Problems 2.3, 2.6, 7.19. Additional problems:

A) Let \((\Omega, \mathcal{F}, \mu)\) be a measure space. Define \(\mathcal{F} = \mathcal{F}^\mu\) as the collection of sets of the type \(B \cup N\), where \(B \in \mathcal{F}\) and there is \(C \in \mathcal{F}\) such that \(\mu(C) = 0\) and \(N \subseteq C\). Prove that \(\mathcal{F}\) is a \(\sigma\)-field and if we define \(\mu(B \cup N) = \mu(B)\) then this is a well-defined function on \(\mathcal{F}\), which is \(\sigma\)-additive and agrees with \(\mu\) on \(\mathcal{F}\).

B) For \(a \geq 0, b \in \mathbb{R}\) and Borel set \(B \subseteq \mathbb{R}\) define \(aB + b = \{ax + b : x \in B\}\). Prove that \(aB + b\) is Borel and \(\lambda(aB + b) = a\lambda(B)\), where \(\lambda\) is Lebesgue measure.

In the following problems, we assume the setting of Example 1.4 presented in class and in lecture notes.

C) Take a finite subset \(\{k_1, k_2, \ldots, k_n\}\) of integers \(\{1, 2, 3, \ldots\}\) such that \(k_1 < k_2 < \ldots < k_n\) and take some numbers \(a_1, a_2, \ldots\) such that \(a_j = 0\) or \(1\). Show that 
\[
\{\omega : \omega_{k_j}(\omega) = a_j \quad \forall j = 1, \ldots, n\}
\]
is an event and compute its probability.

D) For \(\omega = (\omega_1, \omega_2, \ldots)\) define 
\[
X(\omega) = \sum_{k=1}^{\infty} \frac{\omega_{2^k}}{2^k}, \quad Y(\omega) = \sum_{k=1}^{\infty} \frac{\omega_{2^k-1}}{2^k}.
\]
Show that for any numbers \(a, b, c, d\) such that \(0 \leq a \leq b \leq 1\) and \(0 \leq c \leq d \leq 1\) we have 
\[
P\left(\{\omega : (X(\omega), Y(\omega)) \in (a, b] \times (c, d]\}\right) = (b - a)(d - c).
\]