① Let $(X, x_0)$ be a connected smooth $m$-manifold, and $\pi: (Y, y_0) \rightarrow (X, x_0)$ be its universal covering map. Assume there is a smooth structure on $Y$ such that $Y$ is a smooth $m$-manifold and $\pi$ is a smooth covering map. The latter means: for each $p \in X$, there are an open connected neighborhood $U$ of $p$, a discrete space $F$ and a diffeomorphism $\phi: \pi^{-1}(U) \rightarrow U \times F$ such that the following diagram commutes.

\[ \begin{array}{ccc} 
\pi^{-1}(U) & \xrightarrow{\pi} & U \\
\phi \downarrow & & \downarrow \text{proj} \\
U \times F & & 
\end{array} \]

Because $Y$ is a topological covering space of $X$, we knew that there is a right action of $\pi_1(X, x_0)$ on $Y$ as follows.

For each $y \in Y$ and $[\alpha] \in \pi_1(X, x_0)$, let $\tilde{\alpha}$ be a path from $y_0$ to $y$ in $Y$. Then $\tilde{\alpha} \cdot \gamma$ is a path from $y_0$ to $\alpha(y)$. Then $\tilde{\alpha} \cdot (\pi \circ \gamma)$ is also a path from $x_0$ to $\pi(y)$. For each path $\alpha$ in $X$ starting from $x_0$, we denote by $\widetilde{\alpha}$ the lift of $\alpha$ at $y_0$. Then

$y \cdot [\alpha] := \tilde{\alpha} \cdot (\pi \circ \gamma)(1)$. 
By this definition, the action is transitive on each fiber. Indeed, suppose we have \( y, y' \in \pi^{-1}(x) \). Let \( \alpha \) be a path in \( Y \) from \( y \) to \( y' \).

Since \( \pi(y) = \pi(y') = x \), \( \alpha \) is a loop based at \( x \) in \( X \). Let \( \gamma \) be a path from \( y_0 \) to \( y \). Then

\[
\lambda = (\pi \circ \gamma) \cdot (\pi \circ \alpha) \cdot (\pi \circ \gamma)^{-1}
\]

is a loop based at \( x_0 \) in \( X \). Then by definition,

\[
y \cdot [\lambda] = \lambda \cdot (\pi \circ \gamma)(1) = (\pi \circ \gamma) \cdot (\pi \circ \alpha) \cdot (\pi \circ \gamma)^{-1} = \pi \circ (\gamma \cdot \alpha)(1) = (\gamma \cdot \alpha)(1) = y'
\]

This concludes that \( \pi(X, x_0) \) acts transitively on \( Y \). For each \( [\lambda] \in \pi(Y, x_0) \), the map \( f_{[\lambda]} : Y \to Y \), \( y \mapsto y \cdot [\lambda] \) is continuous. Moreover, the inverse of \( f_{[\lambda]} \) is \( f_{[\lambda]^{-1}} \), which is also continuous. Thus \( f_{[\lambda]} \) is a homeomorphism. Assume that \( f_{[\lambda]} \) is smooth for every \( [\lambda] \in \pi(Y, x_0) \). Then \( (f_{[\lambda]})^{-1} = f_{[\lambda]^{-1}} \) is also smooth. Thus \( f_{[\lambda]} \) is a diffeomorphism on \( Y \). Moreover,

\[
\pi \circ f_{[\lambda]}(y) = \pi(\lambda \cdot (\pi \circ \gamma)(1)) = \lambda \cdot (\pi \circ \gamma)(1) = (\pi \circ \gamma)(1) = \pi(y).
\]

Thus \( \pi \circ f_{[\lambda]} = \pi \). \((*)\)
So far, we know that \( \pi_1(X, x_0) \) acts transitively on \( Y \) by diffeomorphisms and \( \pi \circ \sigma(x) = \pi \) for all \( x \in \pi_1(X, x_0) \). From now on, we will not refer to any other properties of \( \pi_1(X, x_0) \). Thus, we should regard \( \pi_1(X, x_0) \) as a group \( G \) acting transitively on \( Y \) by diffeomorphisms. Also, by switching the order of path composition (write \([\gamma]*[\delta]\) instead of \([\delta]*[\gamma]\) for path composition), the right action becomes a left action. Each element \( g \in G \) can be viewed as a diffeomorphism \( g : Y \to Y \), \( g(y) = g \cdot y \). Then (\(*\)) implies \( g \cdot \pi = \pi \).

For each \( n \geq 0 \), let \( \Omega^n(X) \) and \( \Omega^n(Y) \) be the real vector spaces of differential forms on \( X \) and \( Y \) respectively. Since each \( g : Y \to Y \) is a diffeomorphism, it induces a linear isomorphism \( g^* : \Omega^n(Y) \to \Omega^n(X) \). Then a form \( \eta \in \Omega^n(Y) \) is said to be \( G \)-invariant if \( g^*(\eta) = \eta \) for all \( g \in G \). Let \( [\Omega^n(Y)]^G \) be the subset of \( \Omega^n(Y) \) containing all \( G \)-invariant forms.

First, we'll show that \([\Omega^n(Y)]^G\) is a vector subspace of \( \Omega^n(Y) \). Since \( g^*(0) = 0 \) for all \( g \in G \), \( 0 \in [\Omega^n(Y)]^G \). Suppose we have \( c \in \mathbb{R} \), \( \eta_1, \eta_2 \in [\Omega^n(Y)]^G \). Then

\[
g^*(c \eta_1) = c g^*(\eta_1) = c \eta_1, \quad g^*(\eta_2) = \eta_2 \quad \forall g \in G.
\]
Since $\xi$ is a linear map, 
\[ \xi^*(\eta_1 + \eta_2) = \xi^*(\eta_1) + \xi^*(\eta_2) = \eta_1 + \eta_2, \]
\[ \xi^*(c\eta_1) = c\xi^*(\eta_1) = c\eta_1. \]

Thus, $\eta_1 + \eta_2, c\eta_1 \in [\Omega^n(Y)]^G$. This means $[\Omega^n(Y)]^G$ is a vector subspace of $\Omega^n(Y)$.

Since $\pi: Y \to X$ is a smooth map, it induces a linear map $\pi^*: \Omega^n(Y) \to \Omega^n(X)$. We want to show that $\pi^*$ is injective and $\text{Im}(\pi^*) = [\Omega^n(Y)]^G$.

**Show that $\pi^*$ is injective.**

Suppose that $\omega \in \Omega^n(X)$ satisfies $\pi^*(\omega) = 0$. Take $p \in X$ arbitrarily.

We want to show that $\omega_p = 0$. Recall that $\omega_p \in \Omega^n(T_pX)$ is an alternating linear map and linear map from $(T_pX)^n$ to $\mathbb{R}$. Let $v_1, ..., v_n \in T_pX$. We want to show $\omega_p(v_1, ..., v_n) = 0$.

Take $q \in \pi^{-1}(p)$. For any $v_1, ..., v_n \in T_qY$, we have
\[ \pi^*(\omega)_q(v_1, ..., v_n) = 0 \]
Thus, $\omega_p(d\pi_q(v_1), ..., d\pi_q(v_n)) = 0$, where $d\pi_q$ is the linear map between the tangent spaces $d\pi_q: T_qY \to T_pX$. It suffices to show that for each $i = 1, 2, ..., n$, there is $v_i \in T_qY$ such that $v_i = d\pi_q(w_i)$.

Since $\pi$ is a smooth covering map, it is a local diffeomorphism by definition. Thus, $d\pi_q$ is a linear isomorphism. Thus the existence
of \( \nu \) is always guaranteed.

Show that \( \text{Im}(\pi^*) \subseteq [\Sigma^n(Y)]^f \)

For each \( \omega \in \Sigma^n(X) \), we put \( \eta = \pi^*(\omega) \). We want to show that \( \eta \in [\Sigma^n(Y)]^f \). Take any \( g \in \mathcal{G} \), we want to show \( g^*(\eta) = \eta \). We have

\[
g^*(\eta) = g^*(\pi^*(\omega)) = (g^* \circ \pi^*)(\omega) = (\pi^* g)^*(\omega) \quad \text{(by the functorial property of pullbacks)}
\]

\[= \pi^*(\omega) = \eta.
\]

Show that \( [\Sigma^n(Y)]^f \subseteq \text{Im}(\pi^*) \)

Take any \( \eta \in [\Sigma^n(Y)]^f \). We want to find \( \omega \in \Sigma^n(X) \) such that \( \eta = \pi^*(\omega) \). We have

\[
\eta = \pi^*(\omega) \iff \forall q \in Y, \; \eta_q = \pi^*(\omega)_q
\]

\[\iff \forall q \in Y, \; \forall v_1, \ldots, v_n \in ET_q Y, \; \eta_q(v_1, \ldots, v_n) = \pi^*(\omega)_q(v_1, \ldots, v_n)
\]

\[\iff \forall q \in Y, \; \forall v_1, \ldots, v_n \in ET_q Y, \; \eta_q(v_1, \ldots, v_n) = \omega_p(d\pi_q(v_1), \ldots, d\pi_q(v_n)) \quad (\ast)
\]

For each \( p \in X \), we want to define \( \omega_p \in \Sigma^n(T_p X) \) so that \( (\ast) \) is satisfied. Pick any \( q \in \pi^{-1}(p) \). For \( v_1, \ldots, v_n \in ET_p X \), we define

\[
\omega_p(v_1, \ldots, v_n) := \eta_q(d\pi_q^{-1}(v_1), \ldots, d\pi_q^{-1}(v_n)) \quad (1)
\]

If the definition of \( \omega_p \) does not depend on the choice of \( q \) in \( \pi^{-1}(p) \)
then (4) is certainly satisfied.

Suppose \( q, r \in \pi^{-1}(p) \). We'll show that for any \( u_1, \ldots, u_n \in T_pX \),

\[
\eta_q \left( (d\pi_q)^{-1}(u_1), \ldots, (d\pi_q)^{-1}(u_n) \right) = \eta_r \left( (d\pi_r)^{-1}(u_1), \ldots, (d\pi_r)^{-1}(u_n) \right) \quad (2)
\]

Since \( q \) and \( r \) belong to the fiber of \( p \) and \( G \) acts on \( Y \) transitively in each fiber, there exists \( g \in G \) such that \( r = g(q) \).

Since \( \eta \in \Omega^n(Y) \), we have \( \eta = g^* \eta \). Now evaluating both sides at \( q \), we get

\[
\eta_q(u_1, \ldots, u_n) = \eta_{g(q)}(d\eta_{g(q)}(u_1), \ldots, d\eta_{g(q)}(u_n)) \quad \forall u_1, \ldots, u_n \in T_qY \quad (3)
\]

Choose \( u_i = (d\pi_q)^{-1}(u_i) \). Then (3) becomes

\[
\eta_q \left( (d\pi_q)^{-1}(u_1), \ldots, (d\pi_q)^{-1}(u_n) \right) = \eta_r \left( d\eta_q \circ (d\pi_q)^{-1}(u_1), \ldots, d\eta_q \circ (d\pi_q)^{-1}(u_n) \right) \quad (4)
\]

Since \( \pi \circ g = \pi \), we have \( d\pi_r \circ d\eta_q = d\pi_q \). Thus \( d\eta_q \circ (d\pi_q)^{-1} = (d\pi_r)^{-1} \).

Thus, (4) becomes

\[
\eta_q \left( (d\pi_q)^{-1}(u_1), \ldots, (d\pi_q)^{-1}(u_n) \right) = \eta_r \left( (d\pi_r)^{-1}(u_1), \ldots, (d\pi_r)^{-1}(u_n) \right)
\]

Therefore, (2) is proved. By (1), \( \omega_p \) is alternating and multilinear because \( d\eta_q \) is so. Thus \( \omega_p \in \Omega^n(T_pX) \).
we get a section $\omega : X \to \bigcup_{p \in X} \Omega^n(T_pX)$. To show that $\omega \in \Omega^n(X)$, it remains to show that $\omega$ is smooth.

Take $p_0 \in X$ arbitrarily. We'll show that $\omega$ is smooth at $p_0$. Pick $q_0 \in \pi^{-1}(p_0)$. Since $\pi$ is a local diffeomorphism, there are open neighborhoods $V$ of $q_0$ in $Y$, and $U$ of $p_0$ in $X$ such that $\pi|_V : V \to U$ is a diffeomorphism. By shrinking $U$ and $V$ if necessary, we can assume that there are smooth charts $(U, \phi)$ and $(V, \psi)$.

Let $(y^1, \ldots, y^n)$ be the coordinates on $V$, and $(x^1, \ldots, x^n)$ be the coordinates on $U$. Then the coordinate representation of $\pi$, namely $\tilde{\pi} : V' \to U'$, is a diffeomorphism. Thus, for each $i = 1, 2, \ldots, m$, $y^i$ is a smooth map of $x^1, \ldots, x^n$.

$$y^i = h_i(x^1, \ldots, x^n).$$

Then $\tilde{\pi}^{-1}(x^1, \ldots, x^n) = (h_1(x^1, \ldots, x^n), \ldots, h_m(x^1, \ldots, x^n))$.

For each $p \in U$, we put $q = (\pi|_V)^{-1}(p)$. Then
\[(d\pi_q)^{-1} = (\pi_q^*(\omega_{\pi V}))^{-1} = \pi_q^*(\omega_{\pi V})^{-1}(\frac{\partial b_j}{\partial x_i})_{1 \leq i, j \leq n}\]

Thus,
\[(d\pi_q)^{-1}\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial b_j}{\partial x_i} \frac{\partial}{\partial y^j}\] (5)

Since \(\eta\) is smooth on \(V\), for each tuple \(1 \leq j_1 < j_2 < \ldots < j_n \leq m\), there is a smooth function \(\eta_{j_1 \ldots j_n} : V \to \mathbb{R}\) such that
\[\eta_q = \eta_{j_1 \ldots j_n}(q) \, dy^{j_1} \wedge \ldots \wedge dy^{j_n}\]

We have
\[\omega_p = \omega_p \left(\frac{\partial}{\partial x^i}_{\mid p} \ldots \frac{\partial}{\partial x^n}_{\mid p}\right) \, dx^1 \wedge \ldots \wedge dx^n\]

\[\omega_{\eta_{j_1 \ldots j_n}}(p)\]

We want to show that each \(\omega_{\eta_{j_1 \ldots j_n}} : U \to \mathbb{R}\) is smooth.

In (1), we choose \(u_1 = \frac{\partial}{\partial x^1}_{\mid p}, \ldots, \frac{\partial}{\partial x^n}_{\mid p}\). Then (1) gives
\[
\omega_{\eta_{j_1 \ldots j_n}}(p) = \eta_{j_1}(q) \left(\frac{\partial b_{j_1}}{\partial x^1}_{\mid p} \frac{\partial}{\partial y^1}_{\mid q} \ldots \frac{\partial b_{j_n}}{\partial x^n}_{\mid p} \frac{\partial}{\partial y^n}_{\mid q}\right) \\
= \eta_{j_1 \ldots j_n}(q) \left(\frac{\partial b_{j_1}}{\partial x^1}_{\mid p} \ldots \frac{\partial b_{j_n}}{\partial x^n}_{\mid p} \eta_q \left(\frac{\partial}{\partial y^1}_{\mid q} \ldots \frac{\partial}{\partial y^n}_{\mid q}\right)\right) \\
= \frac{\partial b_{j_1}}{\partial x^1}_{\mid p} \ldots \frac{\partial b_{j_n}}{\partial x^n}_{\mid p} \eta_{j_1 \ldots j_n} \circ (\pi_V)^{-1}(p)\]

Thus \(\omega_{j_1 \ldots j_n}\) is smooth on \(U\). Therefore \(\omega\) is smooth and \(\omega \in \Omega^n(M)\).
Let $M$ be a smooth $n$-manifold, and $\Omega^k(M)$ be the space of all differential $k$-forms on $M$. Put $\Omega(M) = \bigoplus_{k=0}^{n} \Omega^k(M)$.

We know that the wedge product $\wedge$ is an operator on $\Omega(M)$ which satisfies the following rules.

(i) If $\omega_1 \in \Omega^p(M)$, $\omega_2 \in \Omega^q(M)$ then $\omega_1 \wedge \omega_2 \in \Omega^{p+q}(M)$.

(ii) Bilinearity:
\[
(\omega_1 + b \omega_2) \wedge \eta = \omega_1 \wedge \eta + b(\omega_2 \wedge \eta), \quad \eta \wedge (\omega_1 + b \omega_2) = \eta \wedge \omega_1 + b(\eta \wedge \omega_2),
\] for all $a, b \in \mathbb{R}$, $\omega_1, \omega_2 \in \Omega^p(M)$, $\eta \in \Omega^q(M)$.

(iii) Associativity:
\[
\omega \wedge (\eta \wedge \xi) = (\omega \wedge \eta) \wedge \xi \quad \forall \omega \in \Omega^p(M), \eta \in \Omega^q(M), \xi \in \Omega^\ell(M).
\]

For each $k \geq 0$, there is a linear map $d : \Omega^k(M) \to \Omega^{k+1}(M)$, which gives rise to a chain complex of real vector spaces:

\[
\begin{array}{cccccc}
0 & \rightarrow & \Omega^0(M) & \xrightarrow{d} & \Omega^1(M) & \xrightarrow{d} & \Omega^2(M) & \xrightarrow{d} & \cdots
\end{array}
\]

Also, $d$ satisfies $d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^k \omega \wedge (d\eta)$ for all $\omega \in \Omega^k(M), \eta \in \Omega^\ell(M)$.

The de Rham cohomology groups were defined as
\[
H^k_{dR}(M) = \frac{\ker(d : \Omega^k(M) \to \Omega^{k+1}(M))}{\operatorname{im}(d : \Omega^{k-1}(M) \to \Omega^k(M))},
\]

Put $H_{dR}(M) = \bigoplus_{k=0}^{\infty} H^k_{dR}(M)$. 

To make $H_{DR}(M)$ a ring, we have to define a product law on it. First, we'll show that $\wedge: \Omega^p(M) \times \Omega^q(M) \rightarrow \Omega^{p+q}(M)$ induces a map $\Lambda: H^p_{DR}(M) \times H^q_{DR}(M) \rightarrow H^{p+q}_{DR}(M)$.

Let $\omega_1, \omega'_1$ be closed $p$-forms, and $\omega_2, \omega'_2$ be closed $q$-forms on $M$. Suppose that $\omega'_1 - \omega_1 = d\eta_1$ for some $\eta_1 \in \Omega^{p-1}(M)$, and $\omega'_2 - \omega_2 = d\eta_2$ for some $\eta_2 \in \Omega^{q-1}(M)$.

We'll show that $\omega'_1 \wedge \omega'_2 - \omega_1 \wedge \omega_2 = d\eta$ for some $\eta \in \Omega^{p+q-1}(M)$.

We have $\omega'_1 = \omega_1 + d\eta_1$ and $\omega'_2 = \omega_2 + d\eta_2$. Then

$$\omega'_1 \wedge \omega'_2 = (\omega_1 + d\eta_1) \wedge (\omega_2 + d\eta_2)$$

$$= (\omega_1 \wedge \omega_2) + (d\eta_1) \wedge \omega_2 + \omega_1 \wedge (d\eta_2) + (d\eta_1) \wedge (d\eta_2) \quad (** \text{(bi-linearity of wedge product)})$$

We have

$$(d\eta_1) \wedge \omega_2 = d(\eta_1 \wedge \omega_2) - (-1)^p \eta_1 \wedge (d\omega_2) \quad (\text{by } (**))$$

$$= d(\eta_1 \wedge \omega_2), \quad \text{(because } d\omega_2 = 0)$$

$$\omega_1 \wedge (d\eta_2) = (-1)^p \left[ d(\omega_1 \wedge \eta_2) - (d\omega_1) \wedge \eta_2 \right] \quad (\text{by } (**))$$

$$= (-1)^p d(\omega_1 \wedge \eta_2) \quad \text{(because } d\omega_1 = 0)$$

$$= d[(-1)^p \omega_1 \wedge \eta_2], \quad \text{(} d \text{ is linear)}$$

$$(d\eta_1) \wedge (d\eta_2) = d(d\eta_1) - (-1)^{p+q} (\eta_1 \wedge d^2 \eta_2) \quad (\text{by } (**))$$

$$= d(\eta_1 \wedge d\eta_2), \quad \text{(because } d^2 \eta_2 = 0).$$
Therefore, \( (**) \) becomes

\[
\omega_1 \wedge \omega_2 = \omega_1 \wedge \omega_2 + d(\eta_1 \wedge \omega_2) + d[(-1)^p \omega_1 \wedge \eta_2] + d(\eta_1 \wedge d\eta_2)
\]

\[
= \omega_1 \wedge \omega_2 + d\eta,
\]

where \( \eta = \eta_1 \wedge \omega_2 + (-1)^p \omega_1 \wedge \eta_2 + \eta_1 \wedge d\eta_2 \in \Omega^{p+q-1}(\mathcal{X}). \)

Note that \( \omega_1 \wedge \omega_2 \) and \( \omega_1 \wedge \omega_2 \) are closed \((p+q)\)-forms because

\[
d(\omega_1 \wedge \omega_2) = \frac{d\omega_1}{\partial_1} \wedge \omega_2 + (-1)^p \omega_1 \wedge \frac{d\omega_2}{\partial_2} = 0,
\]

\[
d(\omega_1 \wedge \omega_2) = \frac{d\omega_1}{\partial_1} \wedge \omega_2 + (-1)^p \omega_1 \wedge \frac{d\omega_2}{\partial_2} = 0.
\]

By what we have showed, \( \wedge \) descends to a map

\[
\tilde{\wedge} : H_{dR}^p(M) \times H_{dR}^q(M) \longrightarrow H_{dR}^{p+q}(M)
\]

\[
[\omega_1] \tilde{\wedge} [\omega_2] := [\omega_1 \wedge \omega_2].
\]

Thus \( \tilde{\wedge} \) is an operator on \( H_{dR}(M) \). To show that \((H_{dR}(M), \tilde{\wedge}, +)\)

is a ring, we must show that \( \tilde{\wedge} \) is associative and distributive over addition.

**Associativity**

Let \( \omega_1 \) be a \( p \)-closed form, \( \omega_2 \) be a \( q \)-closed form, and \( \omega_3 \)

be an \( r \)-closed form. Then

\[
([\omega_1] \tilde{\wedge} [\omega_2]) \tilde{\wedge} [\omega_3] = [\omega_1 \wedge \omega_2] \tilde{\wedge} [\omega_3] = [(\omega_1 \wedge \omega_2) \wedge \omega_3]
\]

\[
= [\omega_1 \wedge (\omega_2 \wedge \omega_3)] \quad \text{(associativity in } \mathcal{X}(M)\text{)}
\]

\[
= [\omega_1] \tilde{\wedge} [\omega_2 \wedge \omega_3]
\]

\[
= [\omega_1] \tilde{\wedge} ([\omega_2] \tilde{\wedge} [\omega_3]).
\]
Distributivity

Let \( \omega_1, \omega_2, \) be \( p \)-closed forms, and \( \omega_3 \) be a \( q \)-form closed \( q \)-form. Then
\[
(\lbrack \omega_1 \rbrack + \lbrack \omega_2 \rbrack) \thicksim \lbrack \omega_3 \rbrack = \lbrack \omega_1 + \omega_2 \rbrack \thicksim \lbrack \omega_3 \rbrack
\]
\[
= \lbrack (\omega_1 + \omega_2) \thicksim \omega_3 \rbrack
\]
\[
= \lbrack (\omega_1 \thicksim \omega_3) + (\omega_2 \thicksim \omega_3) \rbrack \quad \text{(distributivity in } \mathcal{L}(M))
\]
\[
= \lbrack \omega_1 \thicksim \omega_3 \rbrack + \lbrack \omega_2 \thicksim \omega_3 \rbrack
\]  

Therefore, \( (H^{dR}(M), +, \thicksim) \) is a ring.

About the unitality

If \( M = \emptyset \) then \( H^{dR}(M) = \{0\} \). The unit element is \( 0 \).

If \( M \neq \emptyset \) then we define \( \Pi : M \to \mathbb{R} \) to be the constant map \( \Pi(p) = 1 \) for all \( p \in M \). Then \( \Pi \in \omega^0(M) \). Since \( d(\Pi) = 0 \), \( \lbrack \Pi \rbrack \in H^{dR}(M) \) and thus \( \lbrack \Pi \rbrack \in H^{dR}(M) \). For any \( \lbrack \omega \rbrack \in H^{dR}(M) \), we have
\[
\lbrack \Pi \rbrack \thicksim \lbrack \omega \rbrack = \lbrack \Pi \thicksim \omega \rbrack = \lbrack \omega \rbrack, \quad \lbrack \omega \rbrack \thicksim \lbrack \Pi \rbrack = \lbrack \omega \thicksim \Pi \rbrack = \lbrack \omega \rbrack.
\]
Therefore, \( \lbrack \Pi \rbrack \) is the unit element of \( H^{dR}(M) \).