1. (5 points) Solve the following initial value problem.

\[ x^2 y' + 2xy = \ln(x), \quad y(1) = 2 \]

**Solution:** Observing that the left hand side of the equation is \( d(x^2y)/dx \), we integrate both sides with respect to \( x \) as follows:

\[
\int \frac{d(x^2y)}{dx} dx = x^2 y = \int \ln(x) dx = x \ln(x) - x + C \quad \text{ (Integration by parts, with } u = \ln x) 
\]

Solving for \( y \), and using the initial condition

\[
y = \frac{\ln(x) - 1}{x} + \frac{C}{x^2} \quad \text{ and } \quad y(1) = 2 \Rightarrow 2 = -1 + C \Rightarrow C = 3
\]

Therefore, the (unique) solution of this initial value problem is

\[
y = \frac{\ln(x) - 1}{x} + \frac{3}{x^2}
\]

2. (5 points) The population of lions (L) and antelopes (A) are modeled by the following equations

\[
\frac{dA}{dt} = 2A - 0.01AL \\
\frac{dL}{dt} = -0.5L + 0.0001AL
\]

(i) Find the equilibrium solutions and explain their significance.

(ii) Find an expression for \( \frac{dL}{dA} \).

**Solution:**

(i) The equilibrium solutions are the constant solutions, that is, solutions of the form \( A(t) = C_1 \), and \( L = C_2 \) are some constant functions satisfying

\[
\frac{dA}{dt} = 0 \quad \text{ and } \quad \frac{dL}{dt} = 0
\]

In our case, we have

\[
\frac{dA}{dt} = A(2 - 0.01L) = 0 \Rightarrow A = 0 \text{ or } L = 200; \\
\frac{dL}{dt} = L(-0.5 + 0.0001A) = 0 \Rightarrow L = 0 \text{ or } A = 5000,
\]
so the two equilibrium solutions are $A = 0, L = 0$ and $A = 5000, L = 200$. The first, trivial, solution just says that if you start with zero population, nothing changes. The second, non-trivial solution says that if we start with 5000 antelopes and 200 lions, both populations remain constant through time.

(ii) By the chain rule, we have

$$\frac{dL}{dA} = \frac{dL}{dt} \frac{dA}{dt} = -\frac{0.5L + 0.0001AL}{2A - 0.01AL}$$