Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(\mathcal{F}_n)_{n \geq 0}$ be an increasing filtration of $\sigma$-fields of $\mathcal{F}$. Let $\mathcal{F}_n$ be $\mathcal{F}_n$-measurable with $E|\mathcal{F}_n| < \infty$. Suppose that for any bounded stopping times $\tau \leq \sigma$, we have $E\mathcal{F}_\tau \leq E\mathcal{F}_\sigma$. We show that the sequence $(\mathcal{F}_n)$ is a submartingale with respect to $(\mathcal{F}_n)$.

Take any $m \in \{0, 1, 2, \ldots\}$. It suffices to show $E(\mathcal{F}_{m+1} | \mathcal{F}_m) \geq \mathcal{F}_m$.

First, we show the following lemma.

Let $X$ be a random variable on a probability space $(\Omega, \mathcal{F}, P)$ and $EX \mathcal{F}_A > 0$ for every $A \in \mathcal{F}$. Then $X \geq 0$ a.s.

**Proof of the lemma**

Let $B = \{ \omega \in \Omega : X(\omega) < 0 \}$. For each $n \in \mathbb{N}$, we put

$$B_n = \{ \omega \in \Omega : X(\omega) < -\frac{1}{n} \}.$$

Then $B = \bigcup_{n=1}^\infty B_n$. By the hypothesis,

$$0 \leq EX I_{B_n} \leq \left( -\frac{1}{n} \right) EI_{B_n} = -\frac{1}{n} P(B_n).$$

Thus, $P(B_n) = 0$. Then $P(B) \leq \sum_{n=1}^\infty P(B_n) = 0$. This means $P(B) = 0$.

Return to the problem. Let $\gamma = E(\mathcal{F}_{m+1} | \mathcal{F}_m)$, $X = \gamma - \mathcal{F}_m$ and $Y = \mathcal{F}_m$. Because $\gamma$ and $X$ are $\mathcal{F}_m$-measurable, they are random variables on the
probability space $(\Omega, \mathcal{F}, P)$. Our goal is to show $X \geq 0$ a.s. By the above lemma, it suffices to show $EXI_A \geq 0$ for every $A \in \mathcal{F}$. Fix $A \in \mathcal{F}$.

By the definition of conditional expectations,

$$E_{\mathcal{F}} I_A = E_{\mathcal{F}_{m+1}} I_A, \quad (1)$$

which is a finite number because $E|\mathcal{F}_{m+1}| < \infty$. Define two maps $\delta, \tau : \Omega \rightarrow \{0, 1, 2, \ldots\}$,

$$\delta(\omega) = \begin{cases} \text{m+1 if } \omega \in A, \\ m \text{ if } \omega \in A^c = \Omega \setminus A, \end{cases}$$

$$\tau(\omega) = m \quad \forall \omega \in \Omega.$$  

Then $\delta \geq \tau$ in $\Omega$. For each $n \in \{0, 1, 2, \ldots\}$,

$$\{\omega : \tau(\omega) > n\} = \begin{cases} \Omega & \text{if } n < m, \\ \emptyset & \text{if } n \geq m, \end{cases}$$

which belongs to $\mathcal{F}_m$. Thus, $\tau$ is a bounded stopping time.

$$\{\omega : \delta(\omega) > n\} = \begin{cases} \Omega & \text{if } n < m, \\ A & \text{if } n = m, \\ \emptyset & \text{if } n > m, \end{cases}$$

which belongs to $\mathcal{F}_m$. Thus, $\delta$ is also a bounded stopping time. By the hypothesis, $E\mathbb{F}_\delta \geq E\mathbb{F}_\tau$. Then

$$E\mathbb{F}_\delta = E\mathbb{F}_\tau \leq E\mathbb{F}_\delta I_A + E\mathbb{F}_\delta I_A^c = E\mathbb{F}_{m+1} I_A + E\mathbb{F}_m I_A^c.$$  

Then

$$E\mathbb{F}_m I_A + E\mathbb{F}_m I_A^c \leq E\mathbb{F}_{m+1} I_A + E\mathbb{F}_m I_A^c. \quad (2)$$

Because $E|\mathcal{F}_m| < \infty$, $E\mathbb{F}_m I_A \leq E$. Then (2) implies $E\mathbb{F}_m I_A \leq E\mathbb{F}_{m+1} I_A$. We get
\[ EX_t^A = E X_t A - E X^A \]

(2) Let \( X_1, X_2, X_3, \ldots \) be an independent and identically distributed sequence of real-valued random variables. Suppose \( E|X_t| < \infty \) and \( E X_t = 0 \). Put \( S_0 = 0 \) and \( S_n = X_1 + X_2 + \ldots + X_n \) for \( n \in \mathbb{N} \). For \( \varepsilon = \inf \{ n \geq 0 : S_n > 0 \} \), we show that \( E \varepsilon = \infty \). We need the following results.

**Lemma 1:** Let \( \eta_1, \eta_2, \eta_3, \ldots \) be an independent and identically distributed sequence of random variables with \( E \eta_t = 0 \). Put
\[
F_0 = \{ \emptyset, \Omega \}, \quad F_n = \sigma (\eta_1, \eta_2, \ldots, \eta_n) \quad \forall n \in \mathbb{N},
\]
\[
\mathcal{F}_0 = \{ \emptyset, \Omega \}, \quad \mathcal{F}_n = \sigma (\eta_1 + \eta_2 + \ldots + \eta_n) \quad \forall n \in \mathbb{N}.
\]
Then \( (\mathcal{F}_n, \mathcal{F}_n)_{n \geq 0} \) is a martingale.

**Lemma 2:** Let \( (\mathcal{F}_n, \mathcal{F}_n)_{n \geq 0} \) be a martingale. Then \( E \mathcal{F}_n = E \mathcal{F}_0 \) for all \( n \in \mathbb{N} \).

**Doob's theorem** (Problem 18, Fristedt-Gray page 470; Lecture on 02/23/2015 by Professor Krylov)

Let \( (\mathcal{F}_n, \mathcal{F}_n)_{n \geq 0} \) be a submartingale and let \( \tau \) be a stopping time.

Then \( (\mathcal{F}_n, \mathcal{F}_n)_{n \geq 0} \) is also a submartingale.

**Proof of Lemma 1**

It is clear that \( \mathcal{F}_n \) is \( \mathcal{F}_n \)-measurable and \( (\mathcal{F}_n)_{n \geq 0} \) is an increasing sequence of \( \sigma \)-fields. For each \( n \in \{0, 1, 2, \ldots \} \) and \( A \in \mathcal{F}_n \),
\[ E\xi_{n+1}I_A = E(\xi_n + \eta_{n+1})I_A = E\xi_n I_A + E\eta_{n+1}I_A. \quad (1) \]

Since \((\eta_1, \ldots, \eta_n)\) and \(\eta_{n+1}\) are independent, \(I_A\) which is \(F_n\)-measurable and \(\eta_{n+1}\) are independent. Then \(E\eta_{n+1}I_A = E\eta_{n+1}EI_A = 0\). \(I(A) = 0\). Then (1) implies \(E\xi_{n+1}I_A = E\xi_n I_A\). Therefore, \(E(\xi_{n+1} \mid \mathcal{F}_n) = \xi_n\).

**Proof of Lemma 2**

Because \((\xi_n, \mathcal{F}_n)_{n \geq 0}\) is a martingale, \(E(\xi_n \mid \mathcal{F}_0) = \xi_0\) for all \(n \in \mathbb{N}\).

By the definition of conditional expectation, \(E\xi_n I_n = E\xi_0 I_n\). In other words, \(E\xi_n = E\xi_0\).

---

**Return to the problem.** Let

\[ F_0 = \{ \emptyset, \Omega \}, \quad F_n = \sigma(X_1, X_2, \ldots, X_n) \quad \forall n \in \mathbb{N}, \]

\[ S_0 = 0, \quad S_n = |X_1| + |X_2| + \cdots + |X_n| \quad \forall n \in \mathbb{N}, \]

\[ c = E|X_1| \in \mathbb{R}. \]

First, we show that \(\tau\) is a stopping time with respect to the increasing filtration \((F_n)_{n \geq 0}\). For each \(n \in \{0, 1, 2, \ldots\}\),

\[ \{\omega : \tau(\omega) > n\} = \{\omega : S_k(\omega) \leq 0 \quad \forall k = 0, 1, \ldots, n\} \]

\[ = \{\omega : \omega \in S_k^{-1}((-\infty, 0]) \quad \forall k = 0, 1, \ldots, n\} \]

\[ = \bigcap_{k=0}^n S_k^{-1}((-\infty, 0]). \]

Each set \(S_k^{-1}((-\infty, 0])\) is \(F_k\)-measurable, thus is \(F_n\)-measurable. Then
\[ \{ w: \tau(w) > n \} \in \mathcal{F}_n. \]

Next, put \( S_n = 1X_n - c \) and \( Y_0 = 0, Y_n = S_1 + S_2 + \ldots + S_n = S_n - nc \). We show that \((Y_n, F_n)_{n \geq 0}\) is a martingale based on the proof of Lemma 1. It is clear that each \( Y_n \) is \( F_n \)-measurable. For each \( n \in \mathbb{N}, 1, 2, \ldots \), and \( A \in \mathcal{F}_n \),

\[ EY_{n+1}1_A = E(Y_n + S_n)1_A = EY_n1_A + ES_n1_A. \quad (2) \]

Because \((X_i, \ldots, X_n)\) and \( X_{n+1} \) are independent, \( 1_A \) and \( S_{n+1} \) are also independent. Then \( ES_n1_A = ES_n1_A = 0. \; 1_A = 0. \) Then (2) implies \( EY_{n+1}1_A = EY_n1_A \). Therefore, \( E(Y_{n+1}1_{F_n}) = Y_n \). We have showed that \((Y_n, F_n)_{n \geq 0}\) is a martingale.

Then \((Y_n, F_n)_{n \geq 0}\) and \((-Y_n, F_n)_{n \geq 0}\) are submartingales. By Doob's theorem, \((Y_{n+1}, F_n)_{n \geq 0}\) and \((-Y_{n+1}, F_n)_{n \geq 0}\) are submartingales. Then \((Y_{n+1}, F_n)_{n \geq 0}\) is a martingale. By Lemma 2,

\[ EY_{n+1} = EY_n1_{\infty} = EY_0 = E(0) = 0. \]

In other words, \( E(\bar{S}_{n+1} - (n+1)c) = 0 \). Then \( E\bar{S}_{n+1} = c \; E(n+1)c \).

Suppose by contradiction that \( E\tau < \infty \). Because \( 0 \leq n\tau \leq \tau \), \( E\tau < \infty \) and \( \lim_{n \to \infty} (n\tau) = \tau \), by the Dominated Convergence Theorem, we get

\[ \lim_{n \to \infty} E(n\tau) = E\tau. \]
Thus, \( \lim_{n \to \infty} \overline{S}_{n \tau 2} = c \overline{E} \). Because the sequence \((\overline{S}_{n \tau 2})_{n \geq 0}\) is nonnegative increasing and \( \lim_{n \to \infty} \overline{S}_{n \tau 2} = \overline{S} \), by the Monotone Convergence Theorem we get
\[
E \overline{S} = \lim_{n \to \infty} E \overline{S}_{n \tau 2} = c E \overline{S} < \infty.
\]
We have \(1 \leq \overline{S}_{n \tau 2} \leq \overline{S} \), \( E \overline{S} < \infty \) and \( \lim_{n \to \infty} \overline{S}_{n \tau 2} = \overline{S} \). By the Dominated Convergence Theorem,
\[
E \overline{S} = \lim_{n \to \infty} E \overline{S}_{n \tau 2}.
\]
(3)

Applying Lemma 1 for \( \eta_n = X_n \), \( S_n = S_n \), we conclude that \((S_n, F_n)_{n \geq 0}\) is a martingale. Then \((S_n, F_n)_{n \geq 0}\) and \((-S_n, F_n)_{n \geq 0}\) are submartingales. Thus, by Doob's theorem, \((S_{n \tau 2}, F_n)_{n \geq 0}\) and \((-S_{n \tau 2}, F_n)_{n \geq 0}\) are submartingales. Thus, \((S_{n \tau 2}, F_n)_{n \geq 0}\) is a martingale. By Lemma 2,
\[
E \overline{S}_{n \tau 2} = \bar{E} S_{n \tau 2} = \bar{E} S = 0. \text{ Then (3) implies } E \overline{S} = 0.
\]

By the definition of \( \tau \), \( \overline{S}_{\tau (\omega)}(\omega) > 0 \) for every \( \omega \in \Omega \). Then \( P(\overline{S} = 0) = 0 \). On the other hand, the identity \( E \overline{S} = 0 \) and the fact that \( \overline{S} > 0 \) imply \( \overline{S} = 0 \) a.s., which means \( P(\overline{S} = 0) = 1 \). This is a contradiction.

(3) Let \( \lambda > 0 \), \( \lambda + \mu \leq 1 \). Suppose by contradiction that there is a renewal sequence \( X_0, X_1, X_2, \ldots \) such that
\[ X_0 = 1, \]
\[ P(X_1 = 1) = \lambda + \mu, \quad (1) \]
\[ P(X_2 = 1) = \lambda^2 + \mu^2. \quad (2) \]

Define a sequence of random variables \( T_0, T_1, T_2, \ldots \) as follows.

\[ T_0 = 0, \quad T_m = \inf \{ n > T_{m-1} : X_n = 1 \}. \]

For each \( n \in \mathbb{N} \), we put

\[ A_n = \{ \omega : T_1 - T_0 = 1, T_2 - T_1 = 1, \ldots, T_n - T_{n-1} = 1 \}, \]
\[ B_n = \{ \omega : X_1(\omega) = 1, X_2(\omega) = 1, \ldots, X_n(\omega) = 1 \}. \]

We show that \( A_n = B_n \). For each \( \omega \in A_n \), \( T_1(\omega) = 1, T_2(\omega) = 2, \ldots, T_n(\omega) = n \).

By the definition of \( T_m \), we conclude that \( X_1(\omega) = 1, X_2(\omega) = 1, \ldots, X_n(\omega) = 1 \).

Thus \( \omega \in B_n \). For each \( \omega \in B_n \), the definition of \( T_m \) gives us \( T_1(\omega) = 1, T_2(\omega) = 2, \ldots, T_n(\omega) = n \). Then \( \omega \in A_n \).

Now that \( A_n = B_n \), \( P(A_n) = P(B_n) \). In other words,

\[ P(X_1 = 1, X_2 = 1, \ldots, X_n = 1) = P(T_1 - T_0 = 1, T_2 - T_1 = 1, \ldots, T_n - T_{n-1} = 1) \]
\[ = P(T_1 = 1) P(T_2 = 1) \cdots P(T_n = 1) \quad \text{(because } (X_n) \text{ is a renewal sequence}) \]
\[ = P(T_1 = 1)^n. \quad (3) \]

For \( n = 1 \), (3) gives \( P(X_1 = 1) = P(T_1 = 1) \). For \( n = 2 \), (3) gives

\[ P(X_1 = 1, X_2 = 1) = P(T_1 = 1)^2 = P(X_1 = 1)^2. \]
Then \( P(X_1 = 1)^2 = P(X_1 = 1, X_2 = 1) \leq P(X_2 = 1) \).

Substituting (4) and (2) into the above estimate, we get
\[
(\lambda + \mu)^2 \leq \lambda^2 + \mu^2.
\]

This is a contradiction because \( \lambda \) and \( \mu \) are two positive numbers.

4. We determine all \( q \in (0, 1] \) such that the sequence \((1, 0, q, q, q, \ldots)\)
is a potential sequence. That is to determine \( q \in (0, 1] \) such that there
exists a renewal sequence \( X_0, X_1, X_2, \ldots \) such that
\[
\begin{cases}
P(X_0 = 1) = 1, \\
P(X_1 = 1) = 0, \\
P(X_n = 1) = q \quad \forall n \geq 2.
\end{cases}
\]  

Suppose \( q \in (0, 1] \) is such a value. The waiting time \( T_1 \) was defined by
\[
T_1 = \inf \{ n \geq 1 : X_n = 1 \}.
\]

Define
\[
\Psi(s) = \sum_{n=0}^{\infty} P(X_n = 1) s^n, \quad \forall s \in (0, 1),
\]

\[
\Phi(s) = \sum_{n=0}^{\infty} P(T_1 = n) s^n.
\]

By Theorem 4, Friestedt–Gray page 493, \( \Psi(s) = \frac{1}{1 - \Psi(s)} \). This implies
\[
\Psi(s) = \frac{\Psi(s) - 1}{\Psi(s)} \quad \forall s \in (0, 1). \tag{2}
\]

Replacing the data in (1) into the expression of \( \Psi(s) \), we get
\[
\Psi(s) = 1 + \sum_{n=2}^{\infty} q^2 s^n = 1 + \frac{q^2}{1 - s} \quad \forall s \in (0, 1).
\]
Then (2) becomes 

\[ p(s) = \frac{qs^2}{qs^2 - s + 1} \quad \forall s \in \mathbb{C} \setminus \{0\}. \]

Let \( \mu = \frac{1}{q} \in (0, \infty) \). Then 

\[ p(s) = \frac{s^2}{s^2 - \mu s + \mu} \quad \forall s \in \mathbb{C} \setminus \{0\}. \]

Thus, 

\[ \frac{s^2}{s^2 - \mu s + \mu} = \sum_{n=0}^{\infty} P(T_1 = n) s^n \quad \forall s \in \mathbb{C} \setminus \{0\}. \quad (3) \]

Suppose \( \mu \neq 4 \). Then the polynomial \( s^2 - \mu s + \mu \) has two distinct roots \( \alpha_1, \alpha_2 \). They are nonzero, either real or complex. Then 

\[ \text{LHS}(3) = \frac{s^2}{(s - \alpha_1)(s - \alpha_2)} = \frac{s^2}{\alpha_1 - \alpha_2} \left( \frac{1}{\alpha_1^{j+1}} - \frac{1}{\alpha_1^{j+1}} \right) \cdot \frac{1}{A_2} \cdot \frac{1}{A_1}. \]

For \( 0 < s < \min \{ |\alpha_1|, |\alpha_2| \} \), we have the expansion

\[ A_2 = \frac{1}{\alpha_2} \sum_{j=0}^{\infty} \left( \frac{s}{\alpha_2} \right)^j = \sum_{j=0}^{\infty} \frac{s^j}{\alpha_2^{j+1}}. \]

\[ A_1 = \frac{1}{\alpha_1} \sum_{j=0}^{\infty} \left( \frac{s}{\alpha_1} \right)^j = \sum_{j=0}^{\infty} \frac{s^j}{\alpha_1^{j+1}}. \]

Then 

\[ \text{LHS}(3) = \frac{s^2}{\alpha_1 - \alpha_2} \sum_{j=0}^{\infty} \left( \frac{1}{\alpha_2^{j+1}} - \frac{1}{\alpha_1^{j+1}} \right) s^j = \frac{1}{\alpha_1 - \alpha_2} \sum_{j=0}^{\infty} \left( \frac{1}{\alpha_2^{j+1}} - \frac{1}{\alpha_1^{j+1}} \right) s^j. \]

Then (3) becomes 

\[ \frac{1}{\alpha_1 - \alpha_2} \sum_{n=0}^{\infty} \left( \frac{1}{\alpha_1^{n+1}} - \frac{1}{\alpha_2^{n+1}} \right) s^n = \sum_{n=0}^{\infty} P(T_1 = n) s^n \quad \forall 0 < s < \min \{ |\alpha_1|, |\alpha_2| \}. \]
Thus, $P(T_1=0) = 0$, $P(T_1=1) = 0$,
\[ P(T_1=n) = \frac{1}{\alpha_1 - \alpha_2} \left( \frac{1}{\alpha_2^{n-1}} - \frac{1}{\alpha_1^{n-1}} \right) \quad \forall n \geq 2. \quad (4) \]

By (3),
\[ \sum_{n=0}^{\infty} P(T_1=n) = \lim_{s \uparrow 1} \sum_{n=0}^{\infty} P(T_1=n) s^n \quad (\text{Nonzero Convergence Theorem}) \]
\[ = \lim_{s \uparrow 1} \frac{s^2}{s - \mu s + \mu} = 1. \]
Thus, $P(T_1=\infty) = 0$.

Consider $1 \leq \mu < 4$
\[ \alpha_1 = \frac{\mu + \sqrt{4\mu - \mu^2}}{2}, \quad \alpha_2 = \frac{\mu - \sqrt{4\mu - \mu^2}}{2} = \bar{\alpha_1}. \]

Then (4) can be rewritten as
\[ P(T_1=n) = \frac{1}{(\alpha_1 \bar{\alpha_1})^{n-1}} \frac{\alpha_1^{n-1} - \bar{\alpha_1}^{n-1}}{\alpha_1 - \bar{\alpha_1}} = \frac{1}{\mu^{n-1}} \frac{2 \Im \alpha_1^{n-1}}{2 \Im \alpha_1}. \quad \forall n \geq 2. \]

This implies $\Im \alpha_1^{n-1} \geq 0$ for all $n \geq 2$. Write $\alpha_1 = re^{i\theta}$ for $r > 0$, $\theta \in [0, 2\pi)$. Since $\alpha_1 \notin \mathbb{R}$, $\theta \notin \{0, \pi\}$. Thus $\theta \in (0, \pi)$ or $\theta \in (\pi, 2\pi)$.
\[ 0 \leq \Im \alpha_1^{n-1} = \Im r^{n-1} e^{i(n-1)\theta} = r^{n-1} \sin(n-1)\theta \quad \forall n \geq 2. \]
Thus, $\sin n\theta > 0$ for all $n \in \mathbb{N}$. In particular, $\sin \theta > 0$. Thus $\theta \in (0, \pi)$.
Then $\frac{n}{\theta} > 1$. There exists $n_0 \in \mathbb{N}$ such that $\frac{n}{\theta} < n_0 < \frac{2\pi}{\theta}$. Then $\pi < n_0 \theta < 2\pi$. Then $\sin n_0 \theta < 0$, which is a contradiction.
Consider $\mu > 4$

$$
\alpha_1 = \frac{\mu + \sqrt{\mu^2 - 4\mu}}{2}, \quad \alpha_2 = \frac{\mu - \sqrt{\mu^2 - 4\mu}}{2}
$$

Then

$$
P(T_1 = n) = \frac{1}{(\alpha_1 \alpha_2)^{n-1}} \left( \frac{\alpha_1 - \alpha_2}{\alpha_1 \alpha_2} \right) = \frac{1}{\mu^{n-1} \sqrt{\mu^2 - 4\mu}} \left( \frac{(\mu + \sqrt{\mu^2 - 4\mu})^{n-1} - (\mu - \sqrt{\mu^2 - 4\mu})^{n-1}}{2} \right)
$$

We get the distribution of the waiting time $T_1$

$$
P(T_1 = 0) = P(T_1 = 1) = P(T_1 = \infty) = 0,
$$

$$
P(T_1 = n) = \frac{1}{\mu^{n-1} \sqrt{\mu^2 - 4\mu}} \left( \frac{(\mu + \sqrt{\mu^2 - 4\mu})^{n-1} - (\mu - \sqrt{\mu^2 - 4\mu})^{n-1}}{2} \right) \quad \forall n \geq 2
$$

Here $\mu = \frac{1}{q}$.

Consider $\mu = 4$

Then (3) becomes

$$
\frac{s^2}{(s-2)^2} = \sum_{n=0}^{\infty} P(T_1 = n) s^n \quad \forall s \in (0, 1). \quad (6)
$$

Letting $s \uparrow 1$ and using the Monotone Convergence Theorem on the right hand side, we get $1 = \sum_{n=0}^{\infty} P(T_1 = n)$. Thus, $P(T_1 = \infty) = 0$.

We know that

$$
\frac{1}{1-t} = 1 + t + t^2 + \ldots = \sum_{j=0}^{\infty} t^j \quad \forall t \in (-1, 1).
$$

Take the derivative both sides,

$$
\frac{1}{(1-t)^2} = 1 + 2t + 3t^2 + \ldots = \sum_{j=0}^{\infty} (j+1) t^j \quad \forall t \in (-1, 1).
$$
Substituting $t$ with $\frac{s}{2}$, we get

$$\frac{4}{(2-s)^2} = \sum_{j=0}^{\infty} \frac{j+1}{2^j} s^j \quad \forall s \in (0,1).$$

Then

$$\text{LHS}(6) = \sum_{j=0}^{\infty} \frac{j+1}{2^{j+2}} s^{j+2} = \sum_{n=2}^{\infty} \frac{n-1}{2^n} s^n.$$

Then (6) becomes

$$\sum_{n=2}^{\infty} \frac{n-1}{2^n} s^n = \sum_{n=0}^{\infty} p(T_1 = n) s^n \quad \forall s \in (0,1).$$

Therefore,

$$\begin{cases} p(T_1 = 0) = p(T_1 = 1) = p(T_1 = \infty) = 0, \\
p(T_1 = n) = \frac{n-1}{2^n} \quad \forall n \geq 2. \end{cases} \quad (7)$$

So far we have showed that if a renewal sequence exists, $q$ must be in $(0, \frac{1}{4}]$. Also, in that case the waiting time distribution is given by (5) for $q \in (0, \frac{1}{4})$, and by (7) for $q = \frac{1}{4}$.

Now we show that every $q \in (0, \frac{1}{4}]$ is an admissible value. That is, for a fixed $q \in (0, \frac{1}{4}]$ we determine a renewal sequence $X_0, X_1, X_2, \ldots$ whose potential sequence is $(1, 0, q, q, q, \ldots)$. Let $T_1$ be a $\{0, 1, 2, \ldots\}$-valued random variable with distribution given by (5) if $q \in (0, \frac{1}{4})$, or given by (7) if $q = \frac{1}{4}$. Then
\[ \frac{S^2}{S^2 - \mu s + \mu} = \sum_{n=0}^{\infty} P(\bar{T}_n = n) s^n \quad \forall s \in (0, 1), \]

where \( \mu = \frac{1}{\eta} \). Denote

\[ \bar{\Phi}(s) = \sum_{n=0}^{\infty} P(\bar{T}_n = n) s^n = \frac{S^2}{S^2 - \mu s + \mu} \quad \forall s \in (0, 1). \quad (8) \]

This is the probability generating function of \( \bar{T}_1 \). Let \( \bar{Z}_1, \bar{Z}_2, \bar{Z}_3, \ldots \) be an independent and identically distributed sequence of random variables.

Define \( \bar{T}_0 = 0 \) and \( \bar{T}_n = \bar{Z}_1 + \bar{Z}_2 + \cdots + \bar{Z}_n \) for each \( n \geq 2 \). Define a sequence of random variables \( \bar{X}_0, \bar{X}_1, \bar{X}_2, \ldots \) as follows.

\[ \bar{X}_0(\omega) = 1 \quad \forall \omega \in \Omega, \]

\[ \bar{X}_n(\omega) = \begin{cases} 1 & \text{if } n = \bar{T}_m(\omega) \text{ for some } m \geq 0, \\ 0 & \text{otherwise} \end{cases} \quad \forall n \geq 1. \]

\( \bar{X}_n \) is indeed a random variable because

\[ \{ \omega : \bar{X}_n(\omega) = 1 \} = \left\{ \omega : \exists m \geq 0 \text{ such that } n = \bar{T}_m(\omega) \right\} = \left\{ \omega : \exists m \geq 0 \text{ such that } \omega \in \bar{T}_m^{-1}(\{n\}) \right\} = \bigcup_{m=0}^{\infty} \bar{T}_m^{-1}(\{n\}), \]

which is a measurable set. We now show that for \( r \geq 1 \),

\[ \bar{T}_r(\omega) = \inf \left\{ n > \bar{T}_{r-1}(\omega) : \bar{X}_n(\omega) = 1 \right\} \quad \text{a.s.} \]

Put \( A = \{ \omega : \bar{T}_r(\omega) = \bar{T}_{r-1}(\omega) \} \). Then \( A \subset \{ \omega : \bar{Z}_r(\omega) = 0 \} \). Then
\[ P(A) \leq P(Z_r = 0) = P(\tilde{T}_r = 0) = 0. \]

Let \( w \in A^r \). By the definition of \( \tilde{X}_n \), \( \tilde{X}_{\tilde{T}_r}(w) = 1 \). Thus \( \tilde{T}_r(w) \in \{ n > \tilde{T}_{r-1}(w) : \tilde{X}_n(w) = 1 \} \). For each \( n > \tilde{T}_{r-1}(w) \) such that \( \tilde{X}_n(w) = 1 \), there exists \( m > 0 \) such that \( n = \tilde{T}_m(w) \). Because \( \tilde{T}_k \) is an increasing sequence and \( n > \tilde{T}_{r-1}(w), \ m > r-1 \). Then \( m > r \). Then \( \tilde{T}_m(w) \geq \tilde{T}_r(w) \). Then \( n > \tilde{T}_r(w) \). Then
\[ \tilde{T}_r(w) \leq \inf \{ n > \tilde{T}_{r-1}(w) : \tilde{X}_n(w) = 1 \}. \]
Thus, \( \tilde{T}_r(w) = \inf \{ n > \tilde{T}_{r-1}(w) : \tilde{X}_n(w) = 1 \} \) a.s.

For \( n \geq 2 \) and \( k_1, k_2, \ldots, k_n \geq 1 \),
\[ P(\tilde{T}_1 - \tilde{T}_0 = k_1, \tilde{T}_2 - \tilde{T}_1 = k_2, \ldots, \tilde{T}_n - \tilde{T}_{n-1} = k_n) = P(Z_1 = k_1, Z_2 = k_2, \ldots, Z_n = k_n) \]
\[ = P(Z_1 = k_1) P(Z_2 = k_2) \cdots P(Z_n = k_n) \]
\[ = P(\tilde{T}_1 = k_1) P(\tilde{T}_2 = k_2) \cdots P(\tilde{T}_n = k_n). \]
Thus, \( \tilde{X}_0, \tilde{X}_1, \tilde{X}_2, \ldots \) is a renewal sequence. Put
\[ \widetilde{\Phi}(s) = \sum_{n=0}^{\infty} P(\tilde{X}_n = 1) s^n \quad \forall s \in (0,1). \quad (3) \]

By Theorem 4, Fristedt-Gray page 433,
\[ \widetilde{\Phi}(s) = \frac{1}{1 - \widetilde{\Phi}(s)}. \]
Replacing \( \widetilde{\Phi}(s) \) given by (8) into this expression, we get
\[ \tilde{\varphi}(s) = \frac{1}{1 - \frac{s^2}{s^2 - \mu s + \mu}} = 1 + \frac{s^2}{-\mu s + \mu} = 1 + \frac{qs^2}{1-s} \]

\[ = 1 + qs^2 (1 + s + s^2 + s^3 + \ldots) \]

\[ = 1 + \sum_{n=2}^{\infty} q s^n \quad \forall s \in (0,1). \]

Thus,
\[ \sum_{n=0}^{\infty} \mathbb{P}(\tilde{X}_n = 1) s^n = 1 + \sum_{n=2}^{\infty} q s^n \quad \forall s \in (0,1). \]

Then,
\[ \mathbb{P}(\tilde{X}_n = 1) = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n = 1, \\ q & \text{if } n \geq 2. \end{cases} \]

Therefore, \((1,0,q,q,q,\ldots)\) is a potential sequence.