1. Let \( X_1, \ldots, X_n \in L^1(\mathbb{R}, F, P) \). Denote by \( \text{Span}(X_1, \ldots, X_n) \) the linear span of \( \{X_1, X_2, \ldots, X_n\} \) in \( L^2(\mathbb{R}, F, P) \). We show by induction in \( n \in \mathbb{N} \) that \( \text{Span}(X_1, \ldots, X_n) \) is closed in \( L^1(\mathbb{R}, F, P) \).

For \( n = 1 \), \( \text{Span}(X_1) = \{cX_1 : c \in \mathbb{R}\} \). If \( X_1 = 0 \) a.s. then \( \text{Span}(X_1) = \{0\} \), which is closed in \( L^1(\mathbb{R}, F, P) \). Consider the case \( X_1 \) is not almost-surely equal to 0. For each \( m \in \mathbb{N} \), put

\[
A_m = \{\omega \in \Omega : X_1(\omega) > \frac{1}{m}\},
\]

\[
B_m = \{\omega \in \Omega : X_1(\omega) < -\frac{1}{m}\}.
\]

Then \( \bigcup_{m=1}^\infty A_m \cup \bigcup_{m=1}^\infty B_m = \{\omega \in \Omega : X_1(\omega) \neq 0\} \), which has positive measure.

Then

\[
0 < P\left(\bigcup_{m=1}^\infty A_m \cup \bigcup_{m=1}^\infty B_m\right) \leq \sum_{m=1}^\infty P(A_m) + \sum_{m=1}^\infty P(B_m).
\]

This implies there exists \( m_0 \in \mathbb{N} \) such that \( P(A_{m_0}) > 0 \) or \( P(B_{m_0}) > 0 \).

By replacing \( X_1 \) with \( -X_1 \) if necessary, we can assume \( P(A_{m_0}) > 0 \).

Let \( (Z_m) \) be a sequence in \( \text{Span}(X_1) \) that converges to \( Z \in L^1(\mathbb{R}, F, P) \).

Write \( Z_m = c_m X_1 \). Then

\[
\int _\Omega Z_m I_{A_{m_0}} P(dw) = c_m \int _\Omega X_1 I_{A_{m_0}} P(dw), \quad (1)
\]
\[ a = \int \mathbb{I}_{A_{m_0}} P(\omega) \geq \frac{1}{m_0} P(A_{m_0}) > 0. \]

\[ \left| \int \mathbb{I}_{(2m - z)A_{m_0}} P(d\omega) \right| \leq \int \left| 2m - z \right| P(d\omega) = \| 2m - z \|_1 \rightarrow 0 \quad \text{as } m \rightarrow \infty. \]

Thus,
\[ \lim_{m \rightarrow \infty} \int \mathbb{I}_{2m} P(\omega) = \int \mathbb{I}_{A_{m_0}} P(\omega). \]

Then (4) implies
\[ c_m = \frac{1}{a} \int \mathbb{I}_{2m} P(\omega) \longrightarrow \frac{1}{a} \int \mathbb{I}_{A_{m_0}} P(\omega) = c \quad \text{as } m \rightarrow \infty. \]

Then \( Z_m = c_m X_1 \rightarrow c X_1 \) pointwise as \( m \rightarrow \infty. \) For all \( m \) sufficiently large, \( |c_m - c| \leq 1. \) Then \( |2m - cX_1| = |c_m - c| |X_1| \leq |X_1|, \) which is an integrable function. By the Dominated Convergence Theorem, \( Z_m \rightarrow cX_1 \) in \( L^1(\Omega, \mathcal{F}, P). \)

Thus, \( Z = cX_1 \in \text{Span}(X_1). \)

Let \( n \geq 2. \) Suppose \( \text{Span}(X_1, \ldots, X_n) \) is closed in \( L^1(\Omega, \mathcal{F}, P) \) for any \( 1 \leq r < n \) and \( X_1, X_2, \ldots, X_r \in L^1(\Omega, \mathcal{F}, P). \) Take \( X_1, X_2, \ldots, X_r \in L^1(\Omega, \mathcal{F}, P). \)

We show that \( \text{Span}(X_1, \ldots, X_r) \) is closed in \( L^1(\Omega, \mathcal{F}, P). \) Let \( (X_m) \) be a sequence in \( \text{Span}(X_1, \ldots, X_r) \) that converges to some \( X \in L^1(\Omega, \mathcal{F}, P). \)

Because \( \text{Span}(X_1, \ldots, X_r) \) is a finite dimensional vector space, it has a basis \( \{ Y_1, Y_2, \ldots, Y_s \} \) for some \( 0 \leq s \leq n. \) If \( s = 0 \) then \( \text{Span}(X_1, \ldots, X_n) = \{ 0 \}, \) which is closed in \( L^1(\Omega, \mathcal{F}, P). \) If \( s = 1 \) then \( \text{Span}(X_1, \ldots, X_n) = \text{Span}(Y_1), \) which has been proved to be closed in \( L^1(\Omega, \mathcal{F}, P). \) Consider the case \( s \geq 2. \)

Write
\[ \tilde{X}_m = \alpha_{1m} Y_1 + \alpha_{2m} Y_2 + \cdots + \alpha_{5m} Y_5 \]

for \( \alpha_{1m}, \alpha_{2m}, \ldots, \alpha_{5m} \in \mathbb{R} \). For each \( j \in \{1, 2, \ldots, 5\} \), we show that the sequence \((\tilde{X}_m)_m\in\mathbb{N}\) is bounded. Suppose otherwise. Without loss of generality, we can assume the sequence \((\tilde{X}_m)_m\in\mathbb{N}\) is unbounded. By replacing \((\tilde{X}_m)\) with a suitable subsequence, we can assume \(|\tilde{X}_m| \to \infty\) as \(m \to \infty\). Then

\[
Y_s = \frac{1}{\alpha_{5m}} \left( \tilde{X}_m - \alpha_{1m} Y_1 + \cdots - \alpha_{5m-1,m} Y_{s-1} \right)
\]

\[
= \frac{1}{\alpha_{5m}} \tilde{X}_m + \sum_{j=1}^{s-1} \left( -\frac{\alpha_{jm}}{\alpha_{5m}} \right) Y_j \quad \forall m \in \mathbb{N}. \quad (2)
\]

Because \( \tilde{X}_m \to X \) in \( L^1(\Omega, \mathcal{F}, P) \) and \( |\tilde{X}_m| \to \infty \) as \( m \to \infty \), \( \frac{1}{\alpha_{5m}} \tilde{X}_m \to 0 \) in \( L^1(\Omega, \mathcal{F}, P) \). Letting \( m \to \infty \) in (2), we get

\[
Y_s = \lim_{m \to \infty} \sum_{j=1}^{s-1} \left( -\frac{\alpha_{jm}}{\alpha_{5m}} \right) Y_j.
\]

This implies \( Y_s \) belongs to the closure in \( L^1(\Omega, \mathcal{F}, P) \) of \( \text{Span}(Y_1, \ldots, Y_{s-1}) \).

Since \( 1 \leq s-1 < n \), by the induction hypothesis, \( \text{Span}(Y_1, \ldots, Y_{s-1}) \) is closed in \( L^1(\Omega, \mathcal{F}, P) \). Then \( Y_s \in \text{Span}(Y_1, \ldots, Y_{s-1}) \). This is a contradiction because \( Y_1, Y_2, \ldots, Y_s \) are linearly independent.

We have showed that the sequence \((\tilde{X}_m)_m\in\mathbb{N}\) is bounded for every \( j \in \{1, 2, \ldots, 5\} \). \((\tilde{X}_m)\) has a convergent subsequence \((\tilde{X}_{m_k})\). Denote by \( \bar{X}^{(i)} \) the subsequence \((\tilde{X}_{m_k})\) of \((\tilde{X}_m)\). If we regard \( \bar{X}^{(i)} \) instead of \((\tilde{X}_m)\), we can assume \((\tilde{X}_m)\) converges. \((\tilde{X}_m)\) has a convergent subsequence \((\tilde{X}_{2m_k})\). Denote by
the subsequence \((\tilde{X}(n))\) of \(\tilde{X}(1)\). If we regard \(\tilde{X}(n)\) instead of \((X_m)\), we can assume \((X_{n1})\) and \((X_{n2})\) converge. Continue this process \(s\) times. We then conclude that \((\tilde{X}(n))\) has a subsequence such that \((X_{n1}), (X_{n2}), \ldots, (X_{ns})\) converge as \(m \to \infty\). Replace \((\tilde{X}(n))\) by this subsequence. Let \(\tilde{X} = \lim_{m \to \infty} \tilde{X}_m\) for each \(j \in \{1, 2, \ldots, s\}\). Let \(Y = \tilde{X}_1Y_1 + \cdots + \tilde{X}_sY_s\). Then

\[
\tilde{X}_m = \alpha_1Y_1 + \cdots + \alpha_sY_s \to \alpha_1Y_1 + \cdots + \alpha_sY_s = Y \text{ pointwise as } m \to \infty.
\]

\[
|\tilde{X}_m - Y| \leq \sum_{j=1}^{s} |\alpha_j - \alpha_j| |Y_j| \leq \sum_{j=1}^{s} |Y_j| \in L^1(\mathcal{A}, \mathbb{F}, P)
\]

for all \(m\) sufficiently large. By the Dominated Convergence Theorem, \(\tilde{X}_m \to Y\) in \(L^1(\mathcal{A}, \mathbb{F}, P)\). Hence, \(X = Y \in \text{Span} (Y_1, \ldots, Y_s) = \text{Span} (X_1, \ldots, X_s)\).

2. Let \(\mathcal{A} = \bigcup_{n=1}^{\infty} A_n\) be a partition of \(\mathcal{A}\) into disjoint sets \(A_n \in \mathbb{F}\) such that \(P(A_n) > 0\) for every \(n \in \mathbb{N}\). Let \(\mathcal{F} = \sigma (A_n : n \in \mathbb{N})\). Take \(\tilde{f} \in L^1(\mathcal{A}, \mathbb{F}, P)\). We show that

\[
E(\tilde{f}|\mathcal{F}) = \frac{1}{P(A_n)} E(\tilde{f} I_{A_n}).
\]

on \(A_n\) for every \(n \in \mathbb{N}\). Equivalently, we need to show

\[
E(f|\mathcal{F}) = \sum_{n=1}^{\infty} \frac{1}{P(A_n)} E(f I_{A_n}) I_{A_n}. \quad (1)
\]

Denote

\[
X = \sum_{n=1}^{\infty} \frac{1}{P(A_n)} E(f I_{A_n}) I_{A_n},
\]

\[
X_m = \sum_{n=1}^{m} \frac{1}{P(A_n)} E(f I_{A_n}) I_{A_n}.
\]

The indicators \(I_{A_1}, I_{A_2}, I_{A_3}, \ldots\) are \(\mathcal{F}\)-measurable. Thus, \(X_m\) is \(\mathcal{F}\)-measurable.
Because $A_1, A_2, A_3, \ldots$ are pairwise disjoint, $X_m = X$ on $\bigcap_{n=1}^{\infty} A_n$. Hence,

$$X_m = X$$

Because $\bigcap_{n=1}^{\infty} A_n = \emptyset$,

$$X(\omega) = \lim_{n \to \infty} X_m(\omega) \quad \forall \omega \in \emptyset.$$  

Then $X$ is also $\mathcal{F}$-measurable.

Next, we show that $X \in L^1(\Omega, \mathcal{F}, P)$ and $EX = E^\mathcal{F}$.

$$\int |X| P(d\omega) \leq \sum_{n=1}^{\infty} \frac{1}{P(A_n)} E(\mathbb{I}_{\mathbb{I}_{A_n}}) \mathbb{I}_{A_n} P(d\omega)$$

$$= \sum_{n=1}^{\infty} \frac{1}{P(A_n)} E(\mathbb{I}_{\mathbb{I}_{A_n}}) P(A_n)$$

$$= \sum_{n=1}^{\infty} E(\mathbb{I}_{\mathbb{I}_{A_n}})$$

$$= E(\mathbb{I}_{\mathbb{I}_{\bigcap_{n=1}^{\infty} A_n}}). 
$$  \hspace{1cm} (2)

Then

$$\int |X| P(d\omega) \leq \liminf_{m \to \infty} \int |X_m| P(d\omega) \leq \liminf_{m \to \infty} E(\mathbb{I}_{\mathbb{I}_{A_n}})$$

$$= E(\mathbb{I}_{\mathbb{I}_{\bigcap_{n=1}^{\infty} A_n}}) \quad \text{(by Monotone Convergence Theorem)}$$

$$= E|\mathbb{I}_{\bigcap_{n=1}^{\infty} A_n}| < \infty.$$  

While we were deriving (2), we also derived $|X_m| \leq E(\mathbb{I}_{\bigcap_{n=1}^{\infty} A_n}) \leq |\mathbb{I}_{\bigcap_{n=1}^{\infty} A_n}|$.

Thus, by the Dominated Convergence Theorem,

$$EX = \int X P(d\omega) = \lim_{m \to \infty} \int X_m P(d\omega) = \lim_{m \to \infty} \int \sum_{n=1}^{m} \frac{1}{P(A_n)} E(\mathbb{I}_{\mathbb{I}_{A_n}}) \mathbb{I}_{A_n} P(d\omega)$$

$$= \lim_{m \to \infty} \sum_{n=1}^{m} \frac{1}{P(A_n)} E(\mathbb{I}_{\mathbb{I}_{A_n}}) P(A_n)$$
\[
= \lim_{m \to \infty} \sum_{n=1}^{m} E(\delta I_{A_n}) = \lim_{m \to \infty} E(\delta I_{A_{m+1}}) = \mathbb{E}
= E(\delta I_{\mathcal{A}}) \quad \text{(by the Dominated Convergence Theorem)}
= E\delta.
\]

To show that \( X = E(\delta(I_\mathcal{Y})) \), we need to show

\[ E(X I_A) = E(\delta I_A) \quad \forall A \in \mathcal{Y}. \]

Let \( g_0 = \{ A \in \mathcal{Y} : E(X I_A) = E(\delta I_A) \} \). We now show that \( g_0 \) is a \( \mathcal{A} \)-system.

Since \( E(X I_{\phi}) = E(\delta I_{\phi}) = 0 \), \( \phi \in g_0 \). Let \( B_1, B_2, B_3, \ldots \) be a sequence of pairwise disjoint members of \( g_0 \).

\[ E(X I_{B_n}) = E(\delta I_{B_n}) \quad \forall n \in \mathbb{N}. \]

Then

\[ E(X I_{B_n}) = \sum_{n=1}^{\infty} E(X I_{B_n}) = \sum_{n=1}^{\infty} E(\delta I_{B_n}) = E(\delta I_{B_n}) \quad \forall n \in \mathbb{N}. \quad (3) \]

Let \( B = \bigcup_{n=1}^{\infty} B_n \in \mathcal{Y} \). Then \( |X I_{B_n}| \leq |X| \). By the Dominated Convergence Theorem,

\[ \lim_{n \to \infty} E(X I_{B_n}) = E(X I_B). \]

Similarly, \[ \lim_{n \to \infty} E(\delta I_{B_n}) = E(\delta I_B). \]

Letting \( n \to \infty \) in (3), we get \( E(X I_B) = E(\delta I_B) \). Thus \( B \in \mathcal{Y} \).

Take any \( A \in \mathcal{Y} \). Then

\[ E(X I_A) = E(X - X I_A) = EX - E(X I_A), \]
\[ E(\delta I_A) = E(\delta - \delta I_A) = E\delta - E(\delta I_A). \]
Because $E X = E S$ and $E(X|I_A) = E(S|I_A)$, we get $E(X|I_A^c) = E(S|I_A^c)$.

Then $A^c E S$. We have showed that $S_0$ is a $\mathcal{I}$-system.

For each $k \in \mathbb{N}$, $X|I_{A_k} = \frac{1}{P(A_k)} E(S|I_{A_k}) I_{A_k}$ because

$$I_{A_k} I_{A_n} = I_{A_k \cap A_n} = \begin{cases} I_{A_k} & \text{if } n = k, \\ 0 & \text{if } n \neq k. \end{cases}$$

Thus,

$$E(X|I_{A_k}) = \int \frac{1}{P(A_k)} E(S|I_{A_k}) I_{A_k} P(d\omega) = \frac{1}{P(A_k)} E(S|I_{A_k}) P(A_k)$$

$$= E(S|I_{A_k}).$$

This means $A_k \in S_0$. The family $\mathfrak{A} = \{\emptyset, A_1, A_2, A_3, \ldots\}$ is a $\mathfrak{I}$-system because $A_1, A_2, A_3, \ldots$ are pairwise disjoint. Then $\mathfrak{A}$ is a $\mathfrak{I}$-system contained in the $\mathfrak{I}$-system $S_0$. By the Sierpinski Class theorem, $S_0$ contains $\mathfrak{C}(\mathfrak{A}) = \mathfrak{C}(A_1, A_2, A_3, \ldots) = S$. Thus, $S_0 = S$.

3. Let $d \in \mathbb{N}$ and $a_{ij}, b_i \in \mathbb{R}$ for $1 \leq i,j \leq d$. Suppose the function $f: \mathbb{R}^d \to \mathbb{R}$

$$f(x) = \sum_{i,j=1}^{d} a_{ij} x_i x_j + 2 \sum_{i=1}^{d} b_i x_i + c$$

is nonnegative for every $x \in \mathbb{R}^d$. We show that $f$ has a minimum value. Denote by $a = (a_{ij})_{1 \leq i,j \leq d} \in M_d(\mathbb{R})$, $b = (b_i)_{1 \leq i \leq d} \in \mathbb{R}^d$. Denote by $(\cdot, \cdot)$ the usual dot product in $\mathbb{R}^d$. Then

$$f(x) = (ax, x) + 2(b, x) + c. \quad (1)$$

We observe that

$$\sum_{i,j=1}^{d} a_{ij} x_i x_j = \sum_{i,j=1}^{d} \frac{a_{ij} + a_{ji}}{2} x_i x_j.$$

This means if we replace matrix $a$ with $\frac{a + a^T}{2}$, the value of $f(x)$ does
not change. Thus, we can assume \( a \) is symmetric, i.e. \( a_{ij} = a_{ji} \) for all \( 1 \leq i,j \leq d \).

\[
f(tx) = t^2(ax,x) + 2t(b,x) + c \geq 0 \quad \forall t \in \mathbb{R}.
\]

If \((ax,x) < 0\) for some \( x \in \mathbb{R}^d \) then (2) is not satisfied as \( t \to \infty \). Thus, \((ax,x) \geq 0\) for all \( x \in \mathbb{R}^d \). We view LHS(2) as a quadratic polynomial in \( t \). The discriminant must be nonpositive, i.e.

\[
(b,x)^2 - 4c(ax,x) \leq c(ax,x) \quad \forall x \in \mathbb{R}^d.
\]

because \( a \) is symmetric, the transpose matrix of \( a \) is \( a^* = a \). For each \( x \in \ker a^* \),

\[
(b,x)^2 - c(ax,x) = c(a^*x,x) = c(0,x) = 0.
\]

Thus, \((b,x) = 0\). Then \( b \in (\ker a^*)^\perp \). In linear algebra, we know the identity

\[
\text{range}(a) = (\ker a^*)^\perp.
\]

Thus, \( b \in \text{range}(a) \). There exists \( x_0 \in \mathbb{R}^d \) such that \( b = ax_0 \). Then (1) gives

\[
f(x) = (ax,x) + 2(ax_0,x) + c
\]

\[
= (ax,x) + (ax_0,x) + (ax_0,x) + c
\]

\[
= (ax,x) + (ax_0,x) + (x_0,ax)
\]

\[
\geq (ax,x) + (x_0,x_0) - (ax_0,x_0) + c
\]

\[
\geq - (ax_0,x_0) + c.
\]

The equality holds when \( x = -x_0 \). Therefore, \( \min_{x \in \mathbb{R}^d} f(x) = f(-x_0) \).
Consider a random walk with steps \( X_1, X_2, X_3, \ldots \). Suppose \( I(X_1 = 1) = I(X_1 = -1) = \frac{1}{2} \cdot P(X_1 = 1) = \frac{1}{2} - \frac{1}{2} = 0 \). Put \( S_n = X_1 + X_2 + \ldots + X_n \), and \( \xi_n = \left( \frac{S}{4} \right)^n 2^n \). First, we show that \( \xi_n \to 0 \) a.s.

We have \( EX_1 = 1 \cdot P(X_1 = 1) + (-1) \cdot P(X_1 = -1) = \frac{1}{2} - \frac{1}{2} = 0 \). By the Strong Law of Large Numbers, \( \frac{S_n}{n} \to EX_1 = 0 \) a.s. Then

\[
\log \xi_n = \sum_{j=1}^{n} \log 2 - n \log \frac{5}{4} = n \log 2 \left( \frac{S_n}{n} - \frac{\log 5}{\log 2} \right) \rightarrow -\frac{\log 5}{\log 2} < 0
\]

Thus, \( \xi_n = e^{\log \xi_n} \to 0 \) a.s.

Next, we show \( E \sup_{n \in \mathbb{N}} \xi_n = \infty \). Let \( \xi = \sup_{n \in \mathbb{N}} \xi_n \geq 0 \). Suppose by contradiction that \( E \xi < \infty \). Because \( 0 \leq \xi_n \leq \xi \) and \( \xi_n \to 0 \) a.s., by the Dominated Convergence Theorem,

\[
\lim_{n \to \infty} E \xi_n = E \left( \lim_{n \to \infty} \xi_n \right) = E(0) = 0.
\]

On the other hand,

\[
E \xi_n = \left( \frac{5}{4} \right)^n 2^n = \left( \frac{5}{4} \right)^n E(2^{X_1} 2^{X_2} \ldots 2^{X_n})
\]

\[
= \left( \frac{5}{4} \right)^n (E2^{X_i})(E2^{X_i}) \ldots (E2^{X_i}) \quad \text{(because } X_i, X_2, \ldots \text{ are independent)}
\]

\[
= \left( \frac{5}{4} \right)^n (E2^{X_i})^n \quad \text{(because } X_1, X_2, \ldots \text{ have the same distribution)}
\]

\[
= \left( \frac{5}{4} \right)^n \left( 2 \cdot I(X_1 = 1) + 2^{-1} \cdot I(X_1 = -1) \right)^n
\]

\[
= \left( \frac{5}{4} \right)^n \left( \frac{5}{4} \right)^n = 1.
\]
This contradicts (1).