bigcup_{n=1}^{\infty} B_n$ and $N$ is contained in a set of measure 0 in $F$. Thus, $\overline{\mu}(A) = \mu(B) = \mu\left(\bigcup_{n=1}^{\infty} B_n\right)$.

Because $B_i \cap B_j \subset A_i \cap A_j = \emptyset$ for $i \neq j$, $(B_n)$ is a sequence of disjoint sets in $\mathcal{F}$. Thus, $\mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \mu(B_n) = \sum_{n=1}^{\infty} \overline{\mu}(A_n)$. 

Therefore, \( \overline{\mu}(A) = \sum_{n=1}^{\infty} \overline{\mu}(A_n) \).

5. Additional problem \( B \).

For \( a, b \in \mathbb{R} \) and \( B \in \mathcal{B} \), we denote \( aB + b = \{ ax + b : x \in B \} \). Let \( B \) be the Borel \( \sigma \)-field on \( \mathbb{R} \) and \( \lambda \) be the Lebesgue measure on \( B \). Take \( a, b \in \mathbb{R} \). We show that

(i) \( aB + b \in B \quad \forall B \in B \),

(ii) \( \lambda(aB + b) = |a| \lambda(B) \quad \forall B \in B \).

Proof of (i)

If \( a = 0 \), \( aB + b = \{ \emptyset \) if \( B = \emptyset \)

\[ \{ b \} \) if \( B \neq \emptyset \). \]

In either case, \( aB + b \) is a closed subset of \( \mathbb{R} \) and hence is in \( B \).

Consider the case \( a \neq 0 \). Define a function \( \varphi : (\mathbb{R}, B) \to (\mathbb{R}, B) \),

\[ \varphi(x) = \frac{x-b}{a} \]. \ Since \( \varphi \) is continuous, it is measurable. Thus, \( \varphi^{-1}(B) \in B \)

for all \( B \in B \).

\[ \varphi^{-1}(B) = \{ x \in \mathbb{R} : \varphi(x) \in B \} = \{ x \in \mathbb{R} : \frac{x-b}{a} = y \in B \} \]

\[ = \{ ay + b : y \in B \} \]

\[ = aB + b. \]

Therefore, \( aB + b \in B \).

Proof of (ii)

If \( a = 0 \) then \( \lambda(aB + b) = \lambda\{b\} = \lim_{n \to 0} \lambda(b, b + \frac{1}{n}) = \lim_{n \to 0} \frac{1}{n} = 0 = |a| \lambda(B). \)
Consider the case $a \neq 0$. Define $\mu(B) = \frac{\lambda(aB+b)}{|a|}$ for each $B \in \mathcal{B}$.

We first show that $\mu$ is a measure on $(\mathbb{R}, \mathcal{B})$. It is clear that $\mu(B) \geq 0$ for all $B \in \mathcal{B}$.

$$\mu(B) = \frac{\lambda(aB+b)}{|a|} = \frac{\lambda(B)}{|a|} = 0.$$ 

Let $(A_n)$ be a sequence of disjoint sets in $\mathcal{B}$. Let $B_n = aA_n + b$. Then $B_n \cap B_m = \emptyset$ for $m \neq n$. Indeed, suppose there exists $x \in B_n \cap B_m$. Then $\frac{x-b}{a} \in A_n \cap A_m$, which is a contradiction. We have

$$a \left( \bigcup_{n=1}^{\infty} A_n \right) + b = \{ax+b : x \in \bigcup_{n=1}^{\infty} A_n \} = \bigcup_{n=1}^{\infty} \{ax+b : x \in A_n \} = \bigcup_{n=1}^{\infty} (aA_n+b) = \bigcup_{n=1}^{\infty} B_n.$$ 

Thus, according to Part (c), $B_n \in \mathcal{B}$. Thus,

$$\mu \left( \bigcup_{n=1}^{\infty} A_n \right) = \frac{\lambda \left( \bigcup_{n=1}^{\infty} aA_n + b \right)}{|a|} = \frac{\lambda \left( \bigcup_{n=1}^{\infty} B_n \right)}{|a|} = \sum_{n=1}^{\infty} \frac{\lambda(B_n)}{|a|} = \sum_{n=1}^{\infty} \frac{\lambda(aA_n+b)}{|a|} = \sum_{n=1}^{\infty} \mu(A_n).$$

Therefore, $\mu$ is a measure on $(\mathbb{R}, \mathcal{B})$.

Because of the uniqueness of Lebesgue measure on $(\mathbb{R}, \mathcal{B})$, all we need to show next is that $\mu$ is a Lebesgue measure. Problem (3) gives us a method to do so. Accordingly, we have to show that $\mu$ is translation-invariant and $\mu([0,1]) = 1$. For $B \in \mathcal{B}$ and $c \in \mathbb{R},$

$$a(B+c)+b = \{ay+b : y \in B+c \} = \{a(x+c)+b : x \in B \}$$

$$= \{ax + ac + b : x \in B \} = aB + ac + b.$$  \hspace{1cm} (1)

Because $\lambda$ is translation-invariant, $\lambda(aB+ac+b) = \lambda(aB+b).$ \hspace{1cm} (2)
Then
\[ \mu(B+c) = \frac{\mu(a(Btc)+b)}{|a|} \] (1)
\[ \frac{\mu(ab+ac+b)}{|a|} \] (2)
\[ \frac{\mu(ab+b)}{|a|} \]
Thus, \( \mu(a[0,1]+b) \) is translation-invariant. Moreover,
\[ a[0,1]+b = \begin{cases} \{b, a+b\} & \text{if } a > 0, \\ (a+b, b) & \text{if } a < 0. \end{cases} \]
Thus,
\[ \mu(a[0,1]+b) = \begin{cases} \sum (a+b) - b & \text{if } a > 0, \\ b - (a+b) & \text{if } a < 0 . \end{cases} \]
\[ = |a|. \]
Therefore, \( \mu([0,1]) = \frac{\mu(a[0,1]+b)}{|a|} = 1. \)

(6) Additional problem C.

We first repeat what is taught in class about the setting of this problem. We started with the observation that each \( x \in [0,1) \) is written uniquely as
\[ x = 0.a_1 a_2 a_3 \ldots \quad \text{(in base 2)} \]
where each \( a_i \in \{0,1\} \) infinitely many of which are 0. The expression simply means
\[ x = a_1 \frac{1}{2^1} + a_2 \frac{1}{2^2} + a_3 \frac{1}{2^3} + \ldots = \sum_{i=1}^{\infty} a_i \frac{1}{2^i}. \]
Because of the uniqueness of such an expression, we can denote the dependance of each \( a_i \) on \( x \) by writing \( a_i = q_i(x) \). For \( n \in \mathbb{N} \) and \( x \in [0,1] \),
\[ \{x \in [0,1) : a_n(x) = x\} = \bigcup_{a_1, \ldots, a_{n-1} \in \{0,1\}} \{x \in [0,1) : a_1(x) = a_1, \ldots, a_{n-1}(x) = a_{n-1}, a_n(x) = x\} \]
\[ = \bigcup_{a_1, \ldots, a_{n-1} \in \{0,1\}} \left( \sum_{i=1}^{n} \frac{a_i}{2^i} + \frac{x}{2^n}, \sum_{i=1}^{n-1} \frac{a_i}{2^i} + \frac{x+1}{2^n} \right) \quad (1) \]
The observation can be described in probabilistic language as follows. Let
be the probability measure space with Lebesgue measure. For each \( n \in \mathbb{N} \), define a function \( w_n : \Omega \to \{0, 1\}^n \), \( w_n(w) = a_n(w) \). Because of (1), \( w_n \) is a random variable.

For \( a_1, \ldots, a_n \in \{0, 1\} \), \( \{ \omega : w_1(\omega) = a_1, \ldots, w_n(\omega) = a_n \} = \bigcap_{i=1}^{n} \{ \omega : w_i(\omega) = a_i \} = \bigcap_{i=1}^{n} \omega_{a_i}^{-1}(\{ a_i \}) \).

Thus,
\[
P(w_1 = a_1, \ldots, w_n = a_n) = \frac{1}{2^n}. \quad (2)
\]

Now we return to the problem. Let \( k_1 < k_2 < \ldots < k_n \) be positive integers and \( a_1, \ldots, a_n \in \{0, 1\} \). The set
\[
\{ \omega : w_{k_j}(\omega) = a_j \ \forall 1 \leq j \leq n \} = \bigcap_{j=1}^{n} \{ \omega : w_{k_j}(\omega) = a_j \} = \bigcap_{j=1}^{n} \omega_{a_j}^{-1}(\{ a_j \})
\]
is an event because each \( \omega_{a_j}^{-1}(\{ a_j \}) \) is an event. We want to compute its probability. Let \( n = k_n - k_1 \geq 0 \). If \( m = 0 \) then \( k_j = j \) for all \( 1 \leq j \leq n \). Thus,
\[
P(w_{k_j}(\omega) = a_j \ \forall 1 \leq j \leq n) \overset{(2)}{=} \frac{1}{2^n}.
\]

Consider the case \( m > 1 \). Let \( \varsigma_1 < \varsigma_2 < \ldots < \varsigma_m \) be the elements of the set
\[
\{ k_1, k_2, \ldots, k_n \} \setminus \{ k_{\varsigma_1}, k_{\varsigma_2}, \ldots, k_{\varsigma_m} \}.
\]
Then
\[
\{ \omega : w_{k_j}(\omega) = a_j \ \forall 1 \leq j \leq n \} = \bigcup_{a_{\varsigma_1}, \ldots, a_{\varsigma_m} \in \{0, 1\}} \{ \omega : w_{k_j}(\omega) = a_j \ \forall 1 \leq j \leq m, \ w_j(\omega) = a_j \ \forall 1 \leq j \leq n \}.
\]
The sets in the union are pairwise disjoint, each of which has probability \( \frac{1}{2^n} \) because of (2). By the additivity of \( P \),
\[
P(w_{k_j} = a_j \ \forall 1 \leq j \leq n) = \sum_{a_{\varsigma_1}, \ldots, a_{\varsigma_m} \in \{0, 1\}} \frac{1}{2^n} = \frac{2^m}{2^n} = \frac{1}{2^{k_n - k_1}} = \frac{1}{2^n}.
\]
Therefore, in both cases we get
\[
P(w_{k_j} = a_j \ \forall 1 \leq j \leq n) = \frac{1}{2^n}.
\]
Additional problem D.

With the probability space \((\mathcal{S}, \mathcal{B}, \mathbb{P})\) and the random variables \(w_n\) as in Problem 6, we define

\[
X(w) = \sum_{k=1}^{\infty} \frac{w_k(w)}{2^k}, \quad Y(w) = \sum_{k=1}^{\infty} \frac{w_k^{-1}(w)}{2^k}.
\]

Because \(X\) is the (pointwise) limit of a sequence of random variables, it is also a random variable. The same is true for \(Y\). Take \(a,b,c,d \in [0,1]\), \(a \leq b\), \(c \leq d\).

We show that \(\mathbb{P}((X,Y) \in (a,b] \times (c,d]) = (b-a)(d-c)\).

First, we compute \(\mathbb{P}((X,Y) \in (x,1] \times (y,1])\) for \(x,y \in [0,1]\). If \(x = 1\) or \(y = 1\), the event \(\{\omega : (X(\omega), Y(\omega)) \in (x,1] \times (y,1]\}\) is empty and thus has probability 0.

Consider the case \(x,y \in (0,1]\). Write

\[
\begin{align*}
x &= 0, x_1 x_2 \ldots \quad \text{(in base 2)} \\
y &= 0, y_1 y_2 \ldots
\end{align*}
\]

such that there are infinitely many \(x_i\)'s are 0 and infinitely many \(y_i\)'s are 0.

Then \(\{\omega : X(\omega) > x, Y(\omega) > y\} = \bigcup_{m,n=1}^{\infty} \{\omega : w_{2i}(\omega) = x_i, \forall 1 \leq i < n, w_{2j-1}(\omega) = y_j, \forall 1 \leq j < n, w_{2m}(\omega) = x_m, w_{2n-1}(\omega) = y_n\}\)

\[
A_{mn}
\]

If \(x = \beta = 1\) then \(A_{mn} = \emptyset\). If \(x = y = 0\) then

\[
A_{mn} = \{\omega : w_{2i}(\omega) = x_i, \forall 1 \leq i < m, w_{2j-1}(\omega) = y_j, \forall 1 \leq j < n, w_{2m}(\omega) = 1, w_{2n-1}(\omega) = 1\}
\]

According to Problem 6, \(\mathbb{P}(A_{mn}) = \frac{1}{2^{m+n}}\). Combining two cases, we get
\[ P(A_{mn}) = \frac{(1-\alpha_m)(1-\beta_n)}{2^{m+n}}. \]

We see that \( A_{mn} \cap A_{m'n'} = \emptyset \) if \((m,n) \neq (m',n')\). Thus,

\[ P(X > \alpha, Y > \beta) = P\left( \bigcup_{m,n=1}^{\infty} A_{mn} \right) = \sum_{m,n=1}^{\infty} P(A_{mn}) = \sum_{m,n=1}^{\infty} \frac{(1-\alpha_m)(1-\beta_n)}{2^{m+n}} \]

\[ = \sum_{m,n=1}^{\infty} \left( \frac{1-\alpha_m}{2^m} \right) \left( \frac{1-\beta_n}{2^n} \right) = \left( \sum_{m=1}^{\infty} \frac{1-\alpha_m}{2^m} \right) \left( \sum_{n=1}^{\infty} \frac{1-\beta_n}{2^n} \right) \]

\[ = \left( 1 - \frac{1}{2} \right) \left( 1 - \frac{1}{2} \right) = (1-\alpha)(1-\beta). \]

Therefore, \( P(X > \alpha, Y > \beta) = (1-\alpha)(1-\beta) \quad \forall \alpha, \beta \in [0,1]. \tag{1} \)

Let \( A_1 = \{ \omega : X(\omega) \in (a,d], \ Y(\omega) \in (c,1]\} \)

\( A_2 = \{ \omega : X(\omega) \in (b,1], \ Y(\omega) \in (c,1]\} \)

\( A_3 = \{ \omega : X(\omega) \in (a,d], \ Y(\omega) \in (d,1]\} \)

\( A_4 = \{ \omega : X(\omega) \in (b,1], \ Y(\omega) \in (d,1]\} \)

\( A_5 = \{ \omega : X(\omega) \in (a,b], \ Y(\omega) \in (c,d]\} \)

Then \( A_1 = A_5 \cup (A_2 \cup A_3), \ A_5 \cap (A_2 \cup A_3) = \emptyset \) and \( A_2 \cap A_3 = A_4 \). By (1),

\[ P(A_1) = (1-a)(1-c), \ P(A_2) = (1-b)(1-c), \ P(A_3) = (1-a)(1-d), \ P(A_4) = (1-b)(1-d). \]
We have \[ P(A_4) = P(A_1) + P(A_2 \cup A_3) = P(A_5) + P(A_6) + P(A_7) - P(A_2 \cap A_3) \].

Thus, \[ P(A_5) = P(A_4) - P(A_6) - P(A_7) \]
\[ = (1-a)(1-c) - (1-b)(1-c) + (1-b)(1-d) - (1-a)(1-d) \]
\[ = [(1-a) - (1-b)](1-c) + [(1-b) - (1-a)](1-d) \]
\[ = (b-a)(1-c) + (a-b)(1-d) \]
\[ = (b-a)[(1-c) - (1-d)] \]
\[ = (b-a)(d-c). \]

Therefore, \[ P((X,Y) \in (a,b) \times (c,d)] = (b-a)(d-c). \]