1. We show that the function \( r(t) = \sqrt{1-(1-t^2)} \) is not positive definite. Suppose by contradiction that \( r(t) \) is positive definite. For each \( x \in \mathbb{R} \),

\[
\int_{-1}^{1} r(t) e^{ixt} \, dt = \int_{-\infty}^{\infty} r(t) e^{ixt} \, dt = \int_{-\infty}^{\infty} r(t-s) e^{ix(t-s)} \, dt \quad \forall s \in \mathbb{R}.
\]

Thus,

\[
\int_{-1}^{1} r(t) e^{ixt} \, dt = \frac{1}{2} \int_{-1}^{1} \int_{-\infty}^{\infty} r(t-s) e^{ix(t-s)} \, dt \, ds
\]

\[
= \frac{1}{2} \int_{-1}^{1} \int_{-2}^{2} r(t-s) e^{ix} e^{-ixs} \, dt \, ds \quad (1)
\]

(because \( r \) is supported in \([-1, 1]\)).

Let \( N \in \mathbb{N} \). Divide \([-1, 1]\) into \( N \) subintervals of length \( \frac{2}{N} \) by the nodes \( s_j, 0 \leq j \leq N \). Divide \([-2, 2]\) into \( N \) subintervals of length \( \frac{4}{N} \) by the nodes \( t_k, 0 \leq k \leq N \).
Then the double integral (1) is equal to the limit of the Riemann sum as \( N \to \infty \). Thus,

\[
\int_{-1}^{1} r(t) e^{itx} dt = \frac{1}{2} \lim_{N \to \infty} \frac{1}{N^2} \sum_{j,k=0}^{N} r(t_k-s_j) e^{it_k x} e^{-is_j x}.
\]

For each \( N \in \mathbb{N} \), the sum inside the limit is nonnegative because \( r \) is positive definite. Thus, the limit as \( N \to \infty \) is also nonnegative. We get

\[
\int_{-1}^{1} r(t) e^{itx} dt \geq 0 \quad \forall x \in \mathbb{R}.
\]

The real part of the left hand side must be nonnegative. Using the fact that \( r(t) \cos(x t) \) is an even function in \( t \), we get

\[
\int_{0}^{1} r(t) \cos(x t) dt \geq 0 \quad \forall x \in \mathbb{R}.
\]

Taking \( x = \frac{3\pi}{2} \), we get

\[
\int_{0}^{1} \sqrt{1-t^2} \cos\left(\frac{3\pi}{2} t\right) dt \geq 0. \tag{2}
\]

We show that this inequality is false.

For \( t \in (0, \frac{1}{3}) \), \( \sqrt{1-t^2} \leq 1-\frac{t^2}{2} \) and \( \cos\left(\frac{3\pi}{2} t\right) > 0 \).

For \( t \in (\frac{1}{3}, 1) \), \( \sqrt{1-t^2} \geq 1-t^2 \) and \( \cos\left(\frac{3\pi}{2} t\right) < 0 \).

Then

\[
\text{LHS (2)} = \int_{0}^{\frac{1}{3}} \sqrt{1-t^2} \cos\left(\frac{3\pi}{2} t\right) dt + \int_{\frac{1}{3}}^{1} \sqrt{1-t^2} \cos\left(\frac{3\pi}{2} t\right) dt \\
\leq \int_{0}^{\frac{1}{3}} \left(1-\frac{t^2}{2}\right) \cos\left(\frac{3\pi}{2} t\right) dt + \int_{\frac{1}{3}}^{1}(1-t^2) \cos\left(\frac{3\pi}{2} t\right) dt. \tag{3}
\]
Each integrand is the product of a quadratic polynomial and a cosine function. We can compute their indefinite integrals by integrating by parts (twice for each integral). The results are

$$A = \frac{1}{27} \frac{17\pi^2 + 8}{\pi^3} \quad \text{and} \quad B = -\frac{16}{27} \frac{2+\pi^2}{\pi^3}.$$ Then (3) becomes

$$\text{LHS}(2) \leq A + B = \frac{\pi^2 - 24}{27\pi^3} < 0.$$ This contradicts (2).

2. Let $f, g : (0, \infty) \to \mathbb{R}$ be completely monotone functions, i.e. $f$ and $g$ are infinitely differentiable and

$$(-1)^k f^{(k)}(t) > 0 \quad \forall k \in \mathbb{N},$$ $$(-1)^k g^{(k)}(t) > 0 \quad \forall k \in \mathbb{N}.$$ We show that $h = fg$ is also completely monotone. Because $f$ and $g$ are nonnegative, $h$ is also nonnegative. Because $f$ and $g$ are infinitely differentiable, $h$ is also infinitely differentiable. We show by induction on $n \in \mathbb{N}$ the identity

$$h^{(n)}(t) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(t) g^{(n-k)}(t) \quad \forall t \in (0, \infty). \quad (1)$$ We have

$$h'(t) = (fg)'(t) = f'(t)g(t) + f(t)g'(t).$$ Thus, (1) is true for $n = 1$. Suppose (1) is true for some $n \in \mathbb{N}$. Differentiating both sides of (1), we get
\[ h^{(n+1)}(t) = \sum_{k=0}^{n} \binom{n}{k} \left[ f^{(k+1)}(t) g^{(n-k)}(t) + \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(t) g^{(n-k+1)}(t) \right] \]

\[ = \sum_{k=0}^{n} \binom{n}{k} f^{(k+1)}(t) g^{(n-k)}(t) + \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(t) g^{(n-k+1)}(t) \]

\[ = \sum_{j=1}^{n+1} \binom{n+1}{j} f^{(j)}(t) g^{(n+1-j)}(t) + \sum_{j=0}^{n} \binom{n}{j} f^{(j)}(t) g^{(n+1)}(t) \]

\[ = \binom{n+1}{n+1} f^{(n+1)}(t) g^{(0)}(t) + \sum_{j=1}^{n+1} \left[ \frac{\binom{n}{n-j} + \binom{n}{j}}{\binom{n+1}{j}} \right] f^{(j)}(t) g^{(n+1-j)}(t) + \binom{n}{0} f^{(0)}(t) g^{(n+1)}(t) \]

\[ = \binom{n+1}{n+1} f^{(n+1)}(t) g^{(0)}(t) + \binom{n+1}{j} f^{(j)}(t) g^{(n+1-j)}(t). \]

This means (1) is true for \( n+1 \). Thus, it is true for all \( n \in \mathbb{N} \).

Multiplying both sides of (1) by \((-1)^n\), we get

\[ (-1)^n h^{(n)}(t) = \sum_{k=0}^{n} \binom{n}{k} (-1)^n f^{(k)}(t) (-1)^{n-k} g^{(n-k)}(t) \geq 0. \]

Therefore, \( h \) is completely monotone.

3. We give two methods to compute

\[ \lim_{n \to \infty} \sum_{k=0}^{n} \frac{(k+1) n^k}{(1+n)^{k+2}}. \]

Put \( S_n = \sum_{k=0}^{n} \frac{(k+1) n^k}{(1+n)^{k+2}}. \)

Method 1 (Direct Computation)

For fixed \( n \in \mathbb{N} \), we define function \( \Phi: (0, n+1) \to \mathbb{R} \),
\[ F(x) = \frac{1}{1+n} \sum_{k=0}^{n} \left( \frac{x}{1+n} \right)^{k+1}. \]

Then \( S_n = F'(n). \) Put \( q = q(x) = \frac{x}{1+n} \in (0,1). \)

Then
\[
F(x) = \frac{1}{1+n} \sum_{k=0}^{n} q^{k+1} = \frac{1}{1+n} \frac{q-q^{n+2}}{1-q}.
\]

\[
F'(x) = \frac{1}{1+n} \frac{(q^2-(n+3)q^{n+1})(1-q)+q(q^{n+2})}{(1-q)^2}
\]

\[
= \frac{q'}{(1+n)(1-q)^2} \left[ 1-q^{n+1} - \frac{(n+1)(1-q)}{B} q^{n+1} \right]. \tag{1}
\]

At \( x=n, \) \( q' = \frac{1}{1+n} = 1-q. \) Then \( A = B = 1. \) Then (1) gives us

\[
F(n) = 1-2q^{n+1} = 1-2 \left(1-\frac{1}{1+n}\right)^{n+1}.
\]

Using the identity \( \lim_{n \to \infty} \left(1-\frac{1}{n}\right)^n = e^{-1}, \) we get

\[
\lim_{n \to \infty} S_n = 1-2 \lim_{n \to \infty} \left(1-\frac{1}{n+1}\right)^{n+1} = 1-2e^{-1}.
\]

**Method 2 (using moment generating functions)**

Let \( X \) be a probability distribution on \( \mathbb{R} \) whose density function is

\[
f(x) = e^{-x} \mathbb{I}_{[0,\infty)}(x) = \begin{cases} e^{-x} & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}
\]

This is the exponential distribution with parameter \( \theta = 1. \) Its moment
generating function is \( \phi : (0, \infty) \to \mathbb{R} \),
\[
\phi(u) = \int_{\mathbb{R}} e^{-ux} Q(dx) = \int_{-\infty}^{\infty} e^{-ux} f(x) dx = \int_{0}^{\infty} e^{-(u+1)x} dx = \frac{1}{1+u}.
\]

Let \((X_1, X_2)\) be an independent pair of random variables, each having the same distribution \(Q\). Put \(X = X_1 + X_2\). Then the density function of \(X\) is
\[
f_X(x) = (f \ast f)(x) = \int_{-\infty}^{\infty} f(y) f(x-y) dy = \int_{-\infty}^{\infty} e^{-y} e^{-y} I_{y>0} I_{y<x} dy
\]
\[
= e^{-x} \lambda((0,\infty) \cap (-\infty, x)),
\]
where \(\lambda\) is the Lebesgue measure on \(\mathbb{R}\). Thus,
\[
f_X(x) = \begin{cases} 
  xe^{-x} & \text{if } x > 0, \\
  0 & \text{if } x \leq 0.
\end{cases}
\]
Let \(F_X : \mathbb{R} \to [0,1]\) be the distribution function of \(X\). Because \(f_X\) is integrable over \(\mathbb{R}\), \(F_X\) is continuous. The moment generating function of \(X\) is
\[
\phi_X(u) = \phi(u) \phi(u) = (1+u)^{-2} \quad \forall u \in (0, \infty).
\]
The inversion formula says that
\[
F_X(x) = \lim_{n \to \infty} \sum_{0 \leq k \leq nx} \frac{(-n)^k}{k!} \phi_X^{(k)}(n) \quad \forall x \in (0, \infty).
\]
At \(x = 1\),
\[
F_X(1) = \lim_{n \to \infty} \sum_{k=0}^{n} \frac{(-n)^k}{k!} \phi_X^{(k)}(n). \tag{2}
\]
We have
\[
\phi_X^{(k)}(u) = \frac{d^k}{du^k} (1+u)^{-2} = \frac{d^{k-1}}{du^{k-1}} (-2)(1+u)^{-3}
\]
\[
= \frac{d^{k-2}}{du^{k-2}} (2)(-3)(1+u)^{-4}
\]
$$\lim_{n \to \infty} \sum_{k=0}^{n} \frac{n^k}{k!} \frac{(k+1)!}{(1+n)^{k+1}} = \lim_{n \to \infty} S_n.$$ 

Therefore,

$$\lim_{n \to \infty} S_n = F_X(t) = \int_{-\infty}^{t} f_X(x)\,dx = \int_{0}^{t} xe^{-x}\,dx = -xe^{-x}\bigg|_{0}^{t} - \int_{0}^{t} e^{-x}\,dx = -e^{-t} + (1 - e^{-1}) = 1 - 2e^{-1}.$$ 

4. Let $(X_1, X_2, \ldots)$ be an independent and identically distributed sequence of real-valued random variables. Put $S_n = X_1 + X_2 + \ldots + X_n$ and $S = \lim_{n \to \infty} S_n$. 

First, we show that $S$ is equal to a constant in $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm \infty\}$ almost surely. Define a map $g: \prod_{i=1}^{\infty} \overline{\mathbb{R}} \to \overline{\mathbb{R}}$, $g(x_1, x_2, \ldots) = \lim_{n \to \infty} \frac{S_n}{n}$ where $S_n = x_1 + x_2 + \ldots + x_n$. We show that $g$ is measurable. Each $S_n$ can be viewed as a function $s_n: \mathbb{R}^n \to \mathbb{R}$, $s_n(x_1, \ldots, x_n) = x_1 + \ldots + x_n$. It is continuous and, thus, is measurable. For each $a \in \mathbb{R}$,

$$\{x = (x_1, x_2, \ldots); g(x) < a\} = \bigcup_{n=1}^{\infty} \left\{x: \sup_{k \geq n} \frac{S_k}{k} < a\right\}$$

$$= \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \left\{x: \sup_{k \geq n} \frac{S_k}{k} \leq a - \frac{1}{m}\right\}$$

$$= \bigcup_{m=1}^{\infty} \bigcup_{k=1}^{\infty} \left\{x: \frac{S_k}{k} \leq a - \frac{1}{m}\right\}.$$
\[
\bigcup_{m,n=1}^{\infty} \bigcup_{k=0}^{\infty} \mathbb{S}^{-1}_k \left( (-\infty, \phi(a-\frac{1}{m})] \right) \times \mathbb{R} \times \mathbb{R} \times \ldots
\]
measurable in \( \bigcap_{i=1}^{\infty} \mathbb{R} \)

Hence, \( \{ x : g(x) < a \} \) is measurable in \( \bigcap_{i=1}^{\infty} \mathbb{R} \). We have shown that \( g \) is measurable.

Next, we show that \( g(X_1, X_2, \ldots) \) is equal to a constant in \( \mathbb{R} \) almost surely. Let \( \pi \) be a permutation of \( \{1, 2, \ldots, n\} \) for some \( n \in \mathbb{N} \). We have

\[
g(X_{\pi(1)}, \ldots, X_{\pi(n)}, x_{n+1}, \ldots) = \lim_{m \to \infty} \frac{S_m(x_{\pi(1)}, \ldots, x_{\pi(n)}, x_{n+1}, \ldots)}{m}
\]

\[
= \lim_{m \to \infty} \frac{S_m(x_1, \ldots, x_n, x_{n+1}, \ldots)}{m}
\]

\[
= g(X_1, X_2, \ldots, X_n, X_{n+1}, \ldots).
\]

Thus, \( g \) is exchangeable. Define two maps \( f_1, f_2 : \bigcap_{i=1}^{\infty} \mathbb{R} \to \mathbb{R} \),

\[
f_1(x) = \begin{cases} 
g(x) & \text{if } g(x) \in \mathbb{R}, \\ 1 & \text{if } g(x) = \infty, \\ -1 & \text{if } g(x) = -\infty, \end{cases}
\]

\[
f_2(x) = \begin{cases} 
g(x) & \text{if } g(x) \in \mathbb{R}, \\ 2 & \text{if } g(x) = \infty, \\ -2 & \text{if } g(x) = -\infty. \end{cases}
\]

Then \( f_1 \) and \( f_2 \) are measurable and exchangeable maps. By Hewitt–Savage 0-1 Law, \( f_1(X_1, X_2, \ldots) \) and \( f_2(X_1, X_2, \ldots) \) are equal to constants almost surely. Write
\[ Y_1 := f_1(X_1, X_2, \ldots) = c_1 \in \mathbb{R} \text{ a.s.} \]
\[ Y_2 := f_2(X_1, X_2, \ldots) = c_2 \in \mathbb{R} \text{ a.s.} \]

If \( c_1 \notin \{\pm 1\} \) then \( g(X_1, X_2, \ldots) = f_1(X_1, X_2, \ldots) = c_1 \in \mathbb{R} \text{ a.s.} \)

If \( c_2 \notin \{\pm 2\} \) then \( g(X_1, X_2, \ldots) = f_2(X_1, X_2, \ldots) = c_2 \in \mathbb{R} \text{ a.s.} \)

Consider the case \( c_1 \in \{\pm 1\} \) and \( c_2 \in \{\pm 2\} \). Suppose \( c_1 = 1 \). Since \( Y_1 = 1 \text{ a.s.} \)
\[ \{ \omega : Y_1 = -1 \} \text{ is of measure zero. Thus, } \{ \omega : g(X_1, X_2, \ldots) = -1 \} \text{ is of measure zero. Then } \{ \omega : Y_2 = -2 \} \text{ is of measure zero. This implies } c_2 = 2. \]
\[ \{ \omega : g(X_1, X_2, \ldots) \in \mathbb{R} \} \subset \{ \omega : Y_1 = Y_2 \} \]

which is of measure zero because \( Y_1 = 1 \text{ a.s. and } Y_2 = 2 \text{ a.s.} \). Therefore, \( g(X_1, X_2, \ldots) = \infty \text{ a.s.} \). We do similarly to the case \( c_1 = -1 \). In that situation, \( g(X_1, X_2, \ldots) = -\infty \text{ a.s.} \).

We have showed that \( g(X_1, X_2, \ldots) \) is equal to a constant in \( \mathbb{R} \) almost surely.

Write \( \lim_{n \to \infty} \frac{S_n}{n} = c \in \mathbb{R} \text{ a.s.} \). Next, we show that the probability that there exist infinitely many \( n \in \mathbb{N} \) such that \( S_n > cn \) is equal to 0 or 1. But
\[ A = \{ x = (x_1, x_2, \ldots) \in \prod_{i=1}^{\infty} \mathbb{R} : \exists \text{ infinitely many } n \in \mathbb{N} \text{ such that } \frac{S_n}{n} > c \} \]
We show that \( A \) is measurable.
\[ A^c = \{ x \in \prod_{i=1}^{\infty} \mathbb{R} : \exists \text{ finitely many } n \in \mathbb{N} \text{ such that } \frac{S_n}{n} > c \} \]
\[ \{ x \in \bigcap_{i=1}^{\infty} \mathbb{R} : \exists m \in \mathbb{N}, \frac{s_n}{n} \leq c, n \geq m \} \]
\[ = \bigcup_{m=1}^{\infty} \{ x : \frac{s_n}{n} \leq c, n \geq m \} \]
\[ = \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \{ x : \frac{s_n}{n} \leq c \} \]
\[ = \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} s_n^{-1}(-\infty, cnj) \times \mathbb{R} \times \mathbb{R} \times \cdots \]
\[ \text{measurable in } \bigcap_{i=1}^{\infty} \mathbb{R} \]

Thus, \( A \) is measurable. Define a map \( h : \bigcap_{i=1}^{\infty} \mathbb{R} \to \mathbb{R} \), \( h = I_A \). This is a measurable map. Let \( \pi \) be a permutation of \( \{1, 2, \ldots, n\} \) for some \( n \in \mathbb{N} \). Then

\[ (x_1, x_2, \ldots) \in A \iff \exists \text{ infinitely many } m \in \mathbb{N} \text{ such that } \frac{x_1 + \cdots + x_m}{m} > c \]

\[ \iff \exists \text{ infinitely many } m \in \mathbb{N}, m > n \text{ such that } \frac{x_{\pi(1)} + \cdots + x_{\pi(n)} + x_{n+1} + \cdots}{m} > c \]

\[ \iff (x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(n)}, x_{n+1}, \ldots) \in A. \]

Thus, \( h(x_1, x_2, \ldots) = h(x_{\pi(1)}, \ldots, x_{\pi(n)}, x_{n+1}, \ldots) \). In other words, \( h \) is exchangeable.

By Hewitt-Savage 0-1 Law, \( h(X_1, X_2, \ldots) \) is equal to a constant almost surely. This means \( I_A = 1 \) a.s. or \( I_A = 0 \) a.s. Then

\[ \mathbb{P}(X_1, X_2, \ldots) \in A) = \int_{\mathbb{R}^n} I_A(X_1, X_2, \ldots) \mathbb{P}(d\omega) = 1 \text{ or } 0. \]