Let $X$ be a Banach space over $C$, $\mathcal{L}(X)$ be the Banach space of linear continuous map from $X$ to $X$. For each $A \in \mathcal{L}(X)$, we denote

$$\sigma(A) := \{ \lambda \in C : (\lambda I - A) \text{ is not invertible} \}.$$  

We know that $\sigma(A)$ is not empty and is compact. But $\sigma(A) = \max \{ |\lambda| : \lambda \in \sigma(A) \}$

If $X$ is finite dimensional then $\sigma(A)$ is the set of eigenvalues, i.e. the set of roots of the characteristic polynomial. By using the Argument Principle to count the number of roots enclosed in a circle, we can show that the eigenvalues of $A$ depend continuously on $A$. Hence, $\sigma(A)$ depends continuously on $A$. When $X$ is infinite dimensional, however, this conclusion is not true. A counterexample was constructed by Kakutani, found in the book by C. E. Rickart "General theory of Banach algebras" (1960), page 282:

Let $X$ be a separable Hilbert space over $C$, with a complete orthonormal sequence $(f_m)$. For each $m \in \mathbb{N}$, we put

$$\alpha_m = e^{-l}, \text{ where } m = 2^s \text{ and } s \text{ is odd}.$$  

Define a linear map $A : X \rightarrow X$, $A f_m = \alpha_m f_{m+1}$ for all $m \in \mathbb{N}$. Because $\alpha_m \in [0,1]$ for all $m \in \mathbb{N}$, $A$ is continuous and $\|A\| \leq 1$.

We show that $\sigma(A) > 0$.

$$A^2 f_1 = A(\alpha_1 f_2) = \alpha_2 \alpha_3 f_3,$$

$$A^3 f_1 = A(\alpha_1 \alpha_2 f_3) = \alpha_3 \alpha_4 \alpha_5 f_4,$$

$$\cdots$$

$$A^n f_1 = \alpha_1 \alpha_2 \cdots \alpha_n f_{n+1}, \quad \forall n \in \mathbb{N}.$$
Thus, $\|A^n\|=\lambda_1\lambda_2\cdots\lambda_n$. Take $n=2^k-1$. Then
\[
\lambda_1, \lambda_2, \ldots, \lambda_n \in \{\exp(-j) : 0 \leq j < k\}.
\]
We want to count the frequency of occurrence of $\exp(-j)$ in $\lambda_1, \lambda_2, \ldots, \lambda_n$. This is equal to the number of odd numbers $s$ such that $1 \leq 2^j s < 2^k$. Such $s$ lie in the set $\{1, 3, 5, \ldots, 2^{k-j} - 1\}$. This set has $2^{k-j-1}$ elements. Hence, $\exp(-j)$ occurs $2^{k-j-1}$ times in $\lambda_1, \lambda_2, \ldots, \lambda_n$. Thus,
\[
\lambda_1\lambda_2\cdots\lambda_n = \prod_{j=0}^{k-1} (\exp(-j))^{2^{k-j-1}} = \exp\left(\sum_{j=0}^{k-1} j 2^{k-j-1}\right) = \exp\left(-2\sum_{j=0}^{k-1} \frac{j}{2^{k-j+1}}\right).
\]
Then $\|A^n\| = \|A^n\|_n = (\lambda_1\lambda_2\cdots\lambda_n)^{1/n} = \exp\left(-\frac{2}{2^k} \sum_{j=0}^{k-1} \frac{j}{2^{k-j+1}}\right)$.

Using Gelfand's formula and taking the limit as $k \to \infty$, we get
\[
f(A) \geq \exp\left(-\frac{\sum_{j=0}^{\infty} \frac{j}{2^{j+1}}}{2^k}\right) = S > 0.
\]

Now we construct a sequence $(A_k)$ in $\mathcal{L}(X)$ which approximates $A$. For each $k \in \mathbb{N}$, we define a linear map $A_k : X \to X$,

\[
A_k f = \begin{cases}
0 & \text{if } m = 2^k s \text{ with } s \text{ odd}, \\
A_m f & \text{otherwise}.
\end{cases}
\]

Because $A_m \in [0, I]$ for all $m \in \mathbb{N}$, $A_k$ is continuous and $\|A_k\| \leq 1$. We show that $A_k \to A$ in $\mathcal{L}(X)$.

\[
(A - A_k)f = \begin{cases}
A_m f = \exp(-k) f_{m+1} & \text{if } m = 2^k s, \text{ s odd} \\
0 & \text{otherwise}.
\end{cases}
\]

Then $\|A - A_k\| \leq \exp(-k)$. Hence $A_k \to A$ in $\mathcal{L}(X)$.

By the definition of $A_k$ at (*), it is a nilpotent operator. In fact,
$A_k^2 = 0$ because $A_k^m = 0$ for all $m \in \mathbb{N}$. We now show that every nilpotent operator has spectrum equal to $\{0\}$. Let $B \in \mathcal{L}(X)$ be a nilpotent, i.e. $B^n = 0$ for some $n \in \mathbb{N}$. For each $A \in \mathcal{C} \setminus \{0\}$,

\[
(\lambda - B) (\lambda^{n-1} + \lambda^{n-2} B + \cdots + \lambda B^{n-2} + B^{n-1}) = \lambda^n - B^n = \lambda^n I
\]

\[
(\lambda^{n-1} + \lambda^{n-2} B + \cdots + \lambda B^{n-2} + B^{n-1}) (\lambda - B) = \lambda^n - B^n = \lambda^n I
\]

Thus, $(\lambda - B)^{-1} = \frac{1}{\lambda} B \in \mathcal{L}(X)$. Thus, $A \in \mathcal{C} \setminus \sigma(B)$. Then $\mathcal{C} \setminus \{0\} \subset \mathcal{C} \setminus \sigma(B)$, which leads to $\sigma(B) \subset \{0\}$. Because $\sigma(B) \neq \emptyset$, $\sigma(B) = \{0\}$.

By this, we have showed that $\sigma(A_k) = 0$ for all $k \in \mathbb{N}$. Thus, $(A_k)$ is a sequence in $\mathcal{L}(X)$ approaching to $A$ but $\lim_{k \to \infty} \sigma(A_k) = 0 < \sigma(A)$. 