We'll give a definition of the complex Grassmannian $\text{Gr}_c(n,m)$ of $n$-dimensional subspaces of $\mathbb{C}^m$.

Denote by $\text{Gr}_c(n,m)$ the set of all $n$-dimensional subspaces of $\mathbb{C}^m$. If $n = 0$ or $n = m$ then $\text{Gr}_c(0,m) = \{0\}$ and $\text{Gr}_c(m,m) = \{\mathbb{C}^m\}$ respectively. The topology on each of them is simply the unique topology $\{\emptyset, \{\ast\}\}$. The (minimal) atlas is $\mathcal{E} (\{\ast\}, \varphi : * \to \mathbb{C} = \{0\})$. In such a case, $\text{Gr}_c(n,m)$ is a smooth $0$-manifold.

Now we consider $1 \leq n \leq m - 1$. From the topological point of view, we can identify $\mathbb{R}^n$ with $\mathbb{C}^n$ and $\mathbb{R}^{2m}$ with $\mathbb{C}^m$. For each $n$-combination $\mathcal{C}$ of the set $\{1, 2, \ldots, m\}$, we put

\[ W_\mathcal{C} = \{ (x_1, \ldots, x_m) \in \mathbb{C}^m : x_i = 0 \text{ for all indices } i \notin \mathcal{C} \}, \]

\[ W'_\mathcal{C} = \{ (y_1, \ldots, y_m) \in \mathbb{C}^m : y_i = 0 \text{ for all indices } i \in \mathcal{C} \}. \]

If we put $\mathcal{C} = \{1, 2, \ldots, m\} \setminus \mathcal{C}$ then $W'_\mathcal{C} = W_\mathcal{C}$. We see that $\mathbb{C}^m = W_\mathcal{C} \oplus W'_\mathcal{C}$.

Also, we put $\pi_\mathcal{C} : \mathbb{C}^m \to W_\mathcal{C}$,

\[ \pi_\mathcal{C} (x_1, \ldots, x_m) = (y_1, \ldots, y_m), \text{ with } y_i = \begin{cases} x_i & \text{if } i \in \mathcal{C}, \\ 0 & \text{otherwise} \end{cases} \]

Put $U_\mathcal{C} = \{ V \in \text{Gr}_c(n,m) \mid \pi_\mathcal{C} |_V : V \to W_\mathcal{C} \text{ is bijective} \}$. Let $\mathcal{L}(W_\mathcal{C}, W'_\mathcal{C})$
be the vector space over \( \mathbb{C} \) of linear maps from \( W_e \) to \( W_e' \). For each element \( V \in U_e \), each \( x \in V \) is written in a unique way

\[
\pi_e(x) = \frac{\pi_e(x)}{\tilde{\pi}_e(x)}
\]

To \( V \) we associate the function \( f : W_e \rightarrow W_e' \), \( f(\pi_e(x)) = \tilde{\pi}_e(x) \).

Then \( f \) is well-defined because \( \tilde{\pi}_e \mid V \) is bijective. Then \( f \) becomes naturally a linear map. Thus, we obtain a map \( \phi_e : U_e \rightarrow \mathcal{L}(W_e, W_e') \),

\[
V \mapsto f
\]

We'll show that \( \phi_e \) is bijective. For each \( f \in \mathcal{L}(W_e, W_e') \), we have \( \phi_e(V) = f \) if and only if \( f(\pi_e(x)) = \tilde{\pi}_e(x) \) \( \forall x \in V \). Thus \( \{ x = y + f(y) : y \in W_e \} \)

is contained in \( V \). Moreover, \( V' \) is an \( n \)-dimensional subspace of \( \mathbb{C}^m \) because the linear map \( W_e \rightarrow V' \), \( y \mapsto y + f(y) \) is bijective. Thus \( V' \) exists, must be \( V' \). By the definition of \( V' \), we can see that \( \phi_e(V') = f \).

Therefore, \( \phi_e \) is bijective. We know that each \( f \in \mathcal{L}(W_e, W_e') \) associates one-to-one with its matrix representation. Since \( W_e \) is \( n \)-dimensional, and \( W_e' \) is \((m-n)\)-dimensional, the representation matrix is of size \( (m-n) \times n \).

Thus there is a natural bijective linear map from \( \mathcal{L}(W_e, W_e') \) to \( \mathbb{C}^{(m-n) \times n} \). 

\[
U_e \xrightarrow{\phi_e} \mathcal{L}(W_e, W_e') \xrightarrow{\text{bij}} \mathbb{C}^{(m-n) \times n}
\]
Then we have a bijection $\varphi_c: U_c \rightarrow C^{(m-n)n}$. Denote by $S$ the set of all $n$-combinations of $\{1, 2, \ldots, m\}$. Then we have a map family $\{(U_c, \varphi_c)\}_{ces}$. This family has $C_m^k = \frac{m!}{n!(m-n)!}$ members. The topology on $Gr_c(n, m)$ is now defined as the topology generated by the maps $\{(U_c, \varphi_c)\}_{ces}$. This is the coarsest topology on $Gr_c(n, m)$ that makes these maps continuous. Then $\varphi_c$ turns out to be a homeomorphism. Then $Gr_c(n, m)$ becomes a $(m-n)n$-dimensional complex manifold. Then we can check that $\{(U_c, \varphi_c)\}_{ces}$ is a smooth complex structure on $Gr_c(n, m)$, which is called the Grassmannian atlas on $Gr_c(n, m)$.

First we will show that $\bigcup_{ces} U_c = Gr_c(n, m)$. We already have $\bigcup_{ces} U_c \subset Gr_c(n, m)$. Now let $V \subseteq Gr_c(n, m)$, we'll show that $V$ belongs to some $U_c$. Let $\{v_1, \ldots, v_n\}$ be a basis of $V$. Each $v_i$ is a vector in $C^m$. Thus we have an $n \times m$ matrix

$$
\begin{pmatrix}
v_{11} & v_{12} & \cdots & v_{1m} \\
v_{21} & v_{22} & \cdots & v_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
v_{n1} & v_{n2} & \cdots & v_{nm}
\end{pmatrix}
$$

Because $\{v_1, \ldots, v_n\}$ is linearly independent, the above matrix has rank $n$. After doing row reduction, we'll get a row-echelon matrix with exactly
n columns of the form  
\[
\begin{pmatrix}
0 \\
1 \\
\vdots \\
0
\end{pmatrix}
\xleftarrow{j} 
\quad (1 \leq j \leq n)
\]

Let \( C \) be the set of indices of these \( n \) columns in \( \mathfrak{C}_1, \mathfrak{C}_2, \ldots, \mathfrak{C}_m \). Then \( \mathfrak{C} \subseteq \mathfrak{C} \).

Moreover, the projection of \( V \) on each elementary vector \( e_i \), where \( i \in C \), is not zero. Thus the projection of \( V \) on \( C e_i \) is the whole \( C e_i \). Thus the projection of \( V \) on \( W_C = \bigoplus_{i \in C} C e_i \) is the whole \( W_C \). Since \( V \) and \( W_C \) are both \( n \)-dimensional vector spaces, this projection map is bijective. Thus \( V \supseteq W_C \).

2. Let \( M \) be a manifold and \( K, L \) be closed subsets of \( M \). Let \( f, g : M \to \mathbb{R} \) be smooth maps such that \( f = g \) on a open subset \( U \supset \overline{(K \cap L)} \).

We have
\[
U \cup (M \setminus K) \cup (M \setminus L)
\]
\[
= U \cup (M \setminus (K \cup L))
\]
\[
\supset (K \cap L) \cup (M \setminus (K \cup L))
\]
\[
= M.
\]

Thus \( \{ M, M \setminus K, M \setminus L \} \) is an open cover of \( M \). Let \( \{ \psi_1, \psi_2, \psi_3 \} \) be a smooth partition of unity of this cover, such that \( \text{supp} \psi_1 \subseteq U \), \( \text{supp} \psi_2 \subseteq M \setminus K \), \( \text{supp} \psi_3 \subseteq M \setminus L \). We have \( \psi_1(x) + \psi_2(x) + \psi_3(x) = 1 \) for all \( x \in M \). Put \( h : M \to \mathbb{R}, \ h = \psi_3 + g \psi_2 + f \psi_1 \). Then \( h \) is smooth. We'll show that \( h|_K = f|_K \) and \( h|_L = g|_L \).
- For \( x \in K \setminus U \): \( \psi_2(x) = \varphi_1(x) = 0 \). Thus \( h(x) = f(x) \psi_3(x) = f(x) \).

- For \( x \in K \cap U \): \( \varphi_3(x) = 0 \). Then \( h(x) = f(x) \psi_3(x) + f(x) \psi_1(x) \frac{1}{f(x)} \).

\[
= f(x) \left( \frac{\psi_3(x) + \psi_1(x)}{f(x)} \right)
\]

\[
= f(x)
\]

Therefore \( h(x) = f(x) \) for all \( x \in K \).

- For \( x \in L \setminus U \): \( \varphi_3(x) = \varphi_1(x) = 0 \). Thus \( h(x) = g(x) \psi_2(x) = g(x) \).

- For \( x \in L \cap U \): \( \varphi_3(x) = 0 \). Thus \( h(x) = g(x) \varphi_1(x) + f(x) \varphi_2(x) \).

\[
= g(x) \text{ because } x \in U
\]

Then \( h(x) = g(x) \left( \varphi_1(x) + \varphi_2(x) \right) = g(x) \).

Therefore \( h(x) = g(x) \) for all \( x \in L \).

3. Let \( M \) be an \( n \)-dimensional compact manifold, with a smooth structure on it. By restricting the coordinate charts, we can say that each point \( x \in M \) has an open neighborhood that is diffeomorphic to the unit ball \( B^n \) of \( \mathbb{R}^n \). Since \( B^n \) is diffeomorphic to \( \mathbb{R}^n \) by the map \( x \mapsto \frac{x}{\sqrt{1-\|x\|^2}} \), each point \( x \in M \) has an open neighborhood \( U_x \) that is diffeomorphic to \( \mathbb{R}^n \) under the map \( \varphi_x: U_x \to \mathbb{R}^n \). Put \( V_x = \varphi_x^{-1}(B^n) \). We could have assumed that \( \varphi_x(x) = 0 \). Then \( V_x \) is an open neighborhood of \( x \). Then \( \{ V_x \}_{x \in M} \) is an open cover of \( M \). Since \( M \) is compact, there exists a compact subcover that is finite. We call that subcover \( V_1, V_2, \ldots, V_k \), which corresponds to
$U_1, \ldots, U_k$ and the maps $\varphi_i : U_i \rightarrow \mathbb{R}^n$, where $1 \leq i \leq k$. We have $V_i \subset \overline{V_i} = \varphi_i^{-1}(B^n) \subset U_i$. Thus there is a smooth extension of $\varphi_i$, namely $\widetilde{\varphi}_i$, such that $\widetilde{\varphi}_i : M \rightarrow \mathbb{R}^n$ and $\widetilde{\varphi}_i|_{V_i} = \varphi_i$ and $\text{supp} \widetilde{\varphi}_i \subset U_i$.

\[ \begin{array}{c}
U_i \xrightarrow{\varphi_i} \mathbb{R}^n \\
\overline{V}_i \xrightarrow{\widetilde{\varphi}_i} \mathbb{B}^n
\end{array} \]

Put $\omega = (0, 0, \ldots, 0, 1) \in \mathbb{R}^n$ and $D_i = \mathbb{B}^n + (i-1)\omega$ for every $1 \leq i \leq k$. Then each $D_i$ is just a translation of $B^n$ such that $D_i \cap D_j = \emptyset$ if $i \neq j$.

For each $i = 1, \ldots, k$, we define a map $\Psi_i : M \rightarrow \mathbb{R}^n$, $\Psi_i(x) = \widetilde{\varphi}_i(x) + (i-1)\omega$.

Then $\Psi_i$ is still a smooth map. Moreover, for each $x \in V_i$ we have $\Psi_i(x) = \widetilde{\varphi}_i(x) + (i-1)\omega = \varphi_i(x) + (i-1)\omega = D_i$.

Thus $\Psi_i(V_i) \subset D_i$ and $\Psi_i$ is injective on $V_i$.

Now we define the map $f : M \rightarrow \mathbb{R}^{nk}$, $f(x) = (\Psi_1(x), \ldots, \Psi_k(x))$.

Then $f$ is smooth because each component $\Psi_i$ is smooth. Suppose that $f(x) = f(y)$, then for some $x, y \in M$. Since $\{V_1, \ldots, V_k\}$ is an open cover of $M$, $x \in V_i$ and $y \in V_j$ for some $i, j = 1, \ldots, k$. We have $\Psi_i(x) = \Psi_j(y)$ because $f(x) = f(y)$. Moreover, we notice that $\widetilde{\varphi}_i$ could be chosen as $\varphi_i$, $\varphi_i$ is a bump function of $V_i$ and supported by $V_i$. Because $0 \leq f \leq 1$ as a matter of fact, our choice may fail.
for all $z \in M$, we have $|\bar{\Psi}_i(z)| = |\Psi_i(z)| \leq |\Psi_i(z)| < 1$ for all $z \in U_i$. Therefore $|\bar{\Psi}_i(z)| < 1$ for all $z \in M$. Thus $\bar{\Psi}_i(M) \subset B^n$. Therefore $\Psi_i(M) \subset B^n + (i-1)\omega \subset D_i$. Return to the problem, we have $\Psi_i(x) = \Psi_i(y)$ and $\Psi_i(x) \in D_i$, $\Psi_i(y) \in D_i$. Thus $\Psi_i(y) \in D_i \cap D_i$. Thus $x \neq y$. Thus $i = j$. Thus $xy \in E_i$. We know that $\Psi_i$ is injective on $V_i$. Therefore $x = y$. Thus $f$ is injective.

4 Consider the point $p = (3,4,4)$ in Cartesian coordinates on $R^3$. Denote by $(x,y,z)$ the Cartesian coordinates and $(r,\theta,z)$ the cylindrical coordinates. A generic tangent vector at $p$ in Cartesian coordinates is

$$a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y} + c \frac{\partial}{\partial z} : C^\infty(R^3) \to IR$$

$$f \mapsto a \frac{\partial f}{\partial x} |_p + b \frac{\partial f}{\partial y} |_p + c \frac{\partial f}{\partial z} |_p,$$

where $a, b, c \in IR$. To translate this tangent vector into cylindrical coordinates, we'll use the chain rule to find a new $b, c \in IR$ such that

$$a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y} + c \frac{\partial}{\partial z} = a^\prime \frac{\partial}{\partial r} + b^\prime \frac{\partial}{\partial \theta} + c^\prime \frac{\partial}{\partial z}.$$

We have

$$\frac{\partial}{\partial r} = \frac{\partial r}{\partial x} \frac{\partial}{\partial x} + \frac{\partial r}{\partial y} \frac{\partial}{\partial y} + \frac{\partial r}{\partial z} \frac{\partial}{\partial z},$$

$$\frac{\partial}{\partial \theta} = \frac{\partial \theta}{\partial y} \frac{\partial}{\partial y} + \frac{\partial \theta}{\partial z} \frac{\partial}{\partial z} + \frac{\partial \theta}{\partial x} \frac{\partial}{\partial x},$$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial z}.$$
Then
\[ a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y} + c \frac{\partial}{\partial z} = \left( a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y} \right) \frac{\partial}{\partial r} + \left( a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y} \right) \frac{\partial}{\partial \theta} + c \frac{\partial}{\partial z} \]

We have the Jacobian matrix
\[
\begin{bmatrix}
\frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\
\frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\
\frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}
\end{bmatrix}^{-1} = \begin{bmatrix}
\cos \theta & -r \sin \theta \\
\sin \theta & r \cos \theta
\end{bmatrix} = \frac{1}{r} \begin{bmatrix}
r \cos \theta & r \sin \theta \\
-r \sin \theta & r \cos \theta
\end{bmatrix} = \begin{bmatrix}
\frac{x}{r \sqrt{x^2 + y^2}} & \frac{y}{r \sqrt{x^2 + y^2}} \\
\frac{-y}{\sqrt{x^2 + y^2}} & \frac{x}{\sqrt{x^2 + y^2}}
\end{bmatrix}
\]

Because \((x,y,z) = (3,4,1)\), we have
\[
\begin{bmatrix}
\frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\
\frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y}
\end{bmatrix} = \begin{bmatrix}
3/5 & 4/5 \\
-4 & 1
\end{bmatrix}
\]

Thus, \((\ast)\) gives
\[
a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y} + c \frac{\partial}{\partial z} = \left( \frac{3}{5} a + \frac{4}{5} b \right) \frac{\partial}{\partial r} + \left( 4a + b \right) \frac{\partial}{\partial \theta} + c \frac{\partial}{\partial z}
\]

The corresponding vector in cylindrical coordinate is
\[
\text{\textbf{C}}(\mathbb{R}^3) \rightarrow \mathbb{R}
\]
\[
f \mapsto \left( \frac{3}{5} a + \frac{4}{5} b \right) \frac{\partial f}{\partial r} + (4a + b) \frac{\partial f}{\partial \theta} + c \frac{\partial f}{\partial z}
\]

Let \( p \) be a point in a smooth manifold \( M \). We denote by \( T_p(M) \) the set of all tangent vectors of \( M \) at \( p \), i.e. the set of linear maps \( D: \mathcal{C}^\infty(M) \rightarrow \mathbb{R} \) that satisfy the product rule \( D(fg) = (Df)g(p) + (Dg)f(p) \). Then we can define an addition and scalar multiplication on \( T_p(M) \) in a natural way
\[
(D_1 + D_2)(f) := D_1(f) + D_2(f), \quad \forall D_1, D_2 \in T_p(M), f \in \mathcal{C}^\infty(M)
\]
\[
(cD)(f) := c(Df), \quad \forall D \in T_p(M), f \in \mathcal{C}^\infty(M), c \in \mathbb{R}
\]
With these definitions, the operators on $T_p(M)$ satisfy the axioms of modules over $\mathbb{R}$. Then $T_p(M)$ becomes a vector space over $\mathbb{R}$.

Because $M$ is a smooth manifold, there is a chart $(U, \varphi: U \to \mathbb{R}^n)$ containing $p$ such that $\varphi: U \to \mathbb{R}^n$ is a diffeomorphism. We define the following map $\varphi^*: T_p(U) \to T_{\varphi(p)}(\mathbb{R}^n)$,

$$\varphi^*(f)(g) = f(g \circ \varphi) \quad \forall f \in T_p(U), \quad g \in \text{vector of } \mathbb{R}^n \text{ at } \varphi(p).$$

First we show that $\varphi^*$ is well-defined by showing that $\varphi^*(f) \in T_{\varphi(p)}(\mathbb{R}^n)$. The map $g \mapsto f(g \circ \varphi)$ is linear because $f$ is linear.

About the product rule, $f((gh) \circ \varphi) = f((g \circ \varphi)(h \circ \varphi))$

$$= f(g \circ \varphi \circ h \circ \varphi(p)) + f(h \circ \varphi(p)) \circ \varphi(p)$$

$$= f(g \circ \varphi) \circ \varphi(p) + f(h \circ \varphi(p)) \circ \varphi(p).$$

Thus $\varphi^*(f)$ satisfies the product rule. Then $\varphi^*(f) \in T_{\varphi(p)}(\mathbb{R}^n)$.

Secondly, we see that $\varphi^*$ is $\mathbb{R}$-linear because

$$\varphi^*(f_1 + f_2)(g) = (f_1 + f_2)(g \circ \varphi) = f_1(g \circ \varphi) + f_2(g \circ \varphi) = \varphi^* f_1(g) + \varphi^* f_2(g),$$

$$\varphi^*(cf)(g) = (cf)(g \circ \varphi) = c f(g \circ \varphi) = c \varphi^* f(g),$$
Since $U \xrightarrow{\varphi} \mathbb{R}^n$ is a diffeomorphism, there exists a smooth inverse $\mathbb{R}^n \xrightarrow{\psi} U$. We have

$$U \xrightarrow{\varphi} \mathbb{R}^n \xrightarrow{\psi} U,$$

$$\mathcal{T}_p(U) \xrightarrow{\varphi_*} \mathcal{T}_{\varphi(p)}(\mathbb{R}^n) \xrightarrow{\psi_*} \mathcal{T}_p(U).$$

For any $f \in \mathcal{T}_p(U)$ and $g \in C^0(U)$, we have

$$\psi_* \circ \varphi_*(f)(g) = \varphi_*(f)(g \circ \psi) = f \left( (g \circ \psi) \circ \varphi \right) = f(g).$$

Thus $\psi_* \circ \varphi_*(f) = f$. Since $f$ is arbitrary, $\psi_* \circ \varphi_* = id_{\mathcal{T}_p(U)}$. Similarly, by using the chains

$$\mathbb{R}^n \xrightarrow{\psi} U \xrightarrow{\varphi} \mathbb{R}^n,$$

$$\mathcal{T}_{\varphi(p)}(\mathbb{R}^n) \xrightarrow{\psi_*} \mathcal{T}_p(U) \xrightarrow{\varphi_*} \mathcal{T}_{\varphi(p)}(\mathbb{R}^n),$$

we get $\varphi_* \circ \psi_* = id_{\mathcal{T}_{\varphi(p)}(\mathbb{R}^n)}$. Therefore $\varphi_*$ is bijective. Therefore it's a linear isomorphism between $\mathcal{T}_p(U)$ and $\mathcal{T}_{\varphi(p)}(\mathbb{R}^n)$. In the lecture, we have shown that the tangent space of $\mathbb{R}^n$ at any point is an $n$-dimensional vector space with the basis $\left\{ \frac{\partial}{\partial x^i} : i = 1, \ldots, n \right\}$, where $p = (a^1, \ldots, a^n)$ in Cartesian coordinates. Thus $\mathcal{T}_p(U)$ is also an $n$-dimensional vector space over $\mathbb{R}$. Since we can identify $\mathcal{T}_p(M)$ with $\mathcal{T}_p(U)$, $\mathcal{T}_p(M)$ is an $n$-dimensional vector space over $\mathbb{R}$.