Theory of Probability and Measure Theory – Math 8652

Homework #2

1) (Problem 13.23 in the textbook) Let \( \phi \) be the moment generating function of a \([0, \infty)\]-valued random variable \( X \). Show that \( \phi \) is continuous on \((0, \infty)\) and if \( P(X < \infty) = 1 \), then on \([0, \infty)\).

2) Let \( X_1, X_2, \ldots \) be nonnegative independent random variable having the same distribution. Assume that \( P(X_1 > 0) > 0 \). Prove that

\[
\sum_{n=1}^{\infty} X_n = \infty \quad \text{a.s.}
\]

3) (Problem 13.40 in the textbook) Let \( X_n, n = 1, 2, \ldots \), be iid nonnegative random variables and let \( N \) be a \( \{\infty, 1, 2, \ldots\} \)-valued random variable independent of \((X_1, X_2, \ldots)\). Express the moment generating function of \( S = X_1 + \ldots + X_N \) through the moment generating function of \( X_1 \) and the probability generating function of \( N \). Then find \( ES \) and \( \text{Var} S \).

4) (Theorem 14.19 in the textbook) A sequence of distributions on \([0, \infty)\) converges to a distribution \( Q \) on \([0, \infty)\) if and only if the sequence of the corresponding moment generating functions converges pointwise on \([0, \infty)\) to a function \( \phi \) that is continuous at zero. Moreover, \( \phi \) is the moment generating function of \( Q \).