1. For an integer \( n \) and a real number \( R > 0 \), find the effect of the map \( w \mapsto (Rw)^n : S^1 \to C \setminus \{0\} \) on \( \pi_1 \).

Proof. We will follow the following steps:

Step 0: We know that \( \pi_1(S^1, 1) \) is a cyclic group generated by the path \( \gamma : [0, 1] \to S^1, \gamma(t) = e^{2\pi i t} \). If we denote 
\[
\gamma^k = \gamma \circ \cdots \circ \gamma \quad \text{(mod homotopy equivalence)},
\]
then 
\[
\pi_1(S^1, 1) = \{ \gamma^k : k \in \mathbb{Z} \}.
\]

Step 1: Put \( f : S^1 \to C \setminus \{0\}, f(w) = (Rw)^n \). Then \( f(1) = R^n \).

Step 2: \( f^* : \pi_1(S^1, 1) \to \pi_1(C \setminus \{0\}, R^n) \) is defined by 
\[
f^*(\alpha) = f \circ \alpha \quad \forall \alpha \in \pi_1(S^1, 1).
\]

Details: Step 1 Put \( A = \{ w \in C : |w| = R^n \} \). We'll show that 
\[
\pi_1(C \setminus \{0\}, R^n) = \pi_1(A, R^n).
\]
Put $\mathbb{R}^n = \mathbb{R}^n \setminus \{0\}$, $a = \mathbb{R}^n > 0$.

We define $g : C \setminus \{0\} \rightarrow A$ as

$$g(z) = \frac{a \cdot z}{|z|}.$$

Then $g$ is a retraction of $C \setminus \{0\}$ on $A$.

Consider $H : (C \setminus \{0\}) \times [0,1] \rightarrow (C \setminus \{0\})$

$$H(t,s) = sz + (1-s) \frac{a \cdot z}{|z|}.

Then $H$ is a basepoint-preserving homotopy from $id_{C \setminus \{0\}, a}$ to $g$. Therefore each loop $\tilde{\gamma}$ based at $a$ in $C \setminus \{0\}$ will be homotopic to $\tilde{\gamma}'$, based at $a$ in $A$ where

$$\gamma'(t) = H(\gamma(t), 1) = H(\tilde{\gamma}(t), 1).$$

Why? Put $H : [0,1] \times [0,1] \rightarrow X$ defined by

$$H(t,s) = \gamma(H(t), s).$$

Then $H$ is continuous and

$$H(t,0) = H(\gamma(t), 0) = \tilde{\gamma}(t),
$$

$$H(t,1) = H(\gamma(t), 1) = \tilde{\gamma}'(t),
$$

$$H(0,s) = H(\tilde{\gamma}(0), s) = H(a, s) = a$$

because $H$ is basepoint-preserving.

$$H(1,s) = H(\tilde{\gamma}(1), s) = H(a, s) = a$$

therefore $\gamma'$ is path-homotopic to $\gamma$. Thus $\gamma \in \pi_1(A, a)$. Thus

$$\pi_1(C \setminus \{0\}, a) \subseteq \pi_1(A, a).$$

Since $\pi_1(A, a) \subseteq \pi_1(C \setminus \{0\}, a)$, we have

$$\pi_1(C \setminus \{0\}, a) = \pi_1(A, a).$$
Therefore, \( \pi_1 (C \setminus \{0\}, \mathbb{R}^n) = \pi_1 (A, \mathbb{R}^n) \).

The map \( \phi \circ h : S^1 \to A \) is a homeomorphism.

Thus \( \phi \circ h \circ \pi_1 (S^1, 1) \to \pi_1 (A, \mathbb{R}^n) \) is a group isomorphism. Because \( \pi_1 (S^1, 1) = \langle \alpha \rangle \) - cyclic group generated by \( \alpha \), which is the loop \( [0,1] \to S^1 \), \( \alpha(t) = e^{2\pi i t} \), we have

\[ \pi_1 (A, \mathbb{R}^n) = \langle \alpha \rangle \text{-cyclic group generated by } \alpha, \text{ which is the loop } [0,1] \to A, \ x(t) = f_0 \circ \beta(t) = h(e^{2\pi i t}) = R^n e^{2\pi i t}. \text{ Therefore,} \]

\[ \pi_1 (C \setminus \{0\}, \mathbb{R}^n) = \langle \alpha \rangle = \{ \alpha^k : k \in \mathbb{Z} \} \]

Step 2: By definition, \( f_\ast : \pi_1 (S^1, 1) \to \pi_1 (C \setminus \{0\}, \mathbb{R}^n) \) such that

\[ f_\ast(\beta) = f \circ \beta. \text{ Thus } f_\ast(\gamma) = f \circ \gamma \ast \phi. \text{ Particularly,} \]

\[ f_\ast(\gamma)(t) = f(\gamma(t)) = \phi(\gamma(t)) = R^n \gamma(t) = R^n e^{2\pi i nt} \]

Put \( \beta = f_\ast(\gamma). \) We have \( \beta(t) = R^n e^{2\pi i nt}. \) Thus

\[ \beta \bigg|_{[0,\pi]} = \alpha \bigg|_{[0,\pi]} \]

\[ \beta \bigg|_{[\pi, 2\pi]} = \alpha \bigg|_{[\pi, 2\pi]} \]

\[ \beta \bigg|_{[3\pi, 4\pi]} = \alpha \bigg|_{[3\pi, 4\pi]} \]

\[ \beta \bigg|_{[4\pi, 5\pi]} = \alpha \bigg|_{[4\pi, 5\pi]} \]
Therefore, as prove in Problem 3, Homework 4, $\beta$ is path-homotopic to $x \cdot x \cdots x = x^n$. Thus, $f^*(x) = x^n$. Thus,

$$f^*(x^k) = x^{nk} \quad \forall k \in \mathbb{Z}$$

This is the effect of $f$ on $\pi_1$, what we need to find. In the special case $n = 0$, $f^*$ is is the trivial homomorphism. Otherwise, $\text{Im} f^* = \pi_1(A_{\text{away}})$ and $f^*$ is surjective. \(4/4\)

(2) Show that if a polynomial $p(x)$ with complex coefficients has no zeros, then the induced map $\pi_1(S^1, 1) \to \pi_1(C \setminus \{0\}, f(1))$ sends all elements to the identity.

**Proof.** Because $\pi_1(S^1, 1)$ is a cyclic group generated by $r: (0,1) \to S^1$, $r(t) = e^{i\pi t}$, it suffices to show that $f^*(r)$ is path-homotopic to the constant path $\widetilde{f}(t) : [0,1] \to f(1)$. Put $\tilde{r} = f^*(r) = f \circ r$.

![Diagram](image)

We have $f : C \to C$ and $r$ is a curve loop based at 1 in C. Thus, there is a homotopy $H : \Delta^1 \times C \setminus \{0\} \to C$, $H(t,s) = s + (1-s)r(t)$.
between $F$ and the constant map $t \mapsto 1$. Put $\tilde{H} = f \circ H$.

Then $\tilde{H}$ is a homotopy between $\tilde{F} = f \circ \tilde{h}$ and the constant map $t \mapsto f(1)$. Since $f(1)$ is never zero, $\tilde{H} : [0, 1] \times [0, 1] \rightarrow \mathbb{C} \setminus \{0\}$. Thus $\tilde{F}$ and $f(1)$ are path-homotopic in $\mathbb{C} \setminus \{0\}$, which concludes the proof.

Combining Prob. 1, 2. of Homework 6 and Prob. 2 of Homework 4 to show the Fundamental theorem of Algebra:

For $R > 0$, we put $g_R : S^1 \rightarrow \mathbb{C} \setminus \{0\}$

$$g_R(w) = (Rw)^n,$$

and $f_R : S^1 \rightarrow \mathbb{C} \setminus \{0\}$

$$f_R(w) = f(Rw),$$

where $f$ is a polynomial of complex coefficients and of order $n \geq 1$, which Here we assumed that $f$ had no zero in $\mathbb{C}$, and we will try to find a contradiction. For $R$ sufficiently large, Problem 2 in Homework 4 said that there exists a
homotopy between them

\[ S^1 \xrightarrow{H} \mathbb{C} \backslash \{0\} \]

where \( H : S^1 \times [0,1) \to \mathbb{C} \backslash \{0\} \), \( H(x,0) = f_R(x) \) and \( H(x,1) = g_R(x) \).

\[ \gamma(s) = H(d(s)) \]

\[ \gamma(0) = H(d(0)) = f_R(1) \]

\[ \gamma(1) = H(d(1)) = g_R(1) \]

Then there exists a path \( \gamma \) in \( \mathbb{C} \backslash \{0\} \) from \( f_R(1) \) to \( g_R(1) \). Then there exists \( \phi \) such that \( f_R = g_R \circ \phi \).

\[ \pi_1(S^1,1) \xleftarrow{f_R^*} \pi_1(\mathbb{C} \backslash \{0\}, g_R(1)) \]

\[ \pi_1(\mathbb{C} \backslash \{0\}, g_R(1)) \xrightarrow{f_R^*} \pi_1(S^1,1) \]

\[ \phi(x) = \gamma^{-1} \cdot x \cdot \gamma \]

Problem 1 said that \( f_R^* \) is injective (since \( n \geq 1 \)). Thus, \( f_R^* = g_R^* \circ \phi \) is injective. This contradicts problem 2, which says \( f_R^* \) must be trivial on \( \pi_1(S^1,1) \).

3) Show that the fundamental group of the \( n \)-sphere \( S^n \) is trivial for \( n > 1 \) by directly showing that any loop \( \gamma \) is homotopic to the trivial loop.
Proof. Here we will consider in detail the case $n=2$ because it will give us insight, probably sufficient, to deal with any case $n>2$.

Let $\gamma : [0,1] \rightarrow S^2$, $\gamma(0) = \gamma(1)$ be a loop on $S^2$. We will show that $\gamma$ is homotopic to the trivial loop $t \in [0,1] \mapsto \gamma(0)$ by the following steps:

1) If $\text{Im} \gamma = \gamma([0,1]) \neq S^2$ then there exists a $C^\infty \setminus \gamma([0,1])$, or equivalently $\gamma([0,1]) \subset S^2 \setminus \{a\}$. The idea then is to show use the stereographic projection $P$ at pole $a$ the project $\gamma$ onto a plane $\mathbb{R}$, a loop $\tilde{\gamma} \subset \mathbb{R}$ in $\mathbb{R}^2$.

Since $\mathbb{R}^2$ is strongly contractible, $\tilde{\gamma}$ is homotopic to the trivial loop $t \in [0,1] \mapsto \tilde{\gamma}(0)$. Then $\gamma$, as the image of $\tilde{\gamma}$ under $P^{-1}$, is also homotopic to the trivial loop $t \in [0,1] \mapsto \gamma(0)$ on $S^2$.

Note that $S^2$ in our circumstance is a modelling object, or mathematical object. It is described as a set of all points of distance 1 from the origin.
of a certain Cartesian coordinate system \((x_1,x_2,x_3)\).

\[ S^2 = \{(x_1,x_2,x_3) : x_1^2 + x_2^2 + x_3^2 = 1\} \]

Thus \(S^2 \setminus \{a\}\) and \(S^2 \setminus \{(0,0,1)\}\) are really different sets. The stereographic projection is meant for the latter. The real object (object in the real life) that \(S^2\) models is the sphere in our intuition, in which the spheres punctured at any (one) point are topologically the same. To fit our intuition, it is necessary to show that \(S^2 \setminus \{a\}\) and \(S^2 \setminus \{(0,0,1)\}\) are homeomorphic.

2) Find a homeomorphism from \(S^2 \setminus \{a\}\) to \(S^2 \setminus \{(0,0,1)\}\).

To do so, we only need to find a rotation, or a chain of rotations on \(S^2\) that maps \(a\) to \((0,0,1)\).

We will rotate in the \((x_1,x_2)\)-plane first, then rotate in the \((x_2,x_3)\)-plane. First, if \(a = (0,0,1)\), the rotation is just the identity map.

Now if \(a \neq (0,0,1)\) then \(a = (x_1,x_2,x_3)\) has \(x_1^2 + x_2^2 \neq 0\).

If \(a = (0,0,-1)\) then we simply have the symmetry

\[ S^2 \setminus \{a\} \to S^2 \setminus \{(0,0,1)\} \]

\[ w \mapsto -w \]

which is a homeomorphism.
If \( a \neq (0,0,\pm1) \), then \( a = (a_1, a_2, a_3) \) with \( a_1^2 + a_2^2 \neq 0 \). We will find two rotations as follows:

\[
\begin{align*}
A &\rightarrow a' \quad A' \rightarrow a'' \quad (0,1,1) \\
(a_1, a_2, a_3) &\rightarrow (0, \frac{a_2}{\sqrt{a_1^2 + a_2^2}}, a_3) \rightarrow (0,0, a_3')
\end{align*}
\]

Here

\[
A = \begin{pmatrix}
\cos x & -\sin x \\
\sin x & \cos x
\end{pmatrix}
\]

We need \( Aa = a'' \), which is equivalent to \( a_1 \cos x - a_2 \sin x = 0 \). We can choose such an \( x \) that \( \cos x = \frac{a_2}{\sqrt{a_1^2 + a_2^2}} \) and \( \sin x = \frac{a_1}{\sqrt{a_1^2 + a_2^2}} \).

If \( a'' = (0, a_2', a_3) \) is already \( a' = (0,0,1) \), \( A' \) is just the identity map. Otherwise, we perform another rotation

\[
A'' = \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \beta & -\sin \beta \\
0 & \sin \beta & \cos \beta
\end{pmatrix}
\]

We need \( A'a'' = a' \), which is equivalent to \( a_2' \cos \beta - a_3 \sin \beta = 0 \). We can choose \( \beta \) such that \( \cos \beta = \frac{a_3}{\sqrt{a_2^2 + a_3^2}} \) and \( \sin \beta = \frac{a_2}{\sqrt{a_2^2 + a_3^2}} \).

If \( a' = (0,0,a_3') \) is already \( (0,0,1) \) then \( A'' \) is just the identity map. Otherwise, we take another \( 90^\circ \) rotation, or symmetry

\[
A'' = \begin{pmatrix}
1 & 1 & 0 \\
1 & -1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]
Therefore, \( A^nA' A \) is a rotation in \( S^2 \) that maps \( a \) to \((0,0,1)\).

3) If \( \gamma([0,1]) \neq S^2 \) then there exists \( a \in S^2 \) such that \( \gamma([0,1]) - \{a\} \neq S^2 \).

Then there exists a homeomorphism \( \phi : S^2 \setminus \{a\} \to \mathbb{R}^2 \) that maps \( \gamma \) into a loop \( \tilde{\gamma} \) in \( \mathbb{R}^2 \).

Since \( \tilde{\gamma} \sim \tilde{\gamma}(0) \), we get \( \gamma \sim \gamma(0) \).

path-homotopic

4) If \( \gamma([0,1]) = S^2 \), i.e., \( \gamma \) is a sphere-filling curve, then all we need is another \( \gamma' \) in \( S^2 \) such that \( \gamma' \sim \gamma \) and \( \gamma'([0,1]) \neq S^2 \). The strategy is to divide \([0,1]\) into smaller segments and deal with \( \gamma \) on each segment.

5) We define \( U_1 = \{(x_1, x_2, x_3) \in S^2 : x_3 > \frac{1}{2}\} \)

\[ U_2 = \{(x_1, x_2, x_3) \in S^2 : x_3 < \frac{1}{2}\} \]

Then \( \{U_1, U_2\} \) is an open cover of \( S^2 \).
Such that $U_1 \subset S^2$ and $U_2 \subset S^2$. Then $\varphi^{-1}(U_1), \varphi^{-1}(U_2)$ is an open cover of $[0,1]$. Since $[0,1]$ is compact, this cover has a Lebesgue number $\varepsilon = \frac{1}{N}$ for some $N \in \mathbb{N}$, i.e. for every subset $A \subset [0,1]$, if $A$ has the diameter of at most $\frac{1}{N}$, then $A$ is contained in either $\varphi^{-1}(U_1)$ or $\varphi^{-1}(U_2)$. We subdivide $[0,1]$ into $N$ intervals of equal length. For each $k = 0, \ldots, N-1$, we define $\alpha_k$ as follow:

\[
[0,1] \xrightarrow{\frac{k}{N} \pm \frac{\varepsilon}{2}} \left( \frac{k}{N}, \frac{k+1}{N} \right) \xrightarrow{\varphi} \varphi \left( \left[ \frac{k}{N}, \frac{k+1}{N} \right] \right)
\]

Then $\alpha_k(0) = \varphi \left( \frac{k}{N} \right)$, $\alpha_k(1) = \varphi \left( \frac{k+1}{N} \right)$. As proved in Problem 3, Homework 4, $Y = \alpha_0 \cdot \alpha_1 \cdots \alpha_{N-1}$. Because $Y$ is a sphere-filling curve, each $\alpha_k$ may be very "large", i.e. $\alpha_k([0,1])$ may contain an nonempty open subset of $S^2$.

6) For each $k = 0, \ldots, N-1$, we try to replace $\alpha_k$ by a "smaller" path $\alpha'_k$. Because $\alpha_k([0,1]) = \varphi \left( \left[ \frac{k}{N}, \frac{k+1}{N} \right] \right)$, which is contained in either $U_1$ or $U_2$, there exists $a \in S^2$ such that $\alpha_k([0,1]) \subset S^2 \setminus \{a\}$. Then there exists a homeomorphism $\phi_k : S^2 \setminus \{a\} \to \mathbb{R}^2$ which maps $\alpha_k$ to a path $\beta_k$ in $\mathbb{R}^2$. 


\[ \beta_k \text{ is homotopic to the ge-line segment connecting } \beta_k(0) \text{ to } \beta_k(1) \text{ by the following homotopy:} \]

\[ H_t : [0,1] \times [0,1] \to \mathbb{R}^2 \]

\[ H_t(s,t) = \left( t \beta_k(1) + (1-t) \beta_k(0) \right) s + (1-s) \beta_k(t) \]

Then \( \alpha_k \) is homotopic to \( \alpha'_k = \phi_k^{-1} \beta_k \), because the inverse image of a line under the stereographic projection is a circle passing through \( (0,0,1) \) on \( S^2 \) and a circle passing through \( (0,0,1) \) on \( S^2 \) become a circle passing through \( S^2 \) on \( S^2 \) after the backward rotation, \( \alpha'_k(S^2) \) is contained in a circle \( \phi_k \) passing through point \( a \) in \( S^2 \).

7. We have \( \alpha_k \sim \alpha'_k \) for \( k = 0, 1, \ldots, N-1 \). Thus
$\delta = \alpha_1 \cdots \alpha_{n-1} \sim \alpha'_1 \cdots \alpha'_{n-1} = \delta'$.

What we need is to show that $\delta'(p_{10}) \not\equiv S^2$.

We have

$$\delta'(p_{10}) = \bigcup_{k=0}^{n-1} \delta(\alpha_k(p_{10})) = \bigcup_{k=0}^{n-1} \alpha_k$$

We'll show that $\delta'$ is the union of finitely many circles passing through $a$ is not equal to $S^2$. Since we already have a rotation on $S^2$ that maps $a$ to $(0,0,1)$, it suffices to show that the union of finitely many circles passing through $(0,0,1)$ is not equal to $S^2$. Let

$$C = \{ (x_1, x_2, x_3) \in S^2 : x_3 = \frac{1}{2} \}$$

Since $C$ does not pass through $(0,0,1)$, each $C_k$ is none of $C_k$'s. Thus each $C_k$ intersects $C$ at most two points. Thus $C \cap (\bigcup_{k=0}^{n-1} C_k)$ is finite. Suppose by contradiction that $\bigcup_{k=0}^{n-1} C_k = S^2$, then

$$C = C \cap S^2 = C \cap \left( \bigcup_{k=0}^{n-1} C_k \right)$$

which is finite. For each $\nu \in \mathbb{N}$, $\nu > 2$, we have

$$\left( \frac{1}{\lambda}, \frac{3}{4} - \frac{1}{2n}, \frac{1}{2} \right) \in C$$

Thus $C$ must be infinite. This contradiction completes the proof.
A graph is a simplicial complex with only vertices and edges, i.e. where no faces have dimension higher than one. A tree is a graph, with at least one vertex, such that for any vertices \( p \neq q \), there exists a unique sequence \( e_1, e_2, \ldots, e_n \) of edges such that:

- \( e_i \neq e_j \) for \( i \neq j \),
- \( e_i \) and \( e_{i+1} \) always share a common vertex \( p \), and \( p \) is a vertex of \( e_i \), and
- \( q \) is a vertex of \( e_n \).

Show that any tree gives rise to a space with trivial fundamental group.

Rough idea

We will examine the problem with simple cases: finite set of vertices.

Let \( V = \{ v_1, \ldots, v_n \} \) be the set of vertices and \( E \) be the set of edges of the simplicial complex. We consider the case \( V \) finite mainly because we do know that it has a geometric realization in \( \mathbb{R}^n \), which is a very natural candidate for the topological space we are looking for. The following are steps to show that we can find a space that is contractible.

1) Let \( e_i = (0, \ldots, 1, \ldots, 0) \) be the standard basis of \( \mathbb{R}^n \). Consider the map \( v_i \in V \rightarrow e_i \in \mathbb{R}^n \).
This map gives rise to the geometric realization of $(V,E)$ in $\mathbb{R}^k$ as follow:

\[ K = \bigcup_{(u_i, u_j) \in E} [u_i, u_j] \cup \{v_i, v_j\} \cup \{v_i, u_j\} \cup \{u_i, v_j\} \]

where $[u_i, u_j] = \{ t u_i + (1-t) u_j : 0 \leq t \leq 1 \}$.

2) Then $K$ is path-connected. Here is how to show it:

- Each point $v \in K$ is contained in a segment $[u_i, u_j]$. It is path connected to $u_i$ by $t \mapsto u_i + t(v_i - u_i)$.

- Two points $u_i$ and $u_j$ is connected by a path connecting $v_i$ and $v_j$. Each edge in this path in $E$ share a common vertex, thus this path gives rise to a path from $u_i$ to $u_j$ in $K$.

3) We show by induction with respect to $n$ that there is a homotopy $H_n : K \times [0,1] \to K$ between $id_K$ and the constant map $x \mapsto u_i$ (it is specified but arbitrary).

- The base case is $n = 1$

- For $n > 1$; in $V$ there exists one vertex that appears in only one edge. Why? Starting from an arbitrary vertex in $V$, we go to another vertex and keep going and avoid going on an odd edge meeting an already visited vertex. Since there are only finitely many vertices, the process must stop. And it
stops because of either one of two reasons: there is no new edge to follow, or a red edge that there is no new vertex to come. The vertex \( v \) that is the terminal vertex either has no adjacent vertex or there is an edge connecting \( v \) to an odd edge. The second possibility couldn't happen because it will give a cycle in \((V, E)\).

Thus \( v \) appears in only one edge. We can assume \( v = v_n \) and the edge containing \( v \) is \( e = \{v_{n-1}, v_n\} \).

Put \( V' = V \setminus \{v_n\} \) and \( E' = E \setminus \{e\} \). \( K' = K \setminus \{v_{n-1}, v_n\} \), then \((V', E')\) is a graph of \( n-1 \) vertices still satisfying the problem's axioms and \( K' \) is a geometric realization of \((V', E')\) on \( \mathbb{R}^d \) (since we omitted \( v_n \)). Then by induction, there exists a homotopy \( H_{n-1} : K' \times [0, 1] \to K' \), between \( \text{id}_{K'} \) and \( x \in K' \mapsto u_{n-1} \). We define \( H_n : K \times [0, 1] \to K \) as follows:

\[
H_n(x, t) = \begin{cases} 
H_n(x, t) & \text{for } x \in K' \\
(1-t)x + tv_{n-1} & \text{for } x \in \{v_{n-1}, v_n\}
\end{cases}
\]

Since the two formulas agree at \( v_{n-1} \), \( H_n \) is continuous.

What about infinite trees? 3/4