Let $X$ be a topological space. We'll use the Mayer-Vietoris sequence to compute the homology groups of $X \times S^1$.

We put $U' = \{ e^{i2\pi t} \mid \frac{1}{2} < t < 2 \}$, $V = \{ e^{i2\pi t} \mid -\frac{1}{2} < t < 1 \}$.

Then $U = X \times U'$ and $V = X \times V'$ satisfy: $U$ and $V$ are open in $X \times S^1$, and $U \cup V = X \times S^1$. Thus, we have the Mayer-Vietoris sequence

\[
\cdots \rightarrow H_n(U \cup V) \rightarrow H_n(U) \oplus H_n(V) \rightarrow H_n(X \times S^1) \rightarrow H_{n-1}(U \cup V) \rightarrow H_{n-1}(U) \oplus H_{n-1}(V) \rightarrow H_{n-1}(X \times S^1) \rightarrow \cdots
\]

We see that $U \cap V = (X \times U') \cap (X \times V') = X \times (U' \cap V')$. Put
\[ W_1' = \{ e^{i2\pi t} | \frac{1}{2} < t < 1 \} \text{ and } W_2' = \{ e^{i2\pi t} | \frac{3}{2} < t < 2 \}, \] we have

\[ U' \sqcup V' = W_1' \sqcup W_2' \] and this is a disjoint union. Then

\[ U \sqcup V = (X \times W_1') \sqcup (X \times W_2') \]

Thus, \( H_n(U \sqcup V) = H_n(X \times W_1') \oplus H_n(X \times W_2') \) for all \( n > 0 \).

Since \( U' \) is contractible to a point, denoted by \( *_1 \), the following map is a homotopy equivalence:

\[ i_* : U \to X \times \{ *_1 \} \]

\[ (x, a) \mapsto (x, *_1) \text{ for all } x \in X, a \in U' \]

Similarly, we have a homotopy equivalence:

\[ j_* : V \to X \times \{ *_2 \} \]

\[ (x, b) \mapsto (x, *_2) \text{ for all } x \in X, b \in V' \]

We can identify \( X \times \{ *_1 \} \) and \( X \times \{ *_2 \} \) with \( X \) and write \( i' : U \to X \), \( i'_* = pr_x \), and

\[ j' : V \to X \text{, } j'_* = pr_x. \]

Then \( i' \) and \( j' \) induce the following homology homomorphisms

\[ i_* : H_n(U) \to H_n(X), \] and \[ j_* : H_n(V) \to H_n(X), \]

\[ i'_*(x) = pr_x \circ x, \] and \[ j'_*(x) = pr_x \circ x. \]

In other words, for each singular simplex \( \sigma : \Delta^n \to U = X \times U' \), we have

\[ i'_*(\sigma) = pr_x \circ \sigma : \Delta^n \to X. \] The singular explanation applies for \( j'_* \).
The injections $U \rightarrow V$ and $V \rightarrow U$ also induce homology homomorphisms $i_*: H_n(U \cap V) \rightarrow H_n(U)$ and $j_*: H_n(U \cap V) \rightarrow H_n(V)$. For each $z \in H_n(U \cap V)$, $z$ can be written uniquely as a sum $z = z_1 + z_2$ where $z_1 \in H_n(X \times W'_1)$ and $z_2 \in H_n(X \times W'_2)$. We define the map

$$
\varphi: H_n(U \cap V) \rightarrow H_n(X) \oplus H_n(X),
$$

$$
z = z_1 + z_2 \mapsto (pr_X(z_1), pr_X(z_2)),
$$

where the projection $pr_X$ is understood as above.

$$
\xrightarrow{\Delta^n} X \times W'_1 \xrightarrow{pr_X} X
$$

Then by definition, $\varphi$ is an isomorphism. We have the commutative diagram

$$
\begin{array}{ccc}
H_n(U) & \xrightarrow{i_*} & H_n(X) \\
\downarrow{i_*} & & \downarrow{pr_X} \\
H_n(U \cap V) & \xrightarrow{\varphi} & H_n(X) \oplus H_n(X) \\
\downarrow{j_*} & & \downarrow{p} \\
H_n(V) & \xrightarrow{j_*} & H_n(X) \\
\end{array}
$$

where $p(a, b) = a + b$ for all $a, b \in H_n(X)$.

Indeed, for each $z \in H_n(U \cap V)$, we can write $z = z_1 + z_2$ for $z_1 \in H_n(X \times W'_1)$ and $z_2 \in H_n(X \times W'_2)$. Then
Note that $i_*$ and $j_*$ are isomorphisms due to homotopy equivalence. Thus, the Mayer-Vietoris sequence can be replace by the following exact sequence:

$$
\cdots \to H_n(X) \oplus H_n(X) \xrightarrow{\mathbf{f}} H_n(X) \oplus H_n(X) \xrightarrow{\mathbf{g}} H_n(X \times S^1) \to \cdots
$$

$$
\cdots \to H_{n-1}(X) \oplus H_{n-1}(X) \to H_{n-1}(X) \oplus H_{n-1}(X) \to H_{n-1}(X \times S^1) \to \cdots
$$

$$
\cdots \to H_i(X) \oplus H_i(X) \to H_i(X) \oplus H_i(X) \to H_i(X \times S^1) \to \cdots
$$

$$
\cdots \to H_0(X) \oplus H_0(X) \to H_0(X) \oplus H_0(X) \to H_0(X \times S^1) \to 0.
$$

The function $f$ and $g$ are given by $f : H_n(X) \oplus H_n(X) \to H_n(X) \oplus H_n(X)$, 
$$
(a, b) \mapsto (a+b, a+b),
$$
and $g : H_n(X) \oplus H_n(X) \to H_n(X \times S^1)$, 
$$
(a, b) \mapsto a-b.
$$

We have $\text{Im} \mathbf{f} = \{(a+b, a+b) / a, b \in H_n(X) \}$

$$
= \{(a+b)(x_1 + x_2) / a, b \in H_n(X) \}$
}
\[
\sum a(1,1) / a \in H_n(X)^2 \cong H_n(X)
\]

This set subgroup of \(H_n(X) \oplus H_n(X)\) has a direct summand \(\{a(1,0) / a \in H_n(X)^2\}\). Thus,

\[
(H_n(X) \oplus H_n(X)) / \text{Im}f \cong H_n(X). \quad \text{Since } \text{Im}f = \ker g, \text{we have}
\]

\[
\text{Im}g \cong (H_n(X) \oplus H_n(X)) / \ker f = (H_n(X) \oplus H_n(X)) / \text{Im}f \cong H_n(X).
\]

For \(n \neq 1\), we have the sequence,

\[
H_n(X \times S^1) \xrightarrow{h} H_{n-1}(X) \oplus H_{n-1}(X) \xrightarrow{k} H_{n-1}(X) \oplus H_{n-1}(X),
\]

where \(k\) is defined like \(f\), with \(n-1\) instead of \(n\). Specifically,

\[
\ell(a, b) = (a+b, a+b).
\]

Then \(\ker(k) = \{a(1, b) \in H_{n-1}(X) \oplus H_{n-1}(X) \mid a+b = 0\}\)

\[
= \{a(-1, -a) \mid a \in H_{n-1}(X)\}
\]

\[
= \{a(1, 1) \mid a \in H_{n-1}(X)\} \cong H_{n-1}(X)
\]

This subgroup of \(H_{n-1}(X) \oplus H_{n-1}(X)\) has a direct summand \(\{a(0,0) / a \in H_{n-1}(X)^2\}\).

Thus, by the exactness, \(\text{Im}h = \ker(k) \cong H_{n-1}(X)\). Thus,

\[
H_n(X \times S^1) / \ker h \cong \text{Im}h \cong H_{n-1}(X). \quad \text{The exact sequence}
\]

\[
H_n(X) \oplus H_n(X) \xrightarrow{g} H_n(X \times S^1) \xrightarrow{h} H_{n-1}(X) \oplus H_{n-1}(X)
\]

gives us the following short exact sequence

\[
0 \rightarrow \text{Im}g \xrightarrow{\iota} H_n(X \times S^1) \xrightarrow{\text{proj}} H_n(X \times S^1) / \ker h \rightarrow 0
\]

If we can show that this sequence is split, then
\[ H_n(\mathbb{T}) \cong H_n(\mathbb{Z}^2) \oplus \left( H_n(X \times S^1) / \ker \varphi \right) \cong H_n(X) \oplus H_{n-1}(X). \]

To show that the above short exact sequence is split, we should look for a map \( \psi \) such that \( \psi \circ i = \text{id}_{\text{im} \varphi} \).

\[
\begin{array}{c}
0 \rightarrow \text{im} \varphi \xrightarrow{i} H_n(X \times S^1) \xrightarrow{\varphi} H_n(X \times S^1) / \ker \varphi \xrightarrow{\psi} 0
\end{array}
\]

In other words, we should find a function \( \psi : H_n(X \times S^1) \rightarrow \text{im} \varphi \) such that \( \psi(a-b) = (a-b) \) for all \( a, b \in H_n(X) \). It is hard to construct such a function in general case. We will, however, construct it for a specifically simple case, \( X = [0, 1] \subset \mathbb{R} \). Then \( X \times S^1 \) is obtained by edge identification. Just use the retraction \( X \times S^1 \rightarrow X \times [0, 1] \).

The open sets \( U \) and \( V \) are denoted in the picture. The shaded region is \( U \cap V \).

We take two special vertical lines in each connected component of \( U \cap V \).

In general space \( X \), such a line is \( X \times \{1, 3\} \), where \( * \) lies in the intersection \( U \cap V \).
For each cycle $\sigma \in H_n(X \times S^1)$, the two vertical lines will determine two cycles $\sigma_1 \in H_n(U)$ and $\sigma_2 \in H_n(V)$ such that $\sigma = \sigma_1 - \sigma_2$.

The decomposition from $\sigma$ into $(\sigma_1, \sigma_2)$ is denoted by function $\psi: H_n(X \times S^1) \to H_n(U) \oplus H_n(V)$, $\psi(\sigma) = (\sigma_1, \sigma_2)$. Now we'll show that $\psi_i = \id \imath_{\psi_i}$. For $\lambda_1 \in H_n(U), \lambda_2 \in H_n(V)$, we have $g(\lambda_1, \lambda_2) = \lambda_1 - \lambda_2$.

Up to the boundary of a high dimensional domain, we have $\lambda_1 = \lambda'_1$ and $\lambda_2 = \lambda'_2$. Thus $\psi(\lambda_1 - \lambda_2) = \psi(\lambda'_1 - \lambda'_2) = (\lambda'_1, \lambda'_2) = (\lambda_1, \lambda_2)$. Thus $\psi \circ g(\lambda_1, \lambda_2) = (\lambda_1, \lambda_2)$.

Assuming that the sequence $0 \to \imath_{\psi} \to H_n(X \times S^1) \to H_n(X \times S^1) / \ker \psi \to 0$ is also split in general cases, we conclude that $H_n(X \times S^1) \cong H_n(X) \oplus H_{n-1}(X)$ for all $n \geq 1$. For $n = 0$, we have the exact sequence

\[ H_0(X) \oplus H_0(X) \xrightarrow{f} H_0(X) \oplus H_0(X) \xrightarrow{r} H_0(X \times S^1) \to 0 \]
We still have \( \text{Im} f \simeq H_0(X) \) and \((H_0(X) \oplus H_0(X))/\text{Im} f \simeq H_0(X)\).

Since \( \text{Im} f = \ker r \), we have \((H_0(X) \oplus H_0(X))/\ker r \simeq H_0(X)\). Since \( r \) is surjective, \( H_0(X \times S^n) \simeq (H_0(X) \oplus H_0(X))/\ker r \simeq H_0(X)\). In conclusion,

\[
H_n(X \times S^n) \simeq \begin{cases} H_0(X) \\ H_{n-1}(X) \end{cases} \quad \text{for } n > 1, \quad H_0(X) \quad \text{for } n = 0.
\]

2. Let \( A, B, X \) be topological spaces such that \( A \subseteq B \subseteq X \). First, we'll show that there is a short exact sequence of chain complexes:

\[
0 \rightarrow C_\ast(B, A) \rightarrow C_\ast(X, A) \rightarrow C_\ast(X, B) \rightarrow 0
\]

The inclusions \( A \subseteq B \subseteq X \) induces the inclusions of chains \( C_n(A) \subseteq C_n(B) \subseteq C_n(X) \).

Consider the boundary operator \( \partial : C_n(X) \rightarrow C_{n-1}(X) \). We have \( \partial(C_n(B)) = C_{n-1}B \) and \( \partial(C_n(A)) = C_{n-1}(A) \). Therefore, we have a commutative diagram

\[
\begin{array}{ccc}
C_n(B)/C_n(A) & \xrightarrow{\tilde{i}} & C_n(X)/C_n(A) \\
\downarrow & & \downarrow \\
C_n(B)/C_{n-1}(A) & \xrightarrow{\tilde{\iota}} & C_{n-1}(X)/C_{n-1}(A)
\end{array}
\]

where the maps are given by

\[
\begin{array}{ccc}
z + C_n(A) & \xrightarrow{\tilde{\iota}} & z + C_n(A) \\
\downarrow & & \downarrow \\
\partial(z + C_{n-1}(A)) & \rightarrow & \partial(z + C_{n-1}(A))
\end{array}
\]

for all \( z \in C_n(B) \).

Likewise, we have the following commutative diagram
\[ C_n(X)/C_n A \xrightarrow{j} C_n X/C_n B \]

\[ \begin{array}{ccc}
\downarrow & \uparrow & \\
\downarrow & \uparrow & \\
\tilde{\iota} & \tilde{\tau} & \\
C_{n-1} X/C_{n-1} A & \xrightarrow{\tilde{\jmath}} & C_{n-1} X/C_{n-1} B
\end{array} \]

where the maps are given by \[ \tilde{\iota} : \tilde{\iota} \mapsto \tilde{\iota} + C_n A \xrightarrow{j} \tilde{\iota} + C_n B \] for all \( \tilde{\iota} \in C_{n-1} X/C_{n-1} B \).

Also, \( \tilde{\iota} \) and \( j \) defined above are respectively injective and surjective. Thus we have the commutative diagram

\[ \begin{array}{ccc}
0 & \xrightarrow{\iota} & C_n X/C_n A \xrightarrow{j} C_n X/C_n B \xrightarrow{\iota} 0 \\
\downarrow & \downarrow & \downarrow \\
0 & \xrightarrow{\tilde{\iota}} & C_{n-1} X/C_{n-1} A \xrightarrow{\tilde{\jmath}} C_{n-1} X/C_{n-1} B \xrightarrow{\tilde{\iota}} 0
\end{array} \]

By definition, \( C_n B/C_n A = C_n(B, A) \), \( C_n X/C_n A = C_n(X, A) \) and \( C_n X/C_n B = C_n(X, B) \).

Therefore, we obtain an exact sequence of chain complexes:

\[ \begin{array}{ccc}
0 & \xrightarrow{\iota} & C_n(B, A) \xrightarrow{\iota} C_n(X, A) \xrightarrow{\iota} C_n(X, B) \xrightarrow{\iota} 0 \\
\downarrow & \downarrow & \downarrow \\
0 & \xrightarrow{\tilde{\iota}} & C_{n-1}(B, A) \xrightarrow{\tilde{\jmath}} C_{n-1}(X, A) \xrightarrow{\tilde{\iota}} C_{n-1}(X, B) \xrightarrow{\tilde{\iota}} 0 \\
\downarrow & \downarrow & \downarrow \\
0 & \xrightarrow{\iota} & C_0(B, A) \xrightarrow{\iota} C_0(X, A) \xrightarrow{\iota} C_0(X, B) \xrightarrow{\iota} 0
\end{array} \]
Next, whenever we have a short exact sequence of chain complexes
\[0 \to C_*(B,A) \to C_*(X,A) \to C_*(X,B) \to 0,\]
we have a long exact sequence of homology groups (by Zigzag Lemma).

\[\cdots \to H_n(B,A) \to H_n(X,A) \to H_n(X,B) \to \cdots\]

\[\cdots \to H_{n-1}(B,A) \to H_{n-1}(X,A) \to H_{n-1}(X,B) \to \cdots\]

\[\cdots \to H_1(B,A) \to H_1(X,A) \to H_1(X,B) \to \cdots\]

\[\cdots \to H_0(B,A) \to H_0(X,A) \to H_0(X,B) \to 0\]

(3) Let \(X\) be a topological space and \(A, U, V \subset X\) such that \(X = UV\), \(A \subset UV\) and \(U, V\) are open in \(X\). We'll show that there is a Mayer-Vietoris sequence relating \(H_*(X,A), H_*(U,A), H_*(V,A)\) and \(H_*(UV,A)\).

Proof. We see that \(A \subset UV \subset U\). Thus, by the previous exercise, there is an exact sequence of chain complexes
\[0 \to C_*(UV,A) \to C_*(U,A) \to C_*(U,UV) \to 0.\]

Also, we see that \(A \subset V \subset X\). Thus, there is an exact sequence:
\[0 \to C_*(V,A) \to C_*(X,A) \to C_*(X,V) \to 0.\]
Because \( A \subset UNV \subset X \), we have a commutative diagram

\[
\begin{array}{ccc}
A \subset V \subset X \\
\end{array}
\]

of chain complexes:

\[
\begin{array}{ccc}
0 \rightarrow C_*(U \cup V, A) & \rightarrow & C_*(U, A) & \rightarrow & C_*(U, UnV) & \rightarrow & 0 \\
\downarrow i & & \downarrow \gamma & & \downarrow \tilde{\gamma} & & \\
0 \rightarrow C_*(V, A) & \rightarrow & C_*(X, A) & \rightarrow & C_*(X, V) & \rightarrow & 0 \\
\end{array}
\]

where \( \tilde{\gamma} \) is the chain map induced by \( i : U \rightarrow X \). This diagram induces a diagram of homology groups:

\[
\begin{array}{ccc}
H_n(U \cup V, A) & \rightarrow & H_n(U, A) & \rightarrow & H_n(U, UnV) \\
\downarrow i_* & & \downarrow \gamma_* & & \downarrow \tilde{\gamma}_* & & \\
H_n(V, A) & \rightarrow & H_n(X, A) & \rightarrow & H_n(X, V) \\
\end{array}
\]

Moreover, the diagram of chain complexes induces a connecting homomorphism \( d_* \) such that

\[
\begin{array}{ccc}
H_n(U, UnV) & \xrightarrow{\partial_*} & H_{n-1}(U \cup V, A) \\
\downarrow & & \downarrow & & \\
H_n(X, V) & \xrightarrow{\partial_*} & H_n(V, A) \\
\end{array}
\]

commutes.

Therefore, we have a long diagram of homology groups:

\[
\begin{array}{ccc}
\rightarrow \ldots \rightarrow & H_n(U \cup V, A) & \rightarrow H_n(U, A) & \rightarrow H_n(U, UnV) & \xrightarrow{\partial_*} & H_{n-1}(U \cup V, A) & \rightarrow \ldots \\
\downarrow & & \downarrow & & \downarrow i_* & & \\
\rightarrow \ldots \rightarrow & H_n(V, A) & \rightarrow H_n(X, A) & \rightarrow H_n(X, V) & \xrightarrow{\partial_*} & H_{n-1}(V, A) & \rightarrow \ldots
\end{array}
\]
By the previous exercise, the upper and lower rows are exact. We see that $U^c$ is closed in $X$, and $U^c = X \setminus U \subseteq V$. Thus $U^c$ is closed in $V$. Thus by excision, we have $H_n(X, V) \cong H_n(X \setminus U^c, V \setminus U^c)$. Since $X \setminus U^c = X \setminus (X \setminus U) = U$ and $V \setminus U^c = V \setminus U$, we have $H_n(X, V) \cong H_n(U, U \cap V)$. Thus the map in the previous diagram is an isomorphism for all $n \geq 0$. Thus the diagram gives us the following exact sequence.

\[ \cdots \to H_n(U \cap V, A) \to H_n(U, A) \oplus H_n(V, A) \to H_n(X, A) \to \cdots \]

\[ \to H_{n-1}(U \cap V, A) \to H_{n-1}(U, A) \oplus H_{n-1}(V, A) \to H_{n-1}(X, A) \to \cdots \]

\[ \to H_1(U \cap V, A) \to H_1(U, A) \oplus H_1(V, A) \to H_1(X, A) \to \cdots \]

\[ \to H_0(U \cap V, A) \to H_0(U, A) \oplus H_0(V, A) \to H_0(X, A) \to 0 \]

This is the Mayer-Vietoris sequence that we need to find.

5) Let $M$ be a $k$-manifold, $k \geq 0$, and $p \in M$. We will compute the relative homology groups $H_*(M, M \setminus \{p\})$.

For $k=0$, $M$ is just a single point. Thus $M \setminus \{p\} = \emptyset$. Thus $H_n(M, \emptyset) = H_n(M) = \mathbb{Z}$ if $n = 0$,

$0$ if $n > 1$. 

For $k > 0$, we have $H_n(M, M \setminus \{p\}) = \mathbb{Z}$ if $n = 0$, and $0$ if $n > 1$. 

For $k > 1$, we have $H_0(M, M \setminus \{p\}) = \mathbb{Z}$, and $0$ for $n > 0$.
Now we only consider the case $k > 1$, which implies $M \setminus \{p\} \neq \emptyset$. First, we'll prove the following lemma:

Let $f: A \to B$ be a homeomorphism and $C, D$ be subsets of $A$ and $B$ respectively such that $D = f(C)$. Then $H_n(A, C) \cong H_n(B, D)$ for all $n \geq 0$.

Proof The idea is to use the five-lemma. Since $C \subset A$, we have the exact sequence of homology groups:

$$\ldots \to H_n(C) \to H_n(A) \to H_n(A, C) \to H_{n-1}(C) \to H_{n-1}(A) \to \ldots$$

Since $D \subset B$, we have the exact sequence of homology groups:

$$\ldots \to H_n(D) \to H_n(B) \to H_n(B, D) \to H_{n-1}(D) \to H_{n-1}(B) \to \ldots$$

Since $A \xrightarrow{f} B$ and $C \xrightarrow{f} D$, we have the commutative diagram where the vertical maps are induced by $f$:

$$\begin{array}{cccccccc}
\ldots & \to & H_n(C) & \to & H_n(A) & \to & H_n(A, C) & \to & H_{n-1}(C) & \to & H_{n-1}(A) & \to & \ldots \\
&(u) & \downarrow & (v) & \downarrow & (w) & \downarrow & (x) & \downarrow & (y) & \downarrow & (z) & \\
\ldots & \to & H_n(D) & \to & H_n(B) & \to & H_n(B, D) & \to & H_{n-1}(D) & \to & H_{n-1}(B) & \to & \ldots \\
\end{array}$$

The homology groups of index $n - 1$ are replaced by $0$ in case $n = 0$. The group morphisms $(u)$ and $(w)$ are isomorphisms because $C \xrightarrow{f} D$. The morphisms $(v)$ and $(z)$ are also isomorphisms because $A \xrightarrow{f} B$. Thus $(w)$ is an isomorphism by five-lemma. Q.E.D. //
Return to the problem. Since $M$ is a $k$-manifold, $p$ has a Euclidean neighborhood $U$, i.e. $U \cong \mathbb{R}^k$ and $U$ is open in $M$. We have $U^c = M \setminus U^c$. Thus, we can apply the excision property here. We have

$$H_n(M, M \setminus \{p\}) \cong H_n(M \setminus U^c, (M \setminus \{p\}) \setminus U^c) = H_n(U, U \setminus \{p\}).$$

Let $f : U \to \mathbb{R}^k$ be an homeomorphism with $f(p) = q$. Then we can apply the above lemma for $A = U$, $B = \{p\}$, $C = \mathbb{R}^k$, and $D = Eq$. We have

$$H_n(U, U \setminus \{p\}) \cong H_n(\mathbb{R}^k, \mathbb{R}^k \setminus \{q\}).$$

We see that any translation map in $\mathbb{R}^k$ is a homeomorphism. Thus $q$ can be taken zero. Moreover, the map $g : D_k = \{x \in \mathbb{R}^k | \|x\| < 1\} \to \mathbb{R}^k$ with $g(\dot{x}) = \frac{x}{\|x\|}$ is a homeomorphism sending $0$ to $0$. By the above lemma, we have

$$H_n(\mathbb{R}^k, \mathbb{R}^k \setminus \{0\}) \cong H_n(D_k, D_k \setminus \{0\}).$$

Next we'll show that $H_n(D_k, D_k \setminus \{0\}) \cong H_n(D_k, S^{k-1})$ by homotopy invariance of homology groups and the five-lemma.

The map $r : D_k \setminus \{0\} \to S^{k-1}$, $r(\dot{x}) = \frac{x}{\|x\|}$ is a deformation retraction.

Thus, the side inclusion $i : S^{k-1} \to D_k \setminus \{0\}$ is a homotopy equivalence. Since we have the inclusions $D_k \setminus \{0\} \hookrightarrow D_k$ and $S^{k-1} \hookrightarrow D_k$, there is a commutative
diagram where the vertical maps are induced by the inclusion maps.

\[ \cdots \rightarrow H_n(S_{k-1}) \rightarrow H_n(D_k) \rightarrow H_n(D_k, S_{k-1}) \rightarrow H_{n-1}(S_{k-1}) \rightarrow H_{n-1}(D_{k-1}) \rightarrow \cdots \]

(1) \[ \downarrow \]

(2) \[ \downarrow \]

(3) \[ \downarrow \]

(4) \[ \downarrow \]

(5) \[ \downarrow \]

\[ \cdots \rightarrow H_n(D_k \backslash \{p\}) \rightarrow H_n(D_k) \rightarrow H_n(D_k, D_k \backslash \{p\}) \rightarrow H_{n-1}(D_k \backslash \{p\}) \rightarrow H_{n-1}(D_{k-1}) \rightarrow \cdots \]

The homology groups of index $n-1$ will be replaced by 0 in case $n=0$. (2) and (4) are identity maps, (1) and (3) are isomorphisms because $i: S_{k-1} \rightarrow D_{k-1}$ is a homotopy equivalence. Thus, by the five lemma, (3) is an isomorphism. Thus $H_n(D_k, D_k \backslash \{p\}) \cong H_n(D_k, S_{k-1})$. Up to now, we have,

\[ H_n(M, M \backslash \{p\}) \cong H_n(D_k, S_{k-1}) = \begin{cases} \mathbb{Z} & \text{if } n = k \\ 0 & \text{if } n \neq k \end{cases} \]

Combining this result with the case $k=0$, we have

\[ H_n(M, M \backslash \{p\}) = \begin{cases} \mathbb{Z} & \text{if } n = k \\ 0 & \text{if } n \neq k \end{cases}, \quad \text{for all } k \geq 0. \]

Therefore, only $H_k$ is nontrivial where $k$ was given as a "dimension" of $M$. Suppose that $M$ is also an $m$-manifold with $k \neq m$. Then $H_n(M, M \backslash \{p\})$ is nontrivial. This is a contradiction because we knew that only $H_k$ is nontrivial. Therefore, $k$ is a well-defined dimension of a manifold with respect to a given point $p \in M$. To show that $k$ is actually independent
Continue the proof of the lemma of problem 4

\[ V_n \cap A_m = (X \setminus \{x_p : p > n^3\}) \cap A_m = (X \setminus \{x_p : n \leq p < m^3\}) \cap A_m. \]

Since the set \( \{x_p : n \leq p < m^3\} \) is finite, and \( X \) is a T_1 space, this set is closed in \( X \). Thus \( X \setminus \{x_p : n \leq p < m^3\} \) is open in \( X \). Thus \( V_n \cap A_m \) is open in \( A_m \).

Because this is true for all \( m \in \mathbb{N} \), we conclude \( V_n \) is open in \( X \). Since

\[ A \subset X = \bigcup_{k=1}^{\infty} V_k, \]

and \( A \) is compact, there exist \( i_1 < i_2 < \cdots < i_m \) such that \( A \subset \bigcup_{k=1}^{i_m} V_k \). Since the sequence \((V_k)\) is ascending, we have \( A \subset V_{i_m} \). This is a contradiction because we knew that \( x_{i_m} \in A \setminus V_{i_m} \).
of the choice of points \( p, p' \), we need the connectedness of \( M \).

Now assume that \( M \) is connected and \( p, p' \in M \). We'll show that \( H_n(M, M \setminus \{p\}) \cong H_n(M, M \setminus \{p'\}) \) for all \( n \geq 0 \). Since \( M \) is a manifold, it is locally path-connected. Together with the its connectedness, we conclude that \( M \) is path-connected. Thus there exists a path \( \gamma : [0,1] \rightarrow M \) connecting \( p \) and \( p' \). In problem 3, HW 4, we showed that there exists a number \( m \in \mathbb{N} \) and open subsets \( U_0, U_1, \ldots, U_m \) of \( M \) such that \( U_j \cong \mathbb{R}^k \) and \( \gamma \left( \left[ \frac{j}{m}, \frac{j+1}{m} \right] \right) \subseteq U_j \) for all \( 0 \leq j < m \), where \( k \) is the dimension of \( M \) corresponding to point \( p \). Put \( p_j = \gamma \left( \frac{j}{m} \right) \) for all \( 0 \leq j \leq m \). Then \( p = \gamma (0) = p_0 \) and \( p' = \gamma (1) = p_m \). To show that \( H_n(M, M \setminus \{p\}) \cong H_n(M, M \setminus \{p'\}) \), we will show that \( H_n(M, M \setminus \{p_j\}) \cong H_n(M, M \setminus \{p_{j+1}\}) \) for all \( 0 \leq j < m \). Since \( p_j, p_{j+1} \in U_j \) and \( U_j \cong \mathbb{R}^k \), what we need to show is equivalent to show that \( H_n(M, M \setminus \{p_j\}) \cong H_n(M, M \setminus \{p_{j+1}\}) \) provided that there exists an open subset \( U \) of \( M \) such that \( p, p' \in U \) and \( U \cong \mathbb{R}^k \) (homeomorphic). Like before, by excision property, we have

\[
H_n(M, M \setminus \{p\}) \cong H_n(U, U \setminus \{p\})
\]

\[
H_n(M, M \setminus \{p\}) \cong H_n(U, U \setminus \{p\})
\]
Since $U$ and $\mathbb{R}^k$ are homeomorphic, there are corresponding $q$ and $q'$ in $\mathbb{R}^k$ such that $H_n(U, U \setminus \{q\}) \cong H_n(\mathbb{R}^k, \mathbb{R}^k \setminus \{q\})$ and $H_n(U, U \setminus \{q'\}) \cong H_n(\mathbb{R}^k, \mathbb{R}^k \setminus \{q'\})$. The translation $p: x \mapsto x + q' - q$ is a homeomorphism from $\mathbb{R}^k$ to $\mathbb{R}^k$ mapping $q$ to $q'$. Thus,

$$H_n(\mathbb{R}^k, \mathbb{R}^k \setminus \{q\}) \cong H_n(\mathbb{R}^k, \mathbb{R}^k \setminus \{q'\}) = H_n(\mathbb{R}^k, \mathbb{R}^k \setminus \{q'\})$$

We assume the following lemma for a moment:

**Lemma.** Let $X$ be a $T_1$-space and subspaces $A_1 \subset A_2 \subset A_3 \ldots$ be such that $\bigcap_{k=1}^\infty A_k = X$ and a subspace $U$ is closed in $X$ if and only if $U \cap A_n$ is closed in $A_n$ for all $n \geq 1$. Then for every compact set $A \subset X$, there exists $j \in \mathbb{N}$ such that $A \subset A_j$.

Let $X$ be a space satisfying the conditions in the above lemma. First, we will show that each element in $H_k(X)$ is the image of an element in $H_k(A_j)$ for some $j \in \mathbb{N}$. Let $[\alpha] \in H_k(X)$, where $\alpha$ is a cycle in $X$. We have $\alpha = \sum_{k=1}^m n_k \delta_k$ with $d\alpha = 0$ in $C_{k-1}(X)$, where

$$\delta_k: \Delta^k \to X$$

is a singular $k$-simplex for all $k = 1, \ldots, m$. Since $\delta_k$ is continuous and $p(\Delta^k)$ is compact, $\delta_k(\Delta^k)$ is a compact subspace of $X$. Using the
lemma, we conclude that there exists \( j \in \mathbb{N} \) such that \( \sigma_x(A^j) \subset A_j \). Put \( j = \max \{ j_1, j_2, \ldots, j_m \} \). Then \( \sigma_x(A^j) \subset A_j \) for all \( l = 1, \ldots, m \). Thus \( \sigma_x \in C_k(A_j) \). Thus, \( \tilde{z} = \sum_{l=1}^{m} n_l \sigma_x \in C_k(A_j) \).

Since \( \partial \tilde{z} = 0 \) in \( C_{k-1}(X) \) and \( C_{k-1}(A_j) \subset C_{k-1}(X) \), we have \( \partial \tilde{z} = 0 \) on \( C_{k-1}(A_j) \).

Therefore, \([\tilde{z}]\) with respect to \( X \) is the image of \([z]\) with respect to \( A_j \). In other words, the inclusion \( A_j \hookrightarrow X \) induces a homomorphism \( i_*: H_k(A_j) \rightarrow H_k(X) \) such that \( i_*([z]) = [\tilde{z}] \).

Secondly, we'll show that two elements in \( H_k(A_j) \) become the same in \( H_k(X) \) iff there exists \( j' \succ j \) such that their images in \( H_k(A_{j'}) \) coincide.

(\( \Leftarrow \)) Let \([z_1], [z_2] \in H_k(A_j)\), where \( z_1 \) and \( z_2 \) are cycles in \( C_k(A_j) \), such that there exists \( j' \succ j \) with \( i_*([z_1]) = i_*([z_2]) \), where \( i \) is the inclusion \( A_j \hookrightarrow A_{j'} \). Then \( z_1 - z_2 = \partial c \) where \( c \in C_{k+1}(A_{j'}) \). Then \( c \in C_{k+1}(X) \). Then \( z_1 - z_2 \) is equal to a boundary of a \((k+1)\)-chain in \( X \). Thus \([z_1]\) with respect to \( X \) is the same as \([z_2]\) with respect to \( X \).

(\( \Rightarrow \)) Let \([z_1], [z_2] \in H_k(A_j) \) where \( z_1 \) and \( z_2 \) are two cycles in \( C_k(A_j) \) such that \( i_*([z_1]) = i_*([z_2]) \) where \( i \) is the inclusion \( A_j \hookrightarrow X \). Then there exists \( c \in C_{k+1}(X) \) with \( \partial c = z_1 - z_2 \). Thus \( i_*([z_1]) = i_*([z_2]) \).
Such that $\zeta_1 - \zeta_2 = \delta \zeta$. We write $\zeta = \sum_{t=1}^{n} \delta \zeta_t$ where each $\delta \zeta_t$ is a continuous function from $\mathbb{R}^{k+1}$ to $X$. Like before, there exists an index $j' \geq j$ such that $\delta \zeta (d^{+}) \subset \mathbb{A}_{j'}$. Thus $\delta \zeta \in C_{k+1}(\mathbb{A}_{j'})$. Thus $\zeta \in C_{k+1}(\mathbb{A}_{j'})$. Therefore, $\zeta_1 - \zeta_2$ is the boundary of a $(k+1)$-chain in $\mathbb{A}_{j'}$. Thus, $[\zeta_1]$ with respect to $C_{k}(\mathbb{A}_{j'})$ is the same as $[\zeta_2]$ with respect to $C_{k}(\mathbb{A}_{j'})$.

**Proof of the Lemma**

Let $X$ be a space as in the hypothesis. Suppose that a subspace $V$ of $X$ satisfies $V \cap A_n$ is open for all $n \in \mathbb{N}$. Then, if we write $V^c := X \setminus V$ then $V^c \cap A_n = A_n \setminus (A_n \cap V)$, which is open in $A_n$ for all $n \in \mathbb{N}$. Thus, $V^c$ is closed in $X$. Thus $V$ is open in $X$. This means any subspace of $X$ whose intersection with $A_n$ is open in $A_n$ is also open in $X$.

Let $A$ be a compact subspace of $X$. Suppose by contradiction that $A \not\subset A_n$ for all $n \in \mathbb{N}$, then there exists $x_0 \in A \setminus A_n$. We define $V_n = X \setminus \{x_m : m > n\} = X \setminus \{x_n, x_{n+1}, \ldots\}$. Then $V_1 \subset V_2 \subset V_3 \subset \ldots$ and $\bigcup_{k=1}^{\infty} V_k = X$. We'll show that $V_n \cap A_m$ is open in $A_m$ for every $m, n \in \mathbb{N}$. Since $x_p \notin A_p$ and $A_m \subset A_p$ for all $p > m$, we have $x_p \notin A_m$ for all $p > m$. Therefore, (proof continues after page 15 in this Problem set)