Let $M$ be a smooth manifold with atlas $\{(U_i, \phi_i)\}_{i \in I}$. The tangent bundle of $M$ is the set

$$TM = \coprod_{p \in M} T_p M = \{(p, v) : p \in M, v \in T_p M\}.$$ 

We'll give $TM$ a natural smooth structure. Consider the following projection map $\pi : TM \to M$, $\pi(p, v) = p$. For each $p \in U_i$, we have $\phi_i(p) \in \mathbb{R}^n$.

Let $(x^1, \ldots, x^n)$ be the Cartesian coordinates on $\mathbb{R}^n$. Then the set

$$\left\{ \frac{\partial}{\partial x^1}_p, \ldots, \frac{\partial}{\partial x^n}_p \right\}$$

is a basis for $T_p M$. Then for each $v \in T_p M$, we have an $n$-tuple $(a^1, \ldots, a^n) \in \mathbb{R}^n$ such that $v = \frac{\partial}{\partial x^i}_p a^i$. Put $V_i = \pi^{-1}(U_i)$. We can define the map $\Psi_i : V_i \to \mathbb{R}^{2n}$, $\Psi_i(p, v) = (\phi_i(p), a^1, \ldots, a^n)$.

We'll show that $\{(V_i, \Psi_i)\}_{i \in I}$ is a smooth atlas on $TM$. Applying the Smooth Manifold Construction Lemma in Lee, page 1, we need to check 5 points:

(i) For each $i \in I$, $\Psi_i(V_i)$ is an open subset of $\mathbb{R}^{2n}$,

(ii) For each $i, j \in I$, $\Psi_i(V_i \cap V_j)$ is an open subset of $\mathbb{R}^{2n}$,

(iii) If $i, j \in I$ and $V_i \cap V_j \neq \emptyset$, then $\Psi_i \circ \Psi_j^{-1} : \Psi_j(V_i \cap V_j) \to \Psi_i(V_i \cap V_j)$ is a diffeomorphism.
(i) Countably many $V_i$'s cover TM,

(ii) For each $\tilde{p}, \tilde{q} \in TM$ distinct, either there exists $V_i$ such that $\tilde{p} \notin V_i$, or there exists $V_i, V_j$ disjoint with $\tilde{p} \not\in V_i$, $\tilde{q} \not\in V_j$.

Verify (i)

For any $i, j \in I$, we have by definition

$$\Psi_i(V_i \cap V_j) = \{ \Psi_i(p, v) : (p, v) \in V_i \cap V_j \}$$

$$= \{ \Psi_i(p, v) : (p, v) \in \pi^i(U_i \cap U_j) \}$$

$$= \{ \Psi_i(p, v) : \rho \in U_i \cap U_j, v \in \overline{T}_M \}$$

$$= \{ (\Psi_i(p), a^1, \ldots, a^n) : \rho \in U_i \cap U_j, v = \frac{\partial}{\partial x^i} |_\rho ; a^1, \ldots, a^n \in \mathbb{R} \}$$

$$= \{ (\Psi_i(p), a^1, \ldots, a^n) : \rho \in U_i \cap U_j, a^1, \ldots, a^n \in \mathbb{R} \}$$

$$= \Psi_i(U_i \cap U_j) \times \mathbb{R}^n$$

For the case $i=j$, we get $\Psi_i(V_i) = \Psi_i(U_i) \times \mathbb{R}^n$. Since $\Psi_i(U_i)$ is open in $\mathbb{R}^n$, $\Psi_i(V_i)$ is open in $\mathbb{R}^{2n}$.

Verify (ii)

We have showed that $\Psi_i(V_i \cap V_j) = \Psi_i(U_i \cap U_j) \times \mathbb{R}^n$. Since $\Psi_i(U_i \cap U_j)$ is open in $\mathbb{R}^n$, $\Psi_i(V_i \cap V_j)$ is open in $\mathbb{R}^{2n}$.

Verify (iii) Consider the map $\Psi_0 \circ \Psi_1^{-1} : \Psi_i(V_i \cap V_j) \rightarrow \Psi_i(V_i \cap V_j)$.

$$= \Psi_j(U_i \cap U_j) \times \mathbb{R}^n = \Psi_i(U_i \cap U_j) \times \mathbb{R}^n$$

For each $(r, a^1, \ldots, a^n) \in \Psi_j(U_i \cap U_j) \times \mathbb{R}^n$, there is $\rho \in U_i \cap U_j$ such that $r = \psi_0(\rho)$.
Then \( \psi_j^{-1}(r, a^1, \ldots, a^n) = (p, v) \), with \( v = a^i \frac{2}{\partial x^i} \big|_p \in \mathbb{T}_p M \).

Then \( \psi_i \circ \psi_j^{-1}(r, a^1, \ldots, a^n) = \psi_i(p, v) = (\psi_i(p), b^1, \ldots, b^n) \) such that \( v = b^i \frac{2}{\partial x^i} \big|_p \).

Since \( \left\{ \frac{\partial}{\partial x^1} \big|_p, \ldots, \frac{\partial}{\partial x^n} \big|_p \right\} \) is a basis of \( \mathbb{T}_p M \), \( a^l = b^l \) for every \( l \). Thus,

\[ \psi_i \circ \psi_j^{-1}(r, a^1, \ldots, a^n) = (\psi_i(p), a^1, \ldots, a^n) = (\psi_i \circ \psi_j^{-1}(r), a^1, \ldots, a^n). \]

Thus we can rewrite \( \psi_i \circ \psi_j^{-1} : \psi_j(U_i \cap U_j) \times \mathbb{R}^n \rightarrow \psi_i(U_i \cap U_j) \times \mathbb{R}^n, \)

\[ (r, a^1, \ldots, a^n) \mapsto (\psi_i \circ \psi_j^{-1}(r), a^1, \ldots, a^n). \]

Because the map \( r \in \psi_j(U_i \cap U_j) \mapsto \psi_i \circ \psi_j^{-1}(r) \in \psi_i(U_i \cap U_j) \) is a transition map on smooth manifold \( M \), it is smooth. Thus \( \psi_i \circ \psi_j^{-1} \) is also smooth as a map from an open subset of \( \mathbb{R}^{2n} \) to an open subset of \( \mathbb{R}^{2n} \).

Verify (iv)

Because \( M \) is a manifold and \( \{U_i\}_{i \in I} \) is an open cover of \( M \), there exists a countable subcover. Without loss of generality, we can name it \( \{U_1, U_2, \ldots\} \).

Then \( \bigcup_{i=1}^{\infty} V_i = \bigcup_{i=1}^{\infty} \pi^{-1}(U_i) = \pi^{-1}(\bigcup_{i=1}^{\infty} U_i) = \pi^{-1}(M) = TM \). Thus \( \{V_i\}_{i \in \mathbb{N}} \) is a countable open cover of \( TM \).

Verify (v)

Take \( (p, v), (q, w) \in TM \) which are distinct. If \( p = q \) then \( (p, v), (q, w) \) belongs to \( V_i \) where \( p \in U_i \). For the case \( p \neq q \), we can find a neighborhood \( A \)
of p and a neighborhood B of q such that A ∩ B = ∅. This is possible because M is Hausdorff. Then we can choose a chart $U_i \subset A$ which is also a neighborhood of p, and a chart $U_j \subset B$ which is also a neighborhood of q. This is possible because $\{(U_i, \psi_i)\}_{i \in I}$ is a maximal atlas. Then $U_i \cap U_j = \emptyset$.

Thus $V_i \cap V_j = \pi^{-1}(U_i \cap U_j) = \pi^{-1}(\emptyset) = \emptyset$, and $(q, v) \in V_i$, $(q, w) \in V_j$.

Next, let $f : M \to N$ be a smooth map. We'll show that it gives rise to a smooth map $df : TM \to TN$. First, let's give names to coordinate charts on $M, N, TM, TN$. Let the atlas on $M$ be $\{(U_i, \psi_i)\}_{i \in I}$, and the atlas on $N$ be $\{(U_j, \psi_j)\}_{j \in J}$. Put $\pi : TM \to M$, $\pi' : TN \to N$ to be the projection maps. Put $V_i = \pi^{-1}(U_i)$, $V_j = \pi'^{-1}(U_j)$. By the construction of smooth structure on $TM, TN$ as above, the coordinate maps on $TM, TN$ are respectively

$$
\psi_i : V_i \to \mathbb{R}^{2m}, \quad \psi_i(p, v) = (\psi_i(p), a^1, ..., a^m), \quad \text{where } v = \frac{\partial \psi}{\partial \xi}\bigg|_p.
$$

$$
\psi_j : V_j \to \mathbb{R}^{2n}, \quad \psi_j(q, w) = (\psi_j(q), b^1, ..., b^n), \quad \text{where } w = \frac{\partial \psi}{\partial \xi}\bigg|_q.
$$

Here $m = \dim M$, $n = \dim N$, $(x^1, ..., x^m)$ is the Cartesian coordinates on $\mathbb{R}^m$ and $(x^1, ..., x^n)$ is the Cartesian coordinates on $\mathbb{R}^n$. Let $L : \mathbb{R}^m \to \mathbb{R}^n$ be any linear transformation. We can think of $L$ as an $n \times m$ matrix with real coefficients.
Then we define a map \( df : TM \to TN \), such that \( df(p, v) = (f(p), w) \), where \( v = a^i \frac{\partial}{\partial x^i} \bigg|_p \), \( w = b^i \frac{\partial}{\partial x^i} \bigg|_{f(p)} \), and \( \begin{pmatrix} b^1 \\ \vdots \\ b^n \end{pmatrix} = L \begin{pmatrix} a^1 \\ \vdots \\ a^m \end{pmatrix} \).

We'll show that \( df \) is a smooth map. Suppose that \( (p, v) \in V_i \) and \( (f(p), w) \in V'_j \). We'll show that \( \psi'_j \circ df \circ \psi_i^{-1} : \psi_i(V_i) \to \psi'_j(V'_j) \) is smooth.

For each \( (r^1, a^1, \ldots, a^m) \in \psi_i(U_i) \subset \mathbb{R}^m \), we have \( (r^1, a^1, \ldots, a^m) = \psi_i(U_i) \times \mathbb{R}^m \).

Thus there is \( p \in U_i \) such that \( v = \psi_i(p) \). Thus \( \psi_i^{-1}(r^1, a^1, \ldots, a^m) = (p, v) \), where \( v = a^i \frac{\partial}{\partial x^i} \bigg|_p \). Then \( df(p, v) = (f(p), w) \) where

\[
    w = b^i \frac{\partial}{\partial x^i} \bigg|_{f(p)} \bigg|_p + \cdots + b^n \frac{\partial}{\partial x^i} \bigg|_{f(p)} \bigg|_p, \quad \text{and} \quad \begin{pmatrix} b^1 \\ \vdots \\ b^n \end{pmatrix} = L \begin{pmatrix} a^1 \\ \vdots \\ a^m \end{pmatrix}.
\]

Then \( \psi'_j(f(p), w) = (\psi'_j(f(p)), b^1, \ldots, b^n) = (\psi'_j(f(p)), L \begin{pmatrix} a^1 \\ \vdots \\ a^m \end{pmatrix}) \).

Therefore, \( \psi'_j \circ df \circ \psi_i^{-1}(r^1, a^1, \ldots, a^m) = (\psi'_j \circ df \circ \psi_i^{-1}(p), L \begin{pmatrix} a^1 \\ \vdots \\ a^m \end{pmatrix}) \).
Since \( f \) is a smooth map, the map \( r \mapsto \Phi'_j \circ f \circ \Phi_i(r) \) is smooth. Moreover, \( L \) is also smooth because it is a linear map from \( \mathbb{R}^m \) to \( \mathbb{R}^n \). Thus the map \( \Phi'_j \circ df \circ \Phi_i \) is smooth as a map from an open subset of \( \mathbb{R}^{2m} \) to an open subset of \( \mathbb{R}^{2n} \).

2. Given a map \( f : M \to N \) of smooth manifolds, we'll explain why this does not push forward to a natural vector field on \( N \). Let \( Y : M \to TM \) be a (rough) vector field on \( M \); i.e., a map such that \( Y(p) \in T_pX \times T_pM \).

\[
\begin{array}{c}
M \xrightarrow{Y} TM \\
\downarrow f \downarrow \downarrow df \\
N \xrightarrow{Y'} TN
\end{array}
\]

We have proved problem 3 that there exists a natural smooth map \( df \) from \( TM \) to \( TN \). Also, we showed that there can be many different such maps, depending on our choice of the linear transformation \( L : \mathbb{R}^m \to \mathbb{R}^n \). A natural push forward of \( Y \) on \( N \) should be a map \( Y' : N \to TN \) that makes the above diagram commute. That is, \( Y'(f(p)) = df(Y(p)) \) for every \( p \in M \). A problem with this definition is that \( f \) may not be surjective. In such a case, we have not defined \( Y' \) at every point in \( N \). Also, there is
no natural way to define \( Y' \) on \( N \setminus f(M) \). Now even if \( f \) is surjective, the definition \( Y'(f(p)) = df(Y(p)) \) can be illegitimate because \( f \) may not be injective. In such a case, there are more than one tangent vector defined at a point in \( N \).

\[ \begin{array}{c}
\text{M} \\
\downarrow \scriptstyle f \\
\text{N}
\end{array} \]

**Example 1**  \( M = N = \mathbb{R} \) with standard structure, \( f(x) = e^x \). We choose \( Y : \mathbb{R} \to T\mathbb{R}, \ Y(p) = (p, f'(p)) \), where \( f'(p) \) can be identified with the element \( f(p) \frac{df}{dp} \) in \( T_{p}\mathbb{R} \). Then \( Y \) is a vector field on \( \mathbb{R} \) because the first component of \( Y(p) \) is \( p \). Since \( f \) is not surjective, \( Y' \) is not defined (in a natural way everywhere).

**Example 2**  \( M = N = \mathbb{R} \) with standard smooth structure, \( f(x) = x^3 - x \). Again, we choose the vector field \( Y(p) = (p, f'(p)) \). We see that \( f(0) = f(1) = 0 \). By choosing the linear transformation \( L : \mathbb{R} \to \mathbb{R} \) to be the identity map, we have \( df(p, v) = (f(p), v) \) for all \( p \in \mathbb{R} \).

\[
\begin{array}{c}
\mathbb{R} \\
\downarrow f \\
\mathbb{R}' \\
\downarrow df \\
\mathbb{R}' \\
\downarrow f' \\
T\mathbb{R} \\
\downarrow \scriptstyle df \\
T\mathbb{R}
\end{array}
\]

- If we define \( Y'(f(p)) = df(Y(p)) \) \( \forall p \in \mathbb{R} \), then
- At \( p = 0 \) : \( Y'(0) = df(Y(0)) = df(0, f'(0)) = df(0, 1) = (0, -1) \).
At \( p = 1 \), \( \psi'(0) = df_1(\psi(0)) = df(1, f'(0)) = df(1, 2) = (92) \).
Thus, \( \psi'(0) \) is not well-defined.

1. Let \( f: M \rightarrow N \) be a map of smooth manifolds and \( \psi: N \rightarrow T^*N \) be a covector field. We'll show that \( \psi \) pulls back to a natural covector field on \( M \).

\[
\begin{array}{ccc}
M & \xrightarrow{f} & N \\
N & \xrightarrow{\psi} & T^*N
\end{array}
\]

For each \( p \in M \), we have a push forward \( T_pM \xrightarrow{df_p} T_{f(p)}N \), which is called the derivative of \( f \) at \( p \). Because \( \psi_{f(p)} \in T_{f(p)}^*N \), we have the following composition

\[
\begin{array}{ccc}
T_pM & \xrightarrow{df_p} & T_{f(p)}N \\
\downarrow & & \downarrow \psi_{f(p)} \\
\mathbb{R} & & \mathbb{R}
\end{array}
\]

Because \( \psi_{f(p)} \) and \( df_p \) are both linear and continuous, \( \psi_{f(p)} \circ df_p \in T_p^*M \).
Thus, we get a covector field \( \psi: M \rightarrow T^*M \) defined by

\[
\psi_p = \psi_{f(p)} \circ df_p.
\]

In case \( \psi \) is a smooth covector field, we'll show that \( \psi \) is also a smooth covector field. For each \( p \in M \), we'll show that \( \psi \) is smooth at \( p \). Let \( (U, \phi) \) be a coordinate chart containing \( p \) and \( (V, \psi) \) be a coordinate chart containing \( f(p) \) in \( N \), and \( (U, \phi) \) be a coordinate chart containing \( p \) in
For each \( p \in U \), we know that \( \{dx^1|_p, \ldots, dx^n|_p\} \) is a basis of \( T^*_pU \). Thus,

\[
Y_p = \frac{\partial}{\partial x^i}|_p \quad \text{(kth component function)}
\]

To show that \( Y_p \) is smooth in \( U \), we need to show that each component function \( Y_i: U \to \mathbb{R} \) is smooth. We have

\[
Y_i(p) = \frac{\partial}{\partial x^i}|_p \left( \frac{\partial}{\partial y^j}|_p \right) = Y_p \left( \frac{\partial}{\partial x^i}|_p \right) = \omega^{(p)} \circ \frac{\partial f}{\partial x^i}|_p \tag{1}
\]

Let \((J^i_j)^{(p)}\) be the Jacobian matrix of the transformation \((x^i)\) to \((y^j)\).

\[
J^i_j(p) = \frac{\partial y^j}{\partial x^i} (\varphi(p))
\]

Then each function \( J^i_j: U \to \mathbb{R} \) is smooth. By the definition of the derivative of the function \( f \) at \( p \), we get

\[
\frac{\partial f}{\partial x^i}|_p = \frac{\partial y^j}{\partial x^i} |_p \frac{\partial y^j}{\partial y^i} |_{\varphi(p)} = J^j_i(p) \frac{\partial y^j}{\partial y^i} |_{\varphi(p)}. \tag{2}
\]
Now substituting (2) into (1), we get

\[ Y_i(p) = \omega(p) \left( J^i_j(p) \frac{\partial}{\partial y^j} f(p) \right) = J^i_j(p) \omega(p) \left( \frac{\partial}{\partial y^j} f(p) \right). \]  

(3)

For each \( q \in V \), we know that \{dy^1_q, \ldots, dy^n_q\} is a basis of \( T_q^* N \).

Thus \( w_q = w_k(\xi) \cdot dy^k_q \). Since \( w_q \) is smooth, \( w_k: V \to \mathbb{R} \) is also smooth.

Let \( \xi \) be a component function

Then \( \omega(\xi) = w_k(\xi) \cdot dy^k(\xi) \). Then

\[ \omega(\xi) \left( \frac{\partial}{\partial y^j} f(\xi) \right) = w_k(\eta) \cdot dy^k(\eta) \left( \frac{\partial}{\partial y^j} f(\xi) \right) = w_k(\xi) \cdot dy^k(\xi). \]  

(4)

Substituting (4) into (3), we get \( Y_i(p) = J^i_j(p) \cdot w_j(\xi) \). \( \eta \)

Because \( J^i_j: \Psi(U) \to \Psi(V) \), \( w_j: V \to \mathbb{R} \), \( f: M \to N \) are all smooth, \( Y_i \) is smooth on \( U \). In particular, \( Y_i \) is smooth at \( p \).

Let \( M \) be a smooth manifold, \( Y \) be a topological space and \( p: Y \to M \) be a covering map. We'll show that \( Y \) can be given a new smooth structure so that \( p \) is a smooth map. Let \( \{ Y_i \}_{i \in \mathcal{E}} \) be the set of path-connected components of \( Y \).

For each \( i \in \mathcal{E} \), we define \( p_i: Y_i \to M \), \( p_i(y) = p(y) \). First we'll show that \( p_i \) is a covering map. For each \( x \in M \), there exists an open neighborhood of \( x \) in \( U \), which can be assumed to be path-connected because \( M \) is a manifold, such that
there is a discrete space $F$ and a homeomorphism $\varphi: p^{-1}(U) \to U \times F$ such that $p^{-1}(U) \overset{p} \to U$ commutes.

For each $x \in F$, $U \times \{x\}$ is open and connected subspace of $U \times F$. Thus $\varphi^{-1}(U \times \{x\})$ is also path-connected. Thus $\varphi^{-1}(U \times \{x\})$ is contained in exactly one of the path-connected component of $Y$, called $Y_x$. With $i \in I$ fixed, we define

$$F' = \{ x \in F : i_x = i \}$$

We'll show that $\varphi^{-1}(U \times F') = \{ y \in Y_i : \pi(\varphi(y)) \in U \}$. \hspace{1cm} (1)

"$\subseteq$" : Take $y \in \varphi^{-1}(U \times F')$. Then $\varphi(y) \in U \times F'$. Then there is $x \in F'$ such that $\varphi(y) \in U \times \{x\}$. Thus $y \in \varphi^{-1}(U \times \{x\})$. Because $x \in F'$, $y \in \varphi^{-1}(U \times \{x\}) \subseteq Y_i$. Moreover, $\pi(\varphi(y)) \in \pi(U \times \{x\}) \subseteq U$.

"$\supseteq$" : Take $y \in Y_i$ such that $\pi(\varphi(y)) \in U$. Put $\varphi(y) = (z, x)$ for some $z \in U$, $x \in F$. Then $y \in \varphi^{-1}(U \times \{x\})$. Thus since $\varphi^{-1}(U \times \{x\})$ is path-connected in $Y$ and $\varphi^{-1}(U \times \{x\}) \cap Y_y \neq \emptyset$, we have $\varphi^{-1}(U \times \{x\}) \subseteq Y_i$.

Thus $x \in F'$. Thus $y \in \varphi^{-1}(U \times F')$.

The identity (1) shows that $\varphi^{-1}(U \times F') = \{ y \in Y_i : p(y) \in U \}$

$$= \{ y \in Y_i : p(y) \in U \}$$

$$= \rho_i^{-1}(U)$$

Thus $\varphi: \rho_i^{-1}(U) \to U \times F'$ is a homeomorphism.
Then we have the commutative diagram 

\[
\begin{array}{ccc}
p_i^{-1}(U) & \xrightarrow{p_i} & U \\
p_i & \downarrow \quad \downarrow & \\
U \times F_i & \xrightarrow{\pi} & U \times F
\end{array}
\]

Thus \( p : Y \to U \) is a covering map. Suppose that we can give each \( Y_i \) a smooth structure such that \( p_i \) is a smooth map. Then we can introduce an atlas on \( Y \) as a union of the atlases on each \( Y_i \). The compatibility of the coordinate charts on \( Y \) is simply the compatibility of coordinate charts on each component \( Y_i \) because \( Y_i \cap Y_j = \emptyset \) if \( i \neq j \).

To show that \( p \) is smooth on this new atlas, we only need to show that each \( Y_i \) is actually a connected component of \( Y \) (as opposed to path-connected component). To do so, it suffices to show that \( Y \) is locally connected. An open subset of \( Y \). It suffices to show that \( Y \) is locally connected.

Take \( y \in Y \) and put \( x = p(y) \in M \). Then there is an open neighborhood \( U \) of \( x \) in \( M \) such that we have a commutative diagram

\[
\begin{array}{ccc}
p^{-1}(U) & \xrightarrow{p} & U \\
p & \downarrow \quad \downarrow & \\
U \times F & \xrightarrow{\pi} & U \times F
\end{array}
\]

Moreover, by shrinking \( U \) if necessary, we can assume \( U \) is path-connected, and \( U \cong \mathbb{R}^n \).
Let $y \in F$ such that $p(y) \in U \times \{a\}$. Since $U \times \{a\}$ is path-connected, $V = p^{-1}(U \times \{a\})$ is path-connected and containing $y$. Since $U \times \{a\}$ is open in $U \times \{a\}$, $V$ is open in $Y$. We have a homeomorphism $V \xrightarrow{p} U \times \{a\}$. Since $U \cong \mathbb{R}^n$, we get $V \cong \mathbb{R}^n$. Thus $Y$ is locally Euclidean, and hence locally connected.

Therefore, from the beginning, we can assume $Y$ is path-connected. Also we proved that $Y$ is locally looks like $\mathbb{R}^n$. Next we'll show that $Y$ is Hausdorff. Let $y_1 \neq y_2$ be two points in $Y$. We consider 2 cases.

- $p(y_1) \neq p(y_2)$:

  Put $x_1 = p(y_1)$ and $x_2 = p(y_2)$. Since $M$ is Hausdorff, there exist open neighborhoods $U$ of $x_1$ and $V$ of $x_2$ that are disjoint.

  Then $p^{-1}(U)$ is an open neighborhood of $y_1$, and $p^{-1}(V)$ is an open neighborhood of $y_2$, and $p^{-1}(U) \cap p^{-1}(V) = p^{-1}(U \cap V) = \emptyset$.

- $p(y_1) = p(y_2)$:

  Put $x = p(y_1) = p(y_2)$. There exists an open neighborhood $U$ of $x$ in $M$ such that we have the following commutative diagram:
We put $\varphi(y_1) = (z_1, x)$, $\varphi(y_2) = (z_2, x)$.

Suppose by contradiction that $x = y$. Then we know that $x = p(y_1) = \pi(\varphi(y_1)) = z_1$,

$x = p(y_2) = \pi(\varphi(y_2)) = z_2$.

Then $\varphi(y_1) = \varphi(y_2)$, which leads to $y_1 = y_2$ since $\varphi$ is a homeomorphism.

This is a contradiction. Thus $x \neq y$. Then $y_1 \in \varphi^{-1}(U \times \{x\})$ and $y_2 \in \varphi^{-1}(U \times \{x\})$. Then $V_1 \supseteq y_1$ and $V_2 \supseteq y_2$ are open in $Y$ and disjoint. Therefore, $Y$ is Hausdorff.

Next, we'll show that $Y$ is second countable. First, at any point $x \in M$, we have a fundamental group $\pi_1(M, x)$. Since $M$ is an $n$-manifold, it can be covered by a countable number of open sets, each of which is homeomorphic to $\mathbb{R}^n$. Each loop at $x$ runs through a finite number of these open sets due to the compactness of the loop. We can see that two loops running through the same family of open sets are homotopic. Thus, the cardinality of $\pi_1(M, x)$ is at most the cardinality of $\mathbb{R}$ sequences of finite length of elements from a countable set. Thus $\pi_1(M, x)$ is countable.

Next, for each $x \in M$, we'll that the fiber $p^{-1}(x)$ is countable. For each loop $\gamma$ at $x$, we fix $y \in p^{-1}(x)$.
know that $\gamma$ can be lifted to a unique path $\tilde{\gamma}$ in $\tilde{Y}$ such that $\tilde{\gamma}(0) = \gamma$. We define the map $S: \pi_1(M,x) \rightarrow p^{-1}(x)$,

\[ [\gamma] \mapsto \tilde{\gamma}(1). \]

This map is well-defined because $p(\tilde{\gamma}(1)) = \gamma(1) = x$. Thus $\tilde{\gamma}(1) \in p^{-1}(x)$.

Moreover, by the property of path lifting, if $[\gamma] = [\gamma']$ then $\tilde{\gamma}(1) = \tilde{\gamma'}(1)$.

Now we'll show that $S$ is surjective. For each $\tilde{x} \in p^{-1}(x)$, there is a path in $\tilde{Y}$ from $\gamma$ to $\tilde{x}$ because $\tilde{Y}$ is path-connected. We call this path $\eta$. Put $y = p(\gamma)$.

Then $y(0) = y(\eta(0)) = y(\gamma) = x$, $y(1) = y(\eta(1)) = y(\tilde{x}) = x$. Thus $y$ is a loop at $x$ in $\tilde{M}$. Because $\gamma = p \circ \eta$ and $y(0) = \gamma$, $\gamma$ is the lift of $Y$ at $y$. Thus, $S([\gamma]) = y(1) = \tilde{x}$. Thus $S$ is surjective. Since $\pi_1(M,x)$ is countable, so is $p^{-1}(x)$.

We will call an open subset $U$ of $M$ evenly covered if there exists a discrete space $F$ together with a homeomorphism $\varphi: p^{-1}(U) \rightarrow U \times F$ so that the following diagram commutes:

\[ \begin{array}{ccc}
p^{-1}(U) & \xrightarrow{p} & U \\
\varphi \downarrow & & \downarrow \pi \\
U \times F & \xrightarrow{\pi} & \end{array} \]

Put $B = \{ U \subseteq M : U \text{ is open, evenly covered} \}$.

Since $p$ is a covering map, every point of $M$ has an evenly covered open neighborhood. Thus $B$ is a covering of $M$. Moreover, we know that any evenly covered open subset of an evenly covered open subset of $M$ is also evenly covered. Thus, for each open set $V$ of $M$, and for each $x \in V$, we can
find an open, evenly covered neighborhood of \( x \) that is contained in \( F \).

In other words, \( V \) is a union of open, evenly covered subsets. Thus \( B \) is a topological basis of \( M \).

Since \( M \) is second countable, we can extract a countable covered from \( B \), namely \( E = \{ U_i \}_{i \in \mathbb{N}} \). As a space, \( U_i \) is second countable. Thus, each \( U_i \) has an a countable topological basis \( B_i \). For each \( U \) open in \( M \), we have \( U = \bigcup_{i=1}^{\infty} (U_i \cap U) \). Each \( U_i \cap U \) is a union of members in \( B_i \). Thus \( \bigcup_{i=1}^{\infty} B_i \) gives us a basis of \( M \). This is a countable union of countable sets. Thus \( \bigcup_{i=1}^{\infty} B_i \) is countable. By replacing \( P \) by \( \bigcup_{i=1}^{\infty} B_i \), we could have assumed that \( E = \{ U_i \}_{i \in \mathbb{N}} \) which is also a topological basis of \( M \). Thus we obtain a countable basis of \( M \) consisting of evenly covered open subsets of \( M \).

For each \( i \), we have a commutative diagram

\[
\begin{array}{ccc}
p^{-1}(U_i) & \xrightarrow{p} & U_i \\
\psi_i \downarrow & & \downarrow \\
U_i \times F_i & \xrightarrow{\pi} & \end{array}
\]

Fix \( x \in U_i \), we get a diagram

\[
\begin{array}{ccc}
p^{-1}(x) & \xrightarrow{p} & \{x\} \\
\psi_i \downarrow & & \downarrow \\
\{x\} \times F_i & \xrightarrow{\pi} & \end{array}
\]

Thus the cardinality of \( F_i \) is equal to that of \( p^{-1}(x) \), which is countable. Thus, \( p^{-1}(U_i) \) has countably many connected components, which are named \( V_{ij} \).
Technically, the index \( j \) can run through a finite, finite or infinite (countable) index set. However, by allowing \( V_j \) to repeat itself, we can assume that \( j \) runs through \( \mathbb{N} \). What we need is that \( p_j: V_j \rightarrow U \) is a homeomorphism. Now with that definition we have

\[
p^{-1}(U) = \bigcup_{j=1}^{\infty} V_j.
\]

Put \( D = \{ V_j | i,j \in \mathbb{N} \} \). Then \( D \) is countable. We'll show that \( D \) is a basis of \( Y \). Let \( W \) be any open subset of \( Y \) and \( y \in W \). We'll show that there exists \( V_j \) such that \( y \in V_j \subset W \). If we can prove that, we will finish the proof that \( Y \) is second countable.

Put \( x = p(y) \). Because \( p \) is a covering map, there exists \( U \in \mathcal{B} \) such that \( x \in U \). Because \( y \in p^{-1}(U) \), there exists a connected component \( V \) of \( p^{-1}(U) \) that contains \( y \). We have \( p_j: V \rightarrow U \) is a homeomorphism. Thus

\[
p|_{p^{-1}(U)} : V \rightarrow p^{-1}(U) \text{ is also a homeomorphism. By replacing } V \text{ with } p^{-1}(U), U \text{ with } p(V \cap W), \text{ we can assume that } V \subset W. \text{ Now we know that } \mathcal{B} = \{ U_i \}_{i \in \mathbb{N}} \text{ is a topological basis of } M. \text{ Thus } U \text{ is a} \]
union of some of these $U_i$'s. Thus there exists $i \in N$ such that $x \in U_i$. Let

We'll show that $p^{-1}(U_i) = \bigcup_{j \in V_i} V_j$.

Take $z \in p^{-1}(U_i)$. Then $z \in V$ and $p(z) \in U_i$. Thus $z \in V$ and $z \in \bigcup_{j \in V_i} V_j = p^{-1}(U_i)$. Thus

Thus $V \subseteq p^{-1}(U_i)$. Then $p(V) \subseteq p(U_i) = U_i$. Since $U_i$ is connected, $V'$ is also connected. Thus $V'$ is a connected subset of $p^{-1}(U_i)$. Thus $V'$ is contained in some connected component $V_j$. Suppose by contradiction that $V' \neq V_j$. Then there is $\bar{z} \in V_j \setminus V'$. But $p(\bar{z}) = \bar{x} \in U_i$. Then there is $\bar{z}' \in V'$ such that $p(\bar{z}') = \bar{x}$. Then $p(z) = p(\bar{z})$. Because $p_{V_j} : V_j \to U_i$ is a homeomorphism, $z = \bar{z}' \in V'$. This is a contradiction.

Thus $V' = V_j$. Thus $y \in V' = V_j \subseteq V \subseteq W$.

Until now, we have showed that $Y$ is a manifold of the same dimension as $M$. We will introduce an atlas on $Y$ so that the covering map $p : Y \to M$ becomes smooth. We know that $C = \{U_i\}_{i \in N}$ is a topological basis of $M$ consisting of evenly covered, open subset of $M$. In the
definition of $B$ on page 15, we could have include one more constraint, namely, $U_i$ lies in some coordinate chart. If we had done so, now we would have that each $U_i$ lies in some coordinate chart. Now we should assume so. Then there is a homeomorphism $\Psi_i : U_i \to W_i$ from $U_i$ to an open subset of $\mathbb{R}^n$, and $(U_i, \Psi_i)$ is a coordinate chart of $M$. We also define $D = \{ V_{ij} : ij \in \mathbb{N}^2 \}$ where $V_{ij}$ is a connected component of $p^{-1}(U_i)$. We showed that $V_{ij} \xrightarrow{p|_{V_{ij}}} U_i$ is a homeomorphism and $D$ is a topological basis of $Y$.

Define $\Psi_j = \Psi_i \circ (p|_{V_{ij}})$.

$\Psi_j \circ \Psi_i^{-1} : \Psi_k(V_{ij} \cap V_{ik}) \to \Psi_j(V_{ij} \cap V_{ik})$

Thus $(V_{ij}, \Psi_j)$ is a coordinate chart on $Y$. Since $D$ is a basis of $Y$, the family covers $Y$. We'll check the compatibility. Let $ij, kl \in \mathbb{N}$.

For every $x \in \Psi_k(V_{ij} \cap V_{ik})$, we have

$$
\Psi_j \circ \Psi_i^{-1}(x) = \Psi_j \left( \left( \Psi_k \circ p|_{V_{ik}} \right)^{-1} \right) (x) = \Psi_j \left( (p|_{V_{ik}})^{-1} \Psi_k^{-1}(x) \right) = \Psi_i \circ p|_{V_{ij}} \left( (p|_{V_{ik}})^{-1} \Psi_k^{-1}(x) \right) = \Psi_i \circ \Psi_k^{-1}(x).
$$

Since $\Psi_i \circ \Psi_k^{-1}$ is smooth at $x$ (and note that $\Psi_k(V_{ij} \cap V_{ik})$ is open in $Y$ since...
\( \Psi_k \) is a homeomorphism), \( \Psi_j \circ \Psi_k^{-1} \) is smooth at \( x \). Thus the compatibility is now verified. Therefore \( \{(V_{ij}, \Psi_{ij})\}_{ij \in \Omega} \) is an (smooth) atlas on \( Y \). Now we'll show that \( p \) is a smooth map.

Let \( x \in Y \). There is a chart \( (V_{ij}, \Psi_{ij}) \) containing \( x \). Then \( U_i = p(V_{ij}) \) is a chart on \( M \) containing \( p(x) \). By definition, to show that \( p \) is smooth at \( p \), we only need to show that \( \Psi_i \circ p \circ \Psi_j^{-1} : W_i \to W_i \) is smooth at \( \Psi_j^{-1}(x) \).

For any \( z \in W_i \), we have \( \Psi_{ij}^{-1}(z) = (p|_{V_{ij}})^{-1} \circ \Psi_i^{-1}(z) \). Thus

\[
\Psi_i \circ p \circ \Psi_j^{-1}(z) = \underbrace{\Psi_i \circ (p|_{V_{ij}})^{-1}}_{\text{id}} \circ \Psi_i^{-1}(z) = \Psi_i(\Psi_j^{-1}(x)) = z
\]

Thus \( \Psi_i \circ p \circ \Psi_j^{-1} = \text{id}|_{W_i} \), which is smooth at \( x \). Therefore, \( p \) is a smooth map.

5) Let \( M \) be a smooth \( m \)-manifold and \( f : M \to \mathbb{R}^n \setminus \{0\} \) be a smooth function. We define the function \( g : \mathbb{R}^n \setminus \{0\} \to S^{n-1} \), with

\[
g(y^1, \ldots, y^n) = \frac{(y^1, \ldots, y^n)}{\sqrt{(y^1)^2 + \cdots + (y^n)^2}}
\]

where \((y^1, \ldots, y^n)\) are Cartesian coordinates of a point in \( \mathbb{R}^n \setminus \{0\} \). We'll give necessary and sufficient conditions for the smooth map \( g \circ f : M \to S^{n-1} \) to be either a submersion, an immersion or a local diffeomorphism.
Let \((y^1, \ldots, y^n)\) be Cartesian coordinates in \(\mathbb{R}^n\). We denote, for each \(i=1, \ldots, n\),

\[ V_i^+ = \{ (y^1, \ldots, y^n) \in \mathbb{R}^n : y_i > 0 \}, \]

\[ V_i^- = \{ (y^1, \ldots, y^n) \in \mathbb{R}^n : y_i < 0 \}. \]

Then \(\mathbb{R}^n \setminus \{0\} = \bigcup_{i=1}^{n} V_i^+\). In \(V_i^+\), we have a spherical coordinate system \((r, \theta^1, \ldots, \theta^{n-1})\) with \(r > 0\), \(0 < \theta^1, \ldots, \theta^{n-1} < \pi\) whose relation with the Cartesian system is given by

\[
\begin{align*}
y^1 &= r \cos \theta^1 \\
y^2 &= r \sin \theta^1 \cos \theta^2 \\
&\vdots \\
y^{n-1} &= r \sin \theta^1 \cdots \sin \theta^{n-2} \cos \theta^{n-1} \\
y^n &= r \sin \theta^1 \cdots \sin \theta^{n-2} \sin \theta^{n-1}
\end{align*}
\]

\((*)\)

The backward transformation is a bit more complicated; for each point \(y \in V_i^+\) whose Cartesian coordinates are \((y^1, \ldots, y^n)\), we put \(r = \sqrt{(y^1)^2 + \cdots + (y^n)^2} > 0\). Then

\[
\left( \frac{y^1}{r} \right)^2 + \left( \sqrt{\left( \frac{y^2}{r} \right)^2 + \cdots + \left( \frac{y^n}{r} \right)^2} \right)^2 = 1
\]

Thus there exists a unique \(\theta^1 \in (0, \pi)\) such that

\[
\frac{y^1}{r} = \cos \theta^1 \quad \text{and} \quad \sqrt{\left( \frac{y^2}{r} \right)^2 + \cdots + \left( \frac{y^n}{r} \right)^2} = \sin \theta^1
\]

(note that \(\theta^1\) cannot be 0 or \(\pi\) because \(y^n > 0\)). Then

\[
\left( \frac{y^1}{r} \right)^2 + \left( \sqrt{\left( \frac{y^2}{r} \right)^2 + \cdots + \left( \frac{y^n}{r} \right)^2} \right)^2 = \sin^2 \theta^1
\]
Then there exists a unique \( \theta \in (0, \pi) \) such that

\[
\frac{y^2}{r \sin \theta} = \cos \theta \quad \text{and} \quad \sqrt{(\frac{y^3}{r})^2 + \ldots + (\frac{y^n}{r})^2} \frac{1}{\sin \theta} = \sin \theta.
\]

We continue doing so to get the definition of \( \theta^1, \theta^2, \ldots, \theta^{n-2}, \theta^{n-1} \).

In \( V^{-} \), we also have a spherical coordinate system, namely \( (r, \theta^1, \ldots, \theta^n) \) satisfying (\ast) but this time \( r > 0 \), \( 0 < \theta^1, \ldots, \theta^{n-2}, \theta^{n-1} < \pi \), \(-\pi < \theta^n < 0\).

In \( V^{+} \), we label \( y^1, \ldots, y^n \) as \( \bar{y}^1, \ldots, \bar{y}^n \) with \( \bar{y}^1 = \frac{\sin \theta^1}{\sin \theta} y^1, \ldots, \bar{y}^{n-2} = \frac{\sin \theta^{n-2}}{\sin \theta} y^{n-2}, \bar{y}^{n-1} = \frac{\sin \theta^{n-1}}{\sin \theta} y^{n-1}, \bar{y}^n = \frac{1}{\sin \theta} y^n \).

Then we get a spherical coordinate system \( (\bar{r}, \bar{\theta}^1, \ldots, \bar{\theta}^n) \) with

\[
\begin{align*}
\bar{y}^1 &= \bar{r} \cos \bar{\theta}^1, \\
\bar{y}^2 &= \bar{r} \sin \bar{\theta}^1 \cos \bar{\theta}^2, \\
&\vdots \\
\bar{y}^{n-1} &= \bar{r} \sin \bar{\theta}^{n-2} \cos \bar{\theta}^{n-1}, \\
\bar{y}^n &= \bar{r} \sin \bar{\theta}^{n-2} \sin \bar{\theta}^{n-1}.
\end{align*}
\]

where \( \bar{r} > 0 \), \( 0 < \bar{\theta}^1, \ldots, \bar{\theta}^{n-2}, \bar{\theta}^{n-1} < \pi \).

In \( V^{-} \), we also have a spherical coordinate system \( (\bar{r}, \bar{\theta}^1, \ldots, \bar{\theta}^n) \) satisfying the same identities as above, but this time \( \bar{r} > 0 \), \( 0 < \bar{\theta}^1, \ldots, \bar{\theta}^{n-2}, \bar{\theta}^{n-1} < \pi \), \(-\pi < \bar{\theta}^n < 0\).

From now on, we only use spherical coordinates in \( V^{\pm} \) instead of the usual Cartesian coordinates. Up to now, we know that \( \{(V^{\pm}, \mathcal{C}_{V^{\pm}})\}_{i=1,\ldots,n} \) is an atlas on \( \mathbb{R}^n \setminus \{0\} \).

Put \( W^{\pm}_i = V^{\pm}_i \cap S^{n-1} \). Then \( W^{\pm}_i \) is open in \( S^{n-1} \) and \( \bigcup_{i=1}^n W^{\pm}_i = S^{n-1} \).
For each $i = 1, \ldots, n$, we define a map $\Psi_i^+: W_i^+ \to (0, \pi)^{n-1} \subset \mathbb{R}^{n-1}$,

$$\Psi_i^+(\theta^1, \ldots, \theta^{n-1}) = (\theta^1, \ldots, \theta^{n-1}).$$

spherical cords Cartesian cords

Also, we define a map $\Psi_i^-: W_i^- \to (0, \pi) \times (0, \pi) \times \cdots \times (0, \pi) \subset \mathbb{R}^{n-1}$,

$$\Psi_i^-(\theta^1, \ldots, \theta^{n-1}) = (\theta^1, \ldots, \theta^{n-1}).$$

spherical cords Cartesian cords

Then each $\Psi_i^\pm$ is a homeomorphism.

We'll show that $\{(W_i^\pm, \Psi_i^\pm)\}_{i=1, \ldots, n}$ is an atlas on $S^{n-1}$. To do so we only have to check the compatibility of any two charts. By the definition of $\Psi_i^\pm$, we see that $\Psi_i^+$ and $\Psi_j^-$ have disjoint codomains. Also, $\Psi_i^-$ and $\Psi_j^+$ have disjoint codomains if $i \neq j$. Thus we only have to check the compatibility of $\Psi_i^+$ and $\Psi_j^-$. We have

$$\begin{align*}
(\theta^1, \ldots, \theta^{n-1}) \in (0, \pi)^{n-1} \\
\rightarrow (\psi_i^+)^{-1} \rightarrow \begin{cases} 
\overline{y}^1 = \cos \theta^1 \\
\overline{y}^2 = \sin \theta^1 \cos \theta^2 \\
\vdots \\
\overline{y}^{n-1} = \sin \theta^1 \cdots \sin \theta^{n-1} 
\end{cases} \\
\rightarrow (\theta^1, \ldots, \theta^{n-1}) \in (0, \pi)^{n-1}.
\end{align*}$$

Thus $(\psi_i^+) \circ (\psi_j^+)^{-1} = \text{id}$, which is smooth. Therefore $\{(W_i^\pm, \Psi_i^\pm)\}_{i=1, \ldots, n}$ is an atlas on $S^{n-1}$.

Now we take $p \in M$ and put $q = f(p) \in \mathbb{R}^{n+1} \setminus \text{ios}$. Then $q \in V_i^\pm$ for some $i = 1, \ldots, n$. In $V_i^\pm$, we have
\[ g(r, \theta^1, \ldots, \theta^{n-1}) = (1, \theta^1, \ldots, \theta^{n-1}) \]

Spherical coords in \( V_i^\pm \)

Thus, \( \psi_i^\pm \circ g(r, \theta^1, \ldots, \theta^{n-1}) = (\theta^1, \ldots, \theta^{n-1}) \)

Spherical in \( V_i^\pm \) Cartesian in \( \mathbb{R}^{n-1} \)

Denote by \( J_q(g) \) the Jacobian matrix of \( g \) at \( q \). We have

\[
J_q(g) = \begin{pmatrix}
\frac{\partial \theta^1}{\partial r} |_q & \frac{\partial \theta^1}{\partial \theta^1} |_q & \cdots & \frac{\partial \theta^1}{\partial \theta^{n-1}} |_q \\
\frac{\partial \theta^2}{\partial r} |_q & \frac{\partial \theta^2}{\partial \theta^1} |_q & \cdots & \frac{\partial \theta^2}{\partial \theta^{n-1}} |_q \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial \theta^{n-1}}{\partial r} |_q & \frac{\partial \theta^{n-1}}{\partial \theta^1} |_q & \cdots & \frac{\partial \theta^{n-1}}{\partial \theta^{n-1}} |_q
\end{pmatrix}
\]

\( (n-1) \times n \)

\[ \mathbb{R}^n \xrightarrow{\psi} \mathbb{R}^+ \xrightarrow{\text{id}} V_i^\pm \subset \mathbb{R}^{n+1} \xrightarrow{g} \psi_i^\pm \subset \mathbb{R}^{n+1} \xrightarrow{\psi_i^\pm} \mathbb{R}^n \]

Cartesian

any coordinate system
Let \((U, \varphi)\) be any coordinate chart on \(M\) containing \(p\), and \((x^1, \ldots, x^n)\) be any coordinate system in \(\varphi(U) \subset \mathbb{R}^m\). Moreover, we can choose \((U, \varphi)\) such that \(\varphi(U) \subset V_i^\pm\). Then the map \(U \xrightarrow{f} V_i^\pm \xrightarrow{g} W_i^\pm\) pushes forward \(T_p(U) \xrightarrow{df_p} T_q(V_i^\pm) \xrightarrow{dg_q} T_{g(q)}(W_i^\pm)\) with basis \(\left\{ \frac{\partial}{\partial x^1} \bigg|_p, \ldots, \frac{\partial}{\partial x^n} \bigg|_p \right\}\) \(\xrightarrow{df_p}\) basis \(\left\{ \frac{\partial}{\partial \varphi^1} \bigg|_q, \ldots, \frac{\partial}{\partial \varphi^m} \bigg|_q \right\}\) \(\xrightarrow{dg_q}\) basis \(\left\{ \frac{\partial}{\partial \Theta^1} \bigg|_{g(q)}, \ldots, \frac{\partial}{\partial \Theta^m} \bigg|_{g(q)} \right\}\).

We have

\[
df_p \left( \begin{array}{c} \frac{\partial}{\partial x^1} \bigg|_p \\ \vdots \\ \frac{\partial}{\partial x^n} \bigg|_p \end{array} \right) = \left. J_p(f) \right|_p \left( \begin{array}{c} \frac{\partial}{\partial \varphi^1} \bigg|_q \\ \vdots \\ \frac{\partial}{\partial \varphi^m} \bigg|_q \end{array} \right) \tag{**}
\]

where \(J_p(f)\) is the Jacobian matrix of \(f\) at \(p\):

\[
J_p(f) = \begin{pmatrix}
\frac{\partial f_1}{\partial x^1} & \cdots & \frac{\partial f_1}{\partial x^n} \\
\vdots & \ddots & \vdots \\
\frac{\partial f_n}{\partial x^1} & \cdots & \frac{\partial f_n}{\partial x^n}
\end{pmatrix}
\]

and \(f = (f_1, \ldots, f_n)\) spherical coord. in \(V_i^\pm\).

Also, we have

\[
dg_q \left( \begin{array}{c} \frac{\partial}{\partial \Theta^1} \bigg|_q \\ \frac{\partial}{\partial \Theta^2} \bigg|_q \\ \vdots \\ \frac{\partial}{\partial \Theta^m} \bigg|_q \end{array} \right) = \left. J_q(g) \right|_q \left( \begin{array}{c} \frac{\partial}{\partial \Theta^1} \bigg|_{g(q)} \\ \cdots \\ \frac{\partial}{\partial \Theta^m} \bigg|_{g(q)} \end{array} \right) \tag{***}
\]
From (++) and (+++) we get
\[ d g_q \circ d f_p \left( \begin{array}{c}
\frac{\partial}{\partial x^1} |_p \\
\vdots \\
\frac{\partial}{\partial x^n} |_p 
\end{array} \right) = J_p(f)^T J_q(g)^T \left( \begin{array}{c}
\frac{\partial}{\partial y^1} |_{g(q)} \\
\vdots \\
\frac{\partial}{\partial y^{m-1}} |_{g(q)} 
\end{array} \right) \]

Thus \( \text{rank} (d g_q \circ d f_p) = \text{rank} (J_p(f)^T J_q(g)^T) \)

\[ = \text{rank} (J_q(g) J_p(f)) \]

Since the pushing forward is a functor, we have \( d f_p \circ d g_q = d (gof) \), thus \( \text{rank} (d (gof)_p) = \text{rank} (J_q(g) J_p(f)) \).

Because \( J_q(g) \) consists of a zero column on the left and the unit matrix \( \text{In}_{n-1} \) on the right, \( J_q(g) J_p(f) \) is simply taking the last \((n-1)\)-rows of \( J_p(f) \).

Put
\[ A_p(f) = \left( \begin{array}{ccc}
\frac{\partial f_1}{\partial x^1} |_p & \cdots & \frac{\partial f_1}{\partial x^n} |_p \\
\vdots & \ddots & \vdots \\
\frac{\partial f_n}{\partial x^1} |_p & \cdots & \frac{\partial f_n}{\partial x^n} |_p 
\end{array} \right) \]

Then \( J_q(g) J_p(f) = A_p(f) \). Then we have the following conclusions:

- \( gof \) is a submersion at \( p \) \( \iff \) \( \text{rank} A_p(f) = n-1 \),
  \[ m \geq n-1. \]
Thus $g \circ f$ is a submersion iff $m \geq n-1$ and $\text{rank } A_p(f) = n-1$ for all $p \in M$. The coordinates in $\mathbb{R}^n$ charts of $M$ can be chosen arbitrarily, namely $(x^1, \ldots, x^n)$ is not necessarily the Cartesian coordinates.

$g \circ f$ is an immersion at $p$ iff
$$\begin{cases} \text{rank } A_p(f) = m, \\ m \leq n-1 \end{cases}$$
Thus $g \circ f$ is an immersion iff $m \leq n-1$ and $\text{rank } A_p(f) = m$ for all $p \in M$.

$g \circ f$ is a local diffeomorphism at $p$ iff
$$\begin{cases} \text{rank } A^p(f) = m, \\ m = n-1 \end{cases}$$
Thus $g \circ f$ is a local diffeomorphism iff $m = n-1$ and $\text{rank } A_p(f) = m$ for all $p \in M$. 