The Exponential Decay of Fundamental Solutions to Certain Schrödinger Operators

joint with Svitlana Mayboroda

Bruno Poggi

University of Minnesota, Department of Mathematics

Abstract

In 1999, Z. Shen proved, in particular, the point-wise exponential decay of the fundamental solutions to Schrödinger operators of the form $-\Delta + V$ where $V$ lies in a Reverse Hölder class. An open question is whether this result can be extended to the case of operators of the form $-\div A\nabla + V$ where $A$ is a matrix satisfying ellipticity and boundedness conditions. In this paper we investigate this question and provide an affirmative answer. We also study whether an analogous result can be obtained for fundamental solutions to the magnetic Schrödinger operators. For this question, we provide an upper exponential decay bound, but the question of a lower bound remains open.

Introduction

- Let $A = (a_{ij})$ be a real, strongly elliptic matrix, in the sense that there exist $0 < \Lambda < \Lambda < +\infty$ such that
  \[ |\nabla f|^2 \leq \sum_{i,j} a_{ij}(x) |\partial_i f|^2 \leq \Lambda |\nabla f|^2, \]
  and $V \in L^1_{\text{loc}}(\mathbb{R}^n)$. Here, $V$ is called the electric potential. We also denote a magnetic potential by the real vector function $a(x) = (a_1(x), \ldots, a_n(x))$. Consider the fundamental solutions to the following Schrödinger operator $L_\mu := -\div A\nabla + V$, and the magnetic Schrödinger operator
  \[ L_M = \left( \frac{1}{2} \nabla - \alpha \right)^2 + V. \]
- We ask whether
  \[ |u(x, y)| = \left( \frac{1}{2} \nabla - \alpha \right)^d \delta(x-y), \]
  where $\Gamma$ is the fundamental solution to $L_\mu$, with $J$ either $E$ or $M$, and $d\mu$ is an associated distance which captures the behavior of $V$, or $A$ respectively.
- Exponential decay for the fundamental solution is known for the operator $-\Delta + V$ where $V$ lies in a Reverse Hölder class. For instance, see [1]; for power-decay estimates of the fundamental solution to $L_\mu$, $L_M$ respectively.

Set-up and Notation

- We say a function $u \in L^1_{\text{loc}}(\mathbb{R}^n)$, $u > 0$ a.e., belongs to the Reverse Hölder Class $B^p_\mu(\mathbb{R}^n)$ if there exists a constant $C_\mu$ so that for any ball $B \subset \mathbb{R}^n$,
  \[ \frac{1}{|B|} \int_B u^{1/p} \leq C_\mu \left( \frac{1}{|B|} \int_B u \right)^{1/p}. \]

Main Results

Theorem 10. The fundamental solution to the operator $L_\mu$ satisfies for all $x, y \in \mathbb{R}^n$,
\[ \frac{e^{-C_\mu/d(x,y)V}}{|x-y|^{n-2}} \leq \frac{e^{-C_\mu/d(x,y)V}}{|x-y|^{n-2}}, \]
where $C, \varepsilon, \mu$ depend only on $\lambda$, $n$, $C_\mu$, $\nu$.

Theorem 11. The fundamental solution to the operator $L_M$ satisfies
\[ \frac{e^{-C_\mu/d(x,y)V}}{|x-y|^{n-2}} \leq \frac{e^{-C_\mu/d(x,y)V}}{|x-y|^{n-2}}, \]
where $C, \varepsilon$ depend only on $n$ and the constants from (8).

Main Techniques and Tools

The proof technique follows that of [6], with modifications to deal with the structures of the operators $L_\mu$, $L_M$.
- A key fact is an estimate known as a Fefferman-Phong inequality, which we write below for functions $f$ in $C^2(\mathbb{R}^n)$,
\[ \int \frac{m(x, f^2 e^{-V} \partial f \partial f^2 e^{-V}} \leq C \int e^{-V} f f \partial f \partial f \partial f \partial f. \]

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