Let $f: M \to N$ be a map of smooth manifolds. $X$, $Y$ be vector fields on $M$, and $\bar{X}$, $\bar{Y}$ be vector fields on $N$. Suppose that $df_p(X(p)) = \bar{X}(f(p))$, $df_p(Y(p)) = \bar{Y}(f(p))$ for all $p \in M$. We'll show that $df_p([X, Y](p)) = [\bar{X}, \bar{Y}](f(p))$ for all $p \in M$.

Pick any smooth chart $(U, \varphi)$ on $M$ and $(V, \psi)$ on $N$ such that $f(U) \subset V$. Then we have the coordinate representations for the vector fields $X, Y, \bar{X}, \bar{Y}$ as follow.

$$X(p) = a^i(p) \frac{\partial}{\partial x^i} \bigg|_p,$$
$$Y(p) = b^i(p) \frac{\partial}{\partial x^i} \bigg|_p,$$
$$\bar{X}(q) = \bar{a}^j(q) \frac{\partial}{\partial y^j} \bigg|_q,$$
$$\bar{Y}(q) = \bar{b}^j(q) \frac{\partial}{\partial y^j} \bigg|_q,$$

where $a^i, b^i : U \to \mathbb{R}$ are smooth, and $\bar{a}^j, \bar{b}^j : V \to \mathbb{R}$ are smooth.

Then by the definition of $df_p$, we have
\[ d_{f(p)}(X(p)) = a^{i}(p) \left( \frac{\partial}{\partial x^{i}} \right)_{f(p)} = \frac{\partial y^{j}}{\partial x^{i}} \left( \frac{\partial}{\partial y^{j}} \right)_{f(p)} \]

Thus,

\[ d_{f(p)}(X(f(p))) = \bar{a}^{j}(f(p)) \left( \frac{\partial}{\partial y^{j}} \right)_{f(p)} \quad \text{if and only if} \quad \]

\[ \bar{a}^{j}(f(p)) = a^{i}(p) \frac{\partial y^{j}}{\partial x^{i}} \left( \frac{\partial}{\partial y^{j}} \right)_{f(p)} \quad (1) \]

Similarly, by replacing vector field \( X \) by vector field \( Y \) we get

\[ \bar{b}^{j}(f(p)) = b^{i}(p) \frac{\partial y^{j}}{\partial x^{i}} \left( \frac{\partial}{\partial y^{j}} \right)_{f(p)} \quad (2) \]

By the definition of Lie bracket, we have

\[ [X, Y](p) = \left( a^{i}(p) \frac{\partial b^{j}}{\partial x^{i}}(p) - b^{i}(p) \frac{\partial a^{j}}{\partial x^{i}}(p) \right) \left( \frac{\partial}{\partial y^{j}} \right)_{f(p)} \]

Now we apply \( d_{f(p)} \) to both sides. Using the linearity of \( d_{f(p)} \) and the fact that \( d_{f(p)} \left( \frac{\partial}{\partial x^{i}} \right) = \frac{\partial y^{k}}{\partial x^{i}} \left( \frac{\partial}{\partial y^{k}} \right)_{f(p)} \), we get

\[ d_{f(p)}([X, Y](p)) = \left( a^{i}(p) \frac{\partial b^{j}}{\partial x^{i}}(p) - b^{i}(p) \frac{\partial a^{j}}{\partial x^{i}}(p) \right) \frac{\partial y^{k}}{\partial x^{i}} \left( \frac{\partial}{\partial y^{k}} \right)_{f(p)} \quad (3) \]

By the definition of Lie bracket, we have

\[ [\bar{X}, \bar{Y}](q) = \left( \bar{a}^{i}(q) \frac{\partial \bar{b}^{k}}{\partial y^{i}}(q) - \bar{b}^{i}(q) \frac{\partial \bar{a}^{k}}{\partial y^{i}}(q) \right) \left( \frac{\partial}{\partial y^{k}} \right)_{q} \]

Then

\[ [\bar{X}, \bar{Y}](f(p)) = \left( \bar{a}^{i}(f(p)) \frac{\partial \bar{b}^{k}}{\partial y^{i}}(f(p)) - \bar{b}^{i}(f(p)) \frac{\partial \bar{a}^{k}}{\partial y^{i}}(f(p)) \right) \left( \frac{\partial}{\partial y^{k}} \right)_{f(p)} \quad (4) \]
Comparing (3) and (4), we are supposed to show that
\[
\left( \alpha^i(p) \frac{\partial b^j}{\partial x^i}(p) - b^i(p) \frac{\partial \alpha^j}{\partial x^i}(p) \right) \frac{\partial y^k}{\partial x^i}(p) \bigg|_{q(p)} = \bar{a}^i(f(p)) \frac{\partial \bar{b}^k}{\partial y^i}(f(p)) - \bar{b}^i(f(p)) \frac{\partial \bar{a}^k}{\partial y^i}(f(p))
\]
we have
\[
\bar{a}^i(f(p)) \frac{\partial \bar{b}^k}{\partial y^i}(f(p)) \quad (1) \quad \Rightarrow \quad \alpha^i(p) \frac{\partial y^i}{\partial x^k}(q(p)) \quad \frac{\partial \bar{b}^k}{\partial y^i}(f(p))
\]
chain rule
\[
\alpha^i(p) \frac{\partial}{\partial x^k} \left( \bar{b}^k(f(p)) \right)
\]
product rule
\[
\alpha^i(p) b^s(p) \frac{\partial y^k}{\partial x^i}(q(p)) + \alpha^i(p) \frac{\partial b^s}{\partial x^s}(q(p)) \frac{\partial y^k}{\partial x^i}(q(p))
\]
Similarly, by simply switching a and b, we get
\[
\bar{b}^i(f(p)) \frac{\partial \bar{a}^k}{\partial y^i}(f(p)) = \bar{b}^i(p) \frac{\partial \bar{a}^k}{\partial x^i}(q(p)) + \bar{b}^i(p) \frac{\partial \bar{a}^k}{\partial x^i}(q(p)) \frac{\partial y^k}{\partial x^i}(q(p)) \quad (7)
\]
\[
= \alpha^i(p) b^s(p) \frac{\partial y^k}{\partial x^i}(q(p))
\]
Subtracting (7) from (6), we get
\[
\text{RHS}(5) = \alpha^i(p) \frac{\partial b^s}{\partial x^i}(p) \frac{\partial y^k}{\partial x^s}(p) \bigg|_{q(p)} - \bar{b}^i(p) \frac{\partial \bar{a}^s}{\partial x^i}(p) \frac{\partial y^k}{\partial x^s}(p) \bigg|_{q(p)} = \text{LHS}(5).
\]
Let $f: M \to N$ be a smooth map, $X$ be a vector field on $M$ and $c: (a, b) \to M$ a flow line for $X$. Suppose $X'$ is a vector field on $N$ such that $f$ carries $X$ to $X'$. We'll show that $foc: (a, b) \to N$ is also a flow line for $X'$.

$$(a, b) \xrightarrow{c} M \xrightarrow{f} N$$

First we notice that $foc$ is also a smooth map from $(a, b)$ to $N$. Since $c$ is a flow line for $X$, $dc_{b_0} = X(c(b_0))$ for all $t \in (a, b)$. Since $f$ carries $X$ to $X'$, $df_p(X(p)) = X'(f(p))$ for all $p \in M$. To show that $foc$ is a flow line for $N$, we'll show that $d(foc)_{b_0} = X'(foc(b_0))$ for all $t \in (a, b)$. We have

$$d(foc)_{b_0} = df_{c(b_0)} \circ dc_{b_0} \quad \text{(taking derivative is a functor)}$$

$$= df_{c(b_0)}(X(c(b_0))) \quad \text{(c is a flow line for X)}$$

$$= df_p(X(p)) \quad \text{with } p = c(b_0)$$

$$= X'(f(p))$$

$$= X'(foc(b_0)),$$

which completes the proof.

Consider $\mathbb{R}^3$ with the standard smooth structure, and a vector

$$\vec{v} = v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k} \in \mathbb{R}^3.$$
Each point \( p \) in \( \mathbb{R}^3 \) associates with a vector \( \vec{OP} = p_1 \hat{i} + p_2 \hat{j} + p_3 \hat{k} \) where \( O \) is the origin of the chosen Cartesian coordinate system in \( \mathbb{R}^3 \).

We have

\[
\vec{v} \times \vec{p} = \begin{vmatrix} v_0 & v_3 & v_2 \\ v_1 & v_3 & v_1 \\ v_2 & v_1 & v_0 \end{vmatrix} \hat{k}
\]

We define the following map \( X_{\vec{v}} : \mathbb{R}^3 \rightarrow T\mathbb{R}^3 \),

\[
X_{\vec{v}}(p) = \begin{vmatrix} v_2 & v_3 & \frac{2}{\partial x} \\ p_2 & p_3 & \frac{\partial}{\partial x} p_1 \\ v_1 & v_2 & \frac{2}{\partial y} p_1 \end{vmatrix} + \begin{vmatrix} v_1 & v_2 & \frac{2}{\partial z} p_1 \end{vmatrix}.
\]

Then \( X_{\vec{v}}(p) = a^1(p) \frac{\partial}{\partial x} p + a^2(p) \frac{\partial}{\partial y} p + a^3(p) \frac{\partial}{\partial z} p \) where \( a^1, a^2, a^3 : \mathbb{R}^3 \rightarrow \mathbb{R} \),

\[
a^1(p) = v_2 p_3 - v_3 p_2,
\]
\[
a^2(p) = v_3 p_1 - v_1 p_3,
\]
\[
a^3(p) = v_1 p_2 - v_2 p_1.
\]

Since these \( a^i \)'s are linear, they are smooth maps. Thus \( X_{\vec{v}} \) is a smooth vector field on \( \mathbb{R}^3 \). Now let \( \vec{w} = w_1 \hat{i} + w_2 \hat{j} + w_3 \hat{k} \in \mathbb{R}^3 \) be another vector. Then \( X_{\vec{w}}(p) = b^1(p) \frac{\partial}{\partial x} p + b^2(p) \frac{\partial}{\partial y} p + b^3(p) \frac{\partial}{\partial z} p \) where \( b^1, b^2, b^3 : \mathbb{R}^3 \rightarrow \mathbb{R} \),

\[
b^1(p) = w_2 p_3 - w_3 p_2,
\]
\[
b^2(p) = w_3 p_1 - w_1 p_3,
\]
\[
b^3(p) = w_1 p_2 - w_2 p_1.
\]
The Lie bracket of $X_v$ and $X_w$ is $[X_v, X_w](p) = A \frac{\partial}{\partial x} p + B \frac{\partial^2}{\partial y \partial x} p + C \frac{\partial^3}{\partial z \partial y \partial x} p$,

where

$A = a_1(p) \frac{\partial b_1}{\partial x}(p) + a_2(p) \frac{\partial b_1}{\partial y}(p) + a_3(p) \frac{\partial b_1}{\partial z}(p)$

$- b_1(p) \frac{\partial a_1}{\partial x}(p) - b_2(p) \frac{\partial a_1}{\partial y}(p) - b_3(p) \frac{\partial a_1}{\partial z}(p)$

$B = a_1(p) \frac{\partial b_2}{\partial x}(p) + a_2(p) \frac{\partial b_2}{\partial y}(p) + a_3(p) \frac{\partial b_2}{\partial z}(p)$

$- b_1(p) \frac{\partial a_2}{\partial x}(p) - b_2(p) \frac{\partial a_2}{\partial y}(p) - b_3(p) \frac{\partial a_2}{\partial z}(p)$

$C = a_1(p) \frac{\partial b_3}{\partial x}(p) + a_2(p) \frac{\partial b_3}{\partial y}(p) + a_3(p) \frac{\partial b_3}{\partial z}(p)$

$- b_1(p) \frac{\partial a_3}{\partial x}(p) - b_2(p) \frac{\partial a_3}{\partial y}(p) - b_3(p) \frac{\partial a_3}{\partial z}(p)$.

We get

$A = (v_3 p_1 - v_1 p_3)(-w_5) + (v_1 p_2 - v_2 p_1) w_2 - (w_3 p_4 - w_4 p_3)(-v_3)$

$- (w_1 p_2 - w_2 p_1) v_2$

$= v_1 w_3 p_3 + v_1 w_2 p_2 - w_3 w_1 p_3 - v_2 w_4 p_2$

$= (v_1 w_3 - v_3 w_1) p_3 + (v_1 w_2 - v_2 w_2) p_2$

$= \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} p_3 + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} p_2$

Similarly

$B = (v_2 p_3 - v_3 p_2) w_3 + (v_1 p_2 - v_2 p_1) (-w_1)$

$- (w_2 p_3 - w_3 p_2) v_3 - (w_1 p_2 - w_2 p_1) (-v_4)$

$= (v_2 w_3 - v_3 w_3) p_3 - (v_1 w_2 - v_2 w_1) p_4$
\[
= \begin{vmatrix}
  v_2 & v_3 \\
  w_2 & w_3
\end{vmatrix} \rho_3 - \begin{vmatrix}
  v_1 & v_2 \\
  w_1 & w_2
\end{vmatrix} \rho_1.
\]

Similarly,
\[
C = (v_2 \rho_3 - v_3 \rho_2)(w_2) + (v_3 \rho_1 - v_1 \rho_3)w_1
\]
\[
= (w_2 \rho_3 - w_3 \rho_2)(v_1) - (w_3 \rho_1 - w_1 \rho_3)v_1
\]
\[
= v_3 w_2 p_2 + v_1 w_1 p_1 - v_2 w_2 p_2 - v_1 w_3 p_1
\]
\[
= (v_3 w_2 - v_2 w_5) p_2 - (v_1 w_3 - v_3 w_1) p_1
\]
\[
= -\begin{vmatrix}
  v_2 & v_3 \\
  w_2 & w_3
\end{vmatrix} \rho_2 - \begin{vmatrix}
  v_1 & v_3 \\
  w_1 & w_3
\end{vmatrix} \rho_1.
\]

Therefore,
\[
\left[\frac{\partial}{\partial x} \rho, \frac{\partial}{\partial y} \rho, \frac{\partial}{\partial z} \rho\right]
\]
\[
= \left[\begin{vmatrix}
  v_2 & v_3 \\
  w_2 & w_3
\end{vmatrix}, -\begin{vmatrix}
  v_1 & v_3 \\
  w_1 & w_3
\end{vmatrix}\right], \begin{vmatrix}
  v_1 & v_2 \\
  w_1 & w_2
\end{vmatrix}
\]
\[
= \begin{vmatrix}
  v_2 & v_3 \\
  w_2 & w_3
\end{vmatrix} \rho_2 - \begin{vmatrix}
  v_1 & v_3 \\
  w_1 & w_3
\end{vmatrix} \rho_1.
\]

If we identify \( \frac{\partial}{\partial x} \rho \equiv \xi, \frac{\partial}{\partial y} \rho \equiv \eta, \frac{\partial}{\partial z} \rho \equiv \zeta \), we will have

\[
\left[\frac{\partial}{\partial \xi}, \frac{\partial}{\partial \eta}, \frac{\partial}{\partial \zeta}\right](p) = -\left(\begin{vmatrix}
  v_2 & v_3 \\
  w_2 & w_3
\end{vmatrix}, -\begin{vmatrix}
  v_1 & v_3 \\
  w_1 & w_3
\end{vmatrix}\right) \times \rho
\]
\[
= -(\vec{v} \times \vec{w}) \times \rho.
\]

4) Consider \( \mathbb{R}^4 \) as a manifold with the standard smooth structure with atlas consisting of a single chart \((\mathbb{R}^4, (x_1, y_1, x_2, y_2))\).
Consider the vector field \( \mathbf{X} : \mathbb{R}^4 \rightarrow \mathbb{T}\mathbb{R}^4 \) given by

\[
\mathbf{X}(x_1, y_1, x_2, y_2) = -y_1 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial y_1} - y_2 \frac{\partial}{\partial x_2} + x_2 \frac{\partial}{\partial y_2}.
\]

This is a smooth vector field because each component function is smooth as functions from \( \mathbb{R}^4 \) to \( \mathbb{R} \). Pick any point \((x_1, y_1, x_2, y_2) \in \mathbb{R}^4\) and consider the flow line starting at this point. We call such a path a flow line \( c : (a, b) \rightarrow \mathbb{R}^4 \), where \( c(0) = (x_1, y_1, x_2, y_2) \) and \((a, b)\) is a neighborhood of 0 in \( \mathbb{R} \). By definition, we have \( \frac{dc}{dt} = \mathbf{X}(c(t)) \) for all \( t \in (a, b) \).

Write \( c(t) = (x_1(t), y_1(t), x_2(t), y_2(t)) \). We have

\[
\frac{dc}{dt} = x_1'(t) \frac{\partial}{\partial x_1} c(t) + y_1'(t) \frac{\partial}{\partial y_1} c(t) + x_2'(t) \frac{\partial}{\partial x_2} c(t) + y_2'(t) \frac{\partial}{\partial y_2} c(t)
\]

\[
\mathbf{X}(c(t)) = -y_1(t) \frac{\partial}{\partial x_1} c(t) + x_1(t) \frac{\partial}{\partial y_1} c(t) - y_2(t) \frac{\partial}{\partial x_2} c(t) + x_2(t) \frac{\partial}{\partial y_2} c(t).
\]

Thus we get a system of linear differential equations of order 1:

\[
\begin{cases}
    x_1'(t) = -y_1(t) \\
y_1'(t) = x_1(t) \\
x_2'(t) = -y_2(t) \\
y_2'(t) = x_2(t)
\end{cases} \quad \forall t \in (a, b)
\]

with \((x_1(0), y_1(0), x_2(0), y_2(0)) = (x_1, y_1, x_2, y_2)\).
The problem of finding the flow line of \( X \) at \((x_1, y_1, z_1)\) returns to the problem of solving the following differential equations
\[
\begin{aligned}
  x'(t) &= -y(t) \\
  y'(t) &= x(t)
\end{aligned}
\quad \forall t \in (a, b)
\]

with some given initial-value condition. In form of matrix,
\[
\begin{pmatrix}
  x' \\
  y'
\end{pmatrix} =
\begin{pmatrix}
  0 & -1 \\
  1 & 0
\end{pmatrix}
\begin{pmatrix}
  x \\
  y
\end{pmatrix} = \mathbf{A}
\]

We should diagonalize matrix \( \mathbf{A} \). We have

\[
\text{det} \left( \mathbf{xI} - \mathbf{A} \right) = \begin{vmatrix} x & 1 \\ -1 & x \end{vmatrix} = x^2 + 1 = (x - i)(x + i)
\]

Then \( \mathbf{A} \) has two eigenvalues \( \pm i \). We proceed to find the corresponding eigen vectors.

\[
(i) \mathbf{I} - \mathbf{A} = \begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} \sim \begin{pmatrix} 1 & i \\ 0 & 0 \end{pmatrix}
\]

Solution space is \( \{ (x, y) : x + iy = 0 \} = \{ x(1, 0) : x \in \mathbb{C} \} \). Thus an eigen vector corresponding to \( -i \) is \( \mathbf{v}_1 = (1, i) \).

\[
(i) \mathbf{I} - \mathbf{A} = \begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix} \sim \begin{pmatrix} 1 & -i \\ 0 & 0 \end{pmatrix}
\]

Solution space is \( \{ (x, y) : y - ix = 0 \} = \{ x(1, 1) : x \in \mathbb{C} \} \). Thus an eigen vector corresponding to \( i \) is \( \mathbf{v}_2 = (1, -i) \). Put

\[
\mathbf{P} = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}
\]
Then \( p^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \).

We have \( p^{-1} A p = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \). Put \( \begin{pmatrix} u \\ v \end{pmatrix} = p^{-1} \begin{pmatrix} x \\ y \end{pmatrix} \).

We get \( \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -iu \\ iv \end{pmatrix} \).

Thus \( \begin{cases} u' = -iu \\ v' = iv \end{cases} \) or equivalently \( \begin{cases} u' + iv = 0 \\ v' - iv = 0 \end{cases} \).

Thus \( u = u(t) = C_1 e^{-it} \), \( v = v(t) = C_2 e^{it} \) for some constant \( C_1, C_2 \in \mathbb{C} \). Put \( C_u = A_1 + iB_1 \), \( C_v = A_2 + iB_2 \) where \( A_1, A_2, B_1, B_2 \in \mathbb{R} \).

We have \( u(t) = (A_1 + iB_1) \begin{pmatrix} \cos t - i \sin t \end{pmatrix} = (A_1 \cos t + B_1 \sin t) - i(A_1 \sin t - B_1 \cos t) \),

\( v(t) = (A_2 + iB_2) \begin{pmatrix} \cos t + i \sin t \end{pmatrix} = (A_2 \cos t - B_2 \sin t) + i(A_2 \sin t + B_2 \cos t) \).

We have \( \begin{pmatrix} x \\ y \end{pmatrix} = p \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u + v \\ iu - iv \end{pmatrix} \).

Thus \( x = u + v = (A_1 + A_2) \cos t + (B_1 - B_2) \sin t \), \( y = i(u-v) = -[(A_1 - A_2) \sin t + (B_1 - B_2) \cos t] \).
Since \( x(0), y(0) \in \mathbb{R}, \quad A_1 + A_2 + i(B_1 + B_2) = 0 \) and 

\[-(B_1 - B_2) + i(A_1 - A_2) = 0.\]

Thus \( b_1 + b_2 = 0 \) and \( A_1 - A_2 = 0 \). Put \( A = A_1 = A_2 \mathbb{R} \) and \( B = B_1 = -B_2 \). We have

\[
x = 2A \cos t + 2B \sin t, \quad y = 2A \sin t - 2B \cos t
\]

Renaming the constants \( 2A, 2B \) by simply \( A, B \), we get

\[
\begin{align*}
x &= A \cos t + B \sin t \\
y &= -B \cos t + A \sin t
\end{align*}
\]

Thus, the system because

\[
\begin{vmatrix}
a & b \\
-b & a
\end{vmatrix}
\begin{vmatrix}
\cos t \\
\sin t
\end{vmatrix} = 1,
\]

we can determine \( A, B \mathbb{R} \) such that \( x, y \) in \((*)\) satisfy any given initial value condition. Thus, the system

\[
\begin{align*}
x_1'(t) &= -y_1(t) \\
y_1'(t) &= x_1(t) \\
x_2'(t) &= -y_2(t) \\
y_2'(t) &= x_2(t)
\end{align*}
\]

where \( A_1, B_1, A_2, B_2 \) are uniquely determined by the initial condition

\[(x_1(0), y_1(0), x_2(0), y_2(0)) = (x_1, y_1, x_2, y_2).\]

Since these solutions exist for all \( t \in \mathbb{R} \), the flow line \( \phi(x) \) for \( X \) at any point flows infinitely.

(5) Let \( M \) be an \( n \)-dimensional smooth manifold and \( X_1, \ldots, X_n \) be smooth
vector fields on $M$. Each point in $M \times \mathbb{R}$ is written as $(x, t) \in M \times \mathbb{R}$. For each $i = 1, \ldots, n$, there exists an open subset $U_i \subset M \times \mathbb{R}$ with $M \times \{0\} \subset U_i$ and a smooth function $\theta_i : U_i \rightarrow M$, $(x, t) \mapsto \theta_i(x, t)$ such that

1. For each fixed $x$, $(d\theta_i)_e \left( \frac{d}{dt} \right) = X_i(\theta_i(x, t))$,
2. $\theta_i(x, 0) = x$.

Pick a point $p \in M$. Since $(p, 0) \in U_1$, there is $\delta_1 > 0$ such that $\{p\} \times (-\delta_1, \delta_1) \subset U_1$. We define $f_1 : (-\delta_1, \delta_1) \rightarrow M$ such that $f_1(t) = \theta_1(p, t)$ for all $t \in (-\delta_1, \delta_1)$. With this definition, $f_1$ is smooth.

We have $f_1(0) = \theta_1(p, 0) = p$. Thus $(f_1(0), 0) = (p, 0) \in U_2$. Thus there exists $\delta_2 < \delta_1$ such that $f_1(-\delta_2, \delta_2) \times (-\delta_2, \delta_2) \subset U_2$. We define $f_2 : (-\delta_2, \delta_2) \times (-\delta_2, \delta_2) \rightarrow M$ such that $f_2(t_1, t_2) = \theta_2(f_1(t_1), t_2)$ for all $t_1, t_2 \in (-\delta_2, \delta_2)$. Suppose that we found $\delta_k < \delta_{k-1}$ such that $f_{k-1}((\delta_k, \delta_k)^2) \times (\delta_k, \delta_k)$ is contained in $U_k$, we can define $f_k : (-\delta_k, \delta_k)^2 \rightarrow M$, with

$$f_k(t_1, \ldots, t_k) = \theta_k \left( f_{k-1}(t_1, \ldots, t_{k-1}), t_k \right).$$

Then $f_k$ is smooth. We have $f_k(0) = \theta_k(f_{k-1}(0), 0) = \theta_k(p, 0) = p$. Thus, $(f_k(0), 0) = (p, 0) \in U_{k+1}$. Thus there exists $\delta_{k+1} < \delta_k$ such that $f_k((\delta_{k+1}, \delta_{k+1})^2) \times (\delta_{k+1}, \delta_{k+1}) \subset U_{k+1}$. Then we continue the similar argument to define $f_{k+1}, f_k, \ldots, f_1$. Put $\varepsilon = \delta_1$. Then we have $n$ functions $f_j$ with
$f_j : (-\varepsilon, \varepsilon)^j \to M$, $f_j(t_{i_1}, \ldots, t_j) = \theta_j(f_{i_1 \ldots i_{j-1}}(t_{i_1}, \ldots, t_{i_{j-1}}), t_j)$.

Let $\pi_i : U_i \to M$ be the $i$th projection map, i.e. $\pi_i(x, t) = x$, for all $(x, t) \in U_i$.

Then $\pi_i(U_i)$ is an open set in $M$. Since $\rho = \pi_i(p, 0) \in \pi_i(U_i)$, $p \in \bigcap_{i=1}^n \pi_i(U_i)$.

Thus we can find a neighborhood $U$ of $p$ in $M$ such that $\cup_{i=1}^n \pi_i(U_i)$ and that there exists a diffeomorphism $\varphi : U \to \mathbb{R}^n$. Because $f_j(\rho) = \rho \in U$, we can shrink $\varepsilon$ such that $\text{Im} f_j \subset U$ for all $j = 1, \ldots, n$.

The coordinate representation of $f_j$ is $f_j = \varphi \circ f_j$, and $\theta_i|_{\pi_i(U)}$ is $\hat{\theta}_i = \varphi \circ \left( \varphi^{-1}(x_1^i, \ldots, x_n^i), t \right)$. Thus we can view $f_j$ and $\theta_i$ as $\hat{f}_j$ and $\hat{\theta}_i$ and try to prove the same statement of the problem. This point of view allows us to consider $U$ as $\mathbb{R}^n$, and $f_j : (-\varepsilon, \varepsilon)^j \to \mathbb{R}^n$, $\theta_j$ as a function from a neighborhood of $\rho$ in $\mathbb{R}^{n+1}$ to $\mathbb{R}^n$, and $x_1^i, \ldots, x_n^i$ as functions from a neighborhood of $p$ in $\mathbb{R}^n$ to $\mathbb{R}^n$. Put $\theta_i = \varphi_i(\cdot)$.

We have $\theta_i(x_1^i, \ldots, x_n^i, 0) = (x_1^i, \ldots, x_n^i)$. Also, we put $\theta_i = (\theta_i^1, \ldots, \theta_i^n)$, where
each \( \theta_i^k = \theta_i^k(x^1, \ldots, x^n, t) \) is a real-valued function. Then
\[
\theta_i^k(x^1, \ldots, x^n, 0) = x^i
\]

Consequently, \( \frac{\partial \theta_i^k}{\partial x_j} \bigg|_{t=0} = \delta_j^k \). Moreover, \( \frac{\partial \theta_i}{\partial t} (x^1, \ldots, x^n, t) = X_i(\theta_i(x^1, \ldots, x^n, t)) \).

We write \( f_j = (f_{j1}, \ldots, f_{jn}) \) where each \( f_{jk} = (f_{jk})_k(t_1, \ldots, t_j) \) is a real-valued function on \((-\varepsilon, \varepsilon)^j\). Then the differential of \( f_j \) can be viewed as the Jacobian matrix
\[
d f_j = \begin{pmatrix}
\frac{\partial f_{j1}}{\partial t_1} & \cdots & \frac{\partial f_{jn}}{\partial t_1} \\
\vdots & \ddots & \vdots \\
\frac{\partial f_{j1}}{\partial t_j} & \cdots & \frac{\partial f_{jn}}{\partial t_j}
\end{pmatrix}
\]

We want to show that
\[
(df_j)_0 \left( \begin{array}{c}
\frac{\partial}{\partial t_1} \\
\vdots \\
\frac{\partial}{\partial t_j}
\end{array} \right) = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & \ddots & \vdots & \vdots \\
0 & \cdots & 1 & 0
\end{pmatrix}
\begin{pmatrix}
X_1(p) \\
\vdots \\
X_j(p)
\end{pmatrix}
\]

This is equivalent to showing that \( (df_j)_0 \left( \frac{\partial}{\partial t_i} \bigg|_{\theta_i^k} \right) = X_i(p) \) for all \( i = 1, \ldots, j \).

We'll show that \( \left( \frac{\partial (f_{j1})_1}{\partial t_i} \bigg|_{\theta_i^k}, \ldots, \frac{\partial (f_{jn})_n}{\partial t_i} \bigg|_{\theta_i^k} \right) = X_i(p) \ \forall i = 1, \ldots, j \) by induction in \( j \). For \( j = 1 \), \( f_1(t_1) = \theta_1(p, t_1) \). Then
\[ \frac{\partial \theta_i}{\partial t} = \frac{\partial \theta_i}{\partial t} (p, t) = X_i(\theta_i(p, t)) \]

Thus \[ \frac{\partial \theta_i}{\partial t} (0) = X_i(\theta_i(p, 0)) = X_i(p). \]

Suppose that our claim holds for \( j-1 \), i.e. \[ \frac{\partial \theta_{j-1}}{\partial t} \bigg|_{t=t_i} = X_{j}(p) \] for all \( i = 1, \ldots, j-1 \). We'll show that it also holds for \( j \). We have

\[ f_j(t_1, \ldots, t_{j-1}, t_j) = \theta_j\left(f_{j-1}(t_1, \ldots, t_{j-1}), t_j \right). \]

Then

\[ \frac{\partial f_j}{\partial t_j} \bigg|_{t_j=0} = \frac{\partial}{\partial t} \bigg|_{t=0} \theta_j\left(f_{j-1}(t_1, \ldots, t_{j-1}), t \right) = X_j\left(\theta_j\left(f_{j-1}(t_1, \ldots, t_{j-1}), 0 \right)\right) \]

Then

\[ \frac{\partial f_j}{\partial t_j} \bigg|_{t_j=0} = X_j\left(\theta_j\left(f_{j-1}(0, \ldots, 0), 0 \right)\right) = X_j(\theta_j(p, 0)) = X_j(p). \]

For each \( i = 1, 2, \ldots, j-1 \), we write

\[ f_j(t_1, \ldots, t_{j-1}, t_j) = \left(\theta_i\left(f_{i-1}(t_1, \ldots, t_{i-1}), t_j \right), \ldots, \theta_j\left(f_{j-1}(t_1, \ldots, t_{j-1}), t_j \right)\right) \]

\[ \bar{z} = \left(\theta_{i-1}(t_1, \ldots, t_{i-1}), \ldots, \theta_j(t_1, \ldots, t_j)\right) \]

Then

\[ \frac{\partial f_j}{\partial t_i} (t_1, \ldots, t_j) = \left(\frac{\partial}{\partial t_i} \left[\theta_i\left(f_{i-1}(t_1, \ldots, t_{i-1}), t_j \right)\right], \ldots, \frac{\partial}{\partial t_i} \left[\theta_j\left(f_{j-1}(t_1, \ldots, t_{j-1}), t_j \right)\right]\right) \]

\[ = \left(\frac{\partial \theta_i}{\partial x^1} \frac{\partial f_{i-1}}{\partial t_i} + \ldots + \frac{\partial \theta_i}{\partial x^n} \frac{\partial f_{i-1}}{\partial t_i}, \ldots, \frac{\partial \theta_j}{\partial x^1} \frac{\partial f_{j-1}}{\partial t_i} + \ldots + \frac{\partial \theta_j}{\partial x^n} \frac{\partial f_{j-1}}{\partial t_i}\right) \]

Because \[ \frac{\partial \theta_i}{\partial x^i} \bigg|_{t=0} = \delta_i^k, \] we have
\[
\frac{\partial f_j}{\partial t_i} (t_1, \ldots, t_{j-1}, 0) = \left( \frac{\partial f_{j-1}}{\partial t_i} \bigg|_0, \frac{\partial f_{j-1}}{\partial t_2} \bigg|_0, \ldots, \frac{\partial f_{j-1}}{\partial t_n} \bigg|_0 \right).
\]

Thus,

\[
\frac{\partial f_j}{\partial t_i} (0, \ldots, 0, 0) = \left( \frac{\partial f_{j-1}}{\partial t_i} \bigg|_0, \frac{\partial f_{j-1}}{\partial t_2} \bigg|_0, \ldots, \frac{\partial f_{j-1}}{\partial t_n} \bigg|_0 \right) = X_i (p)
\]

by the inductive hypothesis. Therefore, our claim holds for \( j \), and thus holds for all \( 1, 2, \ldots, n \).