Harmonic Analysis
Homework 1
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1 Notation

Throughout, \( \mathbb{N} := \{1, 2, 3, \ldots, \} \), \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \), \( B_r(x) \) is the ball of radius \( r \) with center \( x \) in the understood metric space (usually \( \mathbb{R} \)). Let \( X \) be a compact topological vector space. Then by \( C(X) \) we denote the space of continuous real-valued functions on \( X \). In particular, \( C[0, 1] \) is the space of continuous (hence, uniformly continuous) real-valued functions on \([0, 1]\). We always think of \( C(X) \) as a Banach space endowed with the norm

\[
\|f\|_{C(X)} := \sup_{x \in X} |f(x)|.
\]

Unless otherwise noted, by \( (\cdot, \cdot) \) we denote the \( L^2 \) product of two functions on the measure space understood from context. For a definition of approximate identity, see [3] Definition 1.2.15 (although we use a different order of the axioms). The function \( 1_X : \mathbb{R} \to \{0, 1\} \) is the indicator function of the set \( X \). For \( z \in \mathbb{C} \setminus \{0\} \), it can be written in the form \( z = re^{ix} \) for \( x \in \mathbb{T} \), and we write \( \theta(z) = x \) (i.e. \( \theta(z) \) is the direction of \( z \)).

2 Solutions to Problems

1. Let \( \{f_k\}_{k \in \mathbb{N}} \) be an orthonormal system in \( L^2[0, 1] \). Assume that for each \( k \in \mathbb{N} \), the functions \( f_k \) are continuously differentiable on \([0, 1]\). Show that the family \( \{f_k'\} \) cannot be uniformly bounded.

    Solution. Suppose that the family \( \{f_k'\} \) is uniformly bounded. This means there exists a constant \( C > 0 \) such that

    \[
    \sup_{x \in [0,1]} |f_k'(x)| \leq C, \quad \forall k \in \mathbb{N}.
    \]
For each \( k \in \mathbb{N} \), due to integration by parts we can write
\[
\int_0^1 f_k(y) \, dy = f_k(1) - \int_0^1 y f'_k(y) \, dy
\]
so that by the Triangle Inequality, Cauchy-Bunyakovski-Schwartz Inequality, and the orthonormality of \( \{f_k\} \), we observe
\[
|f_k(1)| \leq \int_0^1 |f_k(y)| \, dy + \int_0^1 y |f'_k(y)| \, dy \leq \|f_k\|_{L^2[0,1]} + \frac{1}{2} C \leq 1 + \frac{1}{2} C, \quad \forall k \in \mathbb{N},
\]
which shows that the sequence \( \{f_k(1)\} \) is bounded. Now fix arbitrary \( x \in [0,1] \). Owing to the Mean Value Theorem, there exists \( \theta_x \in [x,1] \) such that we can write
\[
|f_k(x)| \leq |f_k(1) - f_k(x)| + |f_k(1)| = |f'_k(\theta_x)|(1-x) + |f_k(1)| \leq 1 + \frac{3}{2} C, \quad \forall k \in \mathbb{N},
\]
so that the family \( \{f_k\} \) is equicontinuous.

The last few observations (namely, the equicontinuity and uniform boundedness of \( \{f_k\} \)) imply that \( \{f_k\} \) contains a uniformly convergent subsequence by the Arzelà-Ascoli Theorem, see for example [1] Theorem 7.25. Hence, there exists a function \( f \in C[0,1] \) which is a limit point of \( \{f_k\} \) in \( C[0,1] \). Let \( k_\ell \) be a subsequence such that
\[
f_{k_\ell} \to f \quad \text{in } C[0,1] \text{ as } k \to \infty.
\]
Since
\[
\|f - f_{k_\ell}\|_{L^2[0,1]} = \left( \int_0^1 |f(x) - f_{k_\ell}(x)|^2 \, dx \right)^{1/2} \leq \sup_{x \in [0,1]} |f(x) - f_{k_\ell}(x)| = \|f - f_{k_\ell}\|_{C[0,1]} \to 0
\]
as $\ell \to \infty$, then it follows that $f$ is also a limit point of $\{f_k\}$ in $L^2[0,1]$. On the other hand,
\[
\|f_k - f_\ell\|^2_{L^2[0,1]} = (f_k, f_k) - 2(f_k, f_\ell) + (f_\ell, f_\ell) = 2, \quad \forall k \neq \ell, \ k, \ell \in \mathbb{N},
\]
so that $\{f_k\}$ can have no limit points in $L^2[0,1]$. Thus we have furnished a contradiction. This proves that our initial assumption (1) is untenable, thus the desired result. □

**Remark 2.** Actually, under the same conditions of the problem except for the assumption of (1), it can be shown that the family $\{f'_k\}$ cannot be uniformly bounded in $L^2[0,1]$ either. To see this, it is only a quick modification of the above argument, where instead of using the Mean Value Theorem, we resort to the Fundamental Theorem of Calculus and Cauchy-Bunyakovsky-Schwartz Inequality to control the equicontinuity of the family $\{f_k\}$ by the radius of a ball in $L^2[0,1]$ which contains the family $\{f'_k\}$.

2. Let $D_N(x)$ be the Dirichlet kernel, i.e.
\[
D_N(x) = \sum_{n=-N}^{N} e^{2\pi i n x}.
\]

(a) Show that
\[
D_N(x) = \frac{\sin ((2N + 1)\pi x)}{\sin(\pi x)}. \quad (4)
\]

(b) Show that for any $\delta > 0$ there exists a constant $c_\delta > 0$, such that for any $N \in \mathbb{N}$
\[
\int_{\delta \leq |x| \leq 1/2} |D_N(x)| \, dx \geq c_\delta, \quad (5)
\]
i.e. property $(iii)$ in the definition of approximate identities fails.

(c) Show that
\[
\|D_N\|_1 \approx \log N,
\]
more precisely that there exist constants $c, \ C > 0$, such that $c \log N \leq \|D_N\|_1 \leq C \log N$ holds for all $N$ large enough, i.e. property $(ii)$ in the definition of approximate identities fails.

**Remark:** A more precise estimate is
\[
\frac{4}{\pi^2} \sum_{k=1}^{N} \frac{1}{k} \leq \|D_N\|_1 \leq 2 + \frac{\pi}{4} + \frac{4}{\pi^2} \sum_{k=1}^{N} \frac{1}{k},
\]

but I only ask you to show that $\|D_N\|_1$ grows logarithmically.

**Solution.**

**Proof of (a).** For any $z \in \mathbb{C}$ and $N \in \mathbb{N}$, we can write

$$z^{N+1} - 1 = (z - 1)(z^N + z^{N-1} + \ldots + z + 1)$$

and so if $z \neq 1$,

$$1 + z + \ldots + z^N = \frac{z^{N+1} - 1}{z - 1}. \quad (6)$$

Now say $z = e^{i\theta}$ with $\theta$ not a multiple of $\pi$. Summing identities (6) for $z$ and $\bar{z} = e^{-i\theta}$, we obtain

$$2 + (z + \bar{z}) + (z^2 + \bar{z}^2) + \ldots + (z^N + \bar{z}^N) = \frac{z^{N+1} - 1}{z - 1} + \frac{\bar{z}^{N+1} - 1}{\bar{z} - 1},$$

$$\Rightarrow \sum_{n=-N}^{N} e^{in\theta} = \left[\frac{z^{N+1} - 1}{z - 1} + \frac{\bar{z}^{N+1} - 1}{\bar{z} - 1}\right] - 1 \quad (7)$$

Recalling that

$$\cos \theta = \frac{1}{2} \left[ e^{i\theta} + e^{-i\theta} \right]$$

then from (7) we observe

$$\sum_{n=-N}^{N} e^{in\theta} = \frac{2 \cos(N\theta) - 2 \cos((N+1)\theta)}{2 - 2 \cos \theta},$$

$$\Rightarrow \sum_{n=-N}^{N} e^{in\theta} = \frac{\cos((N+1)\theta) - \cos(N\theta)}{\cos \theta - 1}, \quad (8)$$

and it is easy to verify that

$$\frac{\cos((N+1)\theta) - \cos(N\theta)}{\cos \theta - 1} = \frac{\sin((N+1)\theta) + \sin(N\theta)}{\sin \theta}. \quad (9)$$
Indeed, upon using the angle summation formulas on the expressions containing \((N+1)\theta\) and multiplying so that there is no denominator, it is seen that the above equality is implied by the following equality:

\[
\cos(N\theta) \sin \theta (\cos \theta - 1) - \sin(N\theta) \sin^2 \theta =
\sin(N\theta) \cos \theta (\cos \theta - 1) + \cos(N\theta) \sin \theta (\cos \theta - 1) + \sin(N\theta) (\cos \theta - 1),
\]

and this one is easily seen to be true due to the identity

\[
\cos^2 \theta + \sin^2 \theta = 1.
\]

Hence, due to (8) and (9), whenever \(x \in \mathbb{R}\) is such that \(2x\) is not an integer, we can write

\[
D_N(x) = \sum_{n=-N}^{N} e^{i2\pi x} = \frac{\sin((N+1)2\pi x) + \sin(N2\pi x)}{\sin 2\pi x}. \tag{10}
\]

Then, by noticing that

\[
\sin 2\pi x = 2 \sin(\pi x) \cos(\pi x),
\]

\[
\sin((2N+1)\pi x + \pi x) = \sin((2N+1)\pi x) \cos(\pi x) + \cos((2N+1)\pi x) \sin(\pi x),
\]

\[
\sin(2N\pi x) = \sin((2N+1)\pi x) \cos(\pi x) - \cos((2N+1)\pi x) \sin(\pi x),
\]

from (10), the expression (4) follows. It remains to check the cases when \(2x \in \mathbb{Z}\). Actually, due to the \(2\pi\) periodicity of the map \(\theta \to e^{i\theta}\), it suffices to check \(x = 0\) and \(x = \frac{1}{2}\). We note

\[
D_N(0) = \sum_{n=-N}^{N} e^{2\pi in0} = \sum_{n=-N}^{N} 1 = 2N + 1,
\]

while by L’Hopital’s rule,

\[
\lim_{x \to 0} \frac{\sin ((2N+1)\pi x)}{\sin(\pi x)} = 2N + 1,
\]

so that (4) is verified in this case. Finally, if \(x = \frac{1}{2}\), then we note

\[
D_N(1/2) = \sum_{n=-N}^{N} e^{\pi in} = \sum_{n=-N}^{N} (-1)^n = (-1)^N,
\]

while

\[
\frac{\sin ((2N+1)\frac{1}{2}\pi)}{\sin \left(\frac{1}{2}\pi\right)} = (-1)^N
\]
and this finishes the proof of (a).

**Proof of (b).** Fix \( \delta \in (0, 1/2) \). From (4) it is easy to see that the function \( D_N(x) \) is even, and thus
\[
\int_{\delta}^{1/2} |D_N(x)| \, dx = 2 \int_{\delta}^{1/2} |D_N(x)| \, dx.
\]
Since
\[
|\sin(\pi x)| \leq 1, \quad \forall x \in [\delta, 1/2],
\]
then
\[
\frac{1}{\sin(\pi x)} \geq 1, \quad \forall x \in [\delta, 1/2],
\]
so that
\[
\int_{\delta}^{1/2} |D_N(x)| \, dx = \int_{\delta}^{1/2} \frac{\sin((2N + 1)\pi x)}{\sin(\pi x)} \, dx \geq \int_{\delta}^{1/2} \sin((2N + 1)\pi x) \, dx.
\]

Next, the idea is that since \( \delta < 1/2 \), there exists an \( N \) large enough so that the sum of the areas under all the congruent half arcs of the compressed sine function \( \sin((2N + 1)\pi x) \) between \( \delta \) and \( 1/2 \) stays bounded below by a constant independent of \( N \). To formalize this notion, first note by a change of variables that
\[
\int_{\delta}^{1/2} |\sin((2N + 1)\pi x)| \, dx = \frac{1}{(2N + 1)\pi} \int_{2\delta(2N+1)\pi}^{(2N+1)\pi/2} |\sin(y)| \, dy, \tag{11}
\]
and since \( \delta < 1/2 \), then the above expression is positive for all \( N \in \mathbb{N} \). Moreover, there exists \( N_\delta \in \mathbb{N} \) such that for every \( N \geq N_\delta \),
\[
[2\delta(2N + 1)] + 1 < 2N + 1. \tag{12}
\]
For example, we can pick \( N_\delta = \left\lceil \frac{1}{1-2\delta} \right\rceil \). We deduce that for every \( N \geq N_\delta \),
\[
\int_{2\delta(2N+1)\pi}^{(2N+1)\pi/2} |\sin y| \, dy \geq \int_{[2\delta(2N+1)]\pi/2}^{(2N+1)\pi/2} |\sin y| \, dy = \sum_{k=\lceil 2\delta(2N+1) \rceil}^{2N} \int_{\frac{\pi}{2}k}^{\frac{\pi}{2}(k+1)} |\sin y| \, dy. \tag{13}
\]
Now note that for each \( k \in \mathbb{N} \),
\[
\int_{\frac{\pi}{2}k}^{\frac{\pi}{2}(k+1)} |\sin y| \, dy = \int_{0}^{\pi/2} \sin y \, dy = 1. \tag{14}
\]
Therefore, from (14) and (13), it follows
\[
\int_{\frac{2\delta(N+1)\pi}{2}}^{\frac{(2N+1)\pi}{2}} |\sin y| \, dy \geq 2N + 1 - \lceil 2\delta(N+1) \rceil, \quad \forall N \geq N_\delta
\]

Using this estimate in (11) yields for every \( N \geq N_\delta \),
\[
\frac{1}{2^{N} \pi} \int_{\delta}^{1/2} |\sin \left( (2N + 1)\pi x \right)| \, dx = \frac{2N + 1 - \lceil 2\delta(N+1) \rceil}{(2N+1)\pi} \geq \frac{2N + 1 - 2\delta(N+1) - 1}{(2N+1)\pi} = \frac{2}{\pi} \left( \frac{1}{2} - \delta \right) - \frac{1}{(2N+1)\pi}.
\]

(15)

We readily observe the last term on the right-hand side of (15) approaches 0 as \( N \to \infty \), and moreover the right-hand side of (15) is positive for all \( N \geq N_\delta \) due to (12). As such, there exists a number \( c_1 = c_1(\delta) > 0 \) such that the right-hand side of (15) is bounded below by \( c_1 \). Finally, we take
\[
c_\delta := \min \left\{ 2c_1, \int_{\delta \leq |x| \leq 1/2} |D_N(x)| \, dx, \quad N = 0, 1, \ldots, N_\delta - 1 \right\}
\]

and it’s clear by our construction that \( c_\delta \) satisfies the desired inequality (5). This finishes the proof of (b).

Proof of (c). Identify \( \mathbb{T}^1 \) with the interval \([-1/2, 1/2]\). First fix \( N \in \mathbb{N} \). From (3) it is easily seen that \( D_N \) is continuous on \( \mathbb{T}^1 \), and hence \( D_N \in L^1(\mathbb{T}^1) \). This implies that the right-hand side in (4) is also in \( L^1(\mathbb{T}^1) \). Thus we note
\[
\|D_N\|_1 := \int_{\mathbb{T}^1} |D_N(x)| \, dx = 2 \int_{0}^{1/2} \left| \frac{\sin \left( (2N+1)\pi x \right)}{\sin(\pi x)} \right| \, dx \]
\[
\Rightarrow \|D_N\|_1 = 2 \sum_{k=0}^{2N} \int_{\frac{k\pi}{2(N+1)}}^{\frac{(k+1)\pi}{2(N+1)}} \left| \frac{\sin \left( (2N+1)\pi x \right)}{\sin(\pi x)} \right| \, dx, \quad (16)
\]
where in the right-hand side of (16) we have split the full integral on \([0, 1/2]\) as the sum of the integrals on the pairwise disjoint intervals

\[
I_k := \left( \frac{k}{2(2N + 1)}, \frac{k + 1}{2(2N + 1)} \right), \quad k = 0, \ldots, 2N,
\]

which can be done since

\[
\bigcup_{k=0}^{2N} I_k = (0, 1/2].
\]

Next, on \((0, 1/2]\), the function \(\sin(\pi x)\) is positive and monotone increasing. Thus we observe

\[
\sin \left( \frac{k}{2(2N + 1)} \pi \right) \leq \sin \pi x \leq \sin \left( \frac{k + 1}{2(2N + 1)} \pi \right), \quad \forall x \in I_k.
\]

So, using the second inequality in (17) on (16), we have

\[
\|D_N\|_1 \geq 2 \sum_{k=0}^{2N} \frac{1}{\sin \left( \frac{k + 1}{2(2N + 1)} \pi \right)} \int_{I_k} \left| \sin \left( (2N + 1) \pi x \right) \right| dx
\]

\[
= 2 \sum_{k=0}^{2N} \frac{1}{\sin \left( \frac{k + 1}{2(2N + 1)} \pi \right)} \frac{1}{2N + 1} \int_{\frac{k}{2N + 1}}^{\frac{k + 1}{2N + 1}} |\sin y| dy
\]

\[
= \frac{4}{\pi^2} \sum_{k=0}^{2N} \frac{1}{\sin \left( \frac{k + 1}{2(2N + 1)} \pi \right)} \frac{1}{k + 1} \geq \frac{4}{\pi^2} \sum_{k=0}^{2N} \frac{1}{k + 1},
\]

where on the first equality we did a change of variables \(y = (2N + 1)\pi x\) on each \(I_k\), on the second equality we used (14), and on the last inequality we observed that since

\[
x = \int_{0}^{x} 1 dy \geq \int_{0}^{x} \cos y dy = \sin x, \quad \forall x \geq 0
\]

then

\[
\frac{k + 1}{2(2N + 1)} \pi \geq 1, \quad \forall k = 0, \ldots, 2N.
\]

Therefore, we have shown

\[
\|D_N\|_1 \geq \frac{4}{\pi^2} \sum_{k=1}^{2N + 1} \frac{1}{k}, \quad \forall N \in \mathbb{N}.
\]
To see that the above estimate implies at least logarithmic growth, note that
\[
\sum_{k=1}^{2N+1} \frac{1}{k} = \sum_{k=1}^{2N+1} \int_{1}^{\frac{1}{k}} dx \geq \sum_{k=1}^{2N+1} \int_{1}^{\frac{1}{k}} \frac{1}{x} dx = \int_{1}^{\frac{1}{2N+1}} \frac{1}{x} dx = \ln(2N+2) \geq \ln 2 + \ln N
\]
true for each \( N \geq 1 \). This shows that
\[
\|D_N\|_1 \geq \frac{4}{\pi^2} \ln 2 + \frac{4}{\pi^2} \ln N, \quad \forall N \geq 1.
\] (19)

Now we concentrate on an upper bound for the growth of \( \|D_N\|_1 \). We use (16), the first inequality in (17) for \( k = 1, \ldots, 2N \), and the fact that
\[
|D_N(x)| \leq \sum_{n=-N}^{N} |e^{2\pi inx}| = 2N + 1 = D_N(0) \quad \forall x \in \mathbb{R},
\]
to see that
\[
\|D_N\|_1 \leq 2 \int_{I_0} \left| \frac{\sin ((2N+1)\pi x)}{\sin(\pi x)} \right| dx + 2 \sum_{k=1}^{2N} \frac{1}{\sin \left( \frac{k}{2(2N+1)} \pi \right)} \int_{I_k} \left| \frac{\sin ((2N+1)\pi x)}{\sin(\pi x)} \right| dx
\]
\[
\leq 2(2N+1)|I_0| + \frac{4}{\pi^2} \sum_{k=1}^{2N} \frac{k}{2(2N+1)\pi} \frac{1}{\sin \left( \frac{k}{2(2N+1)} \pi \right)} \int_{\frac{\pi}{2k}}^{\frac{\pi}{2}} |\sin y| dy
\]
\[
= 1 + \frac{4}{\pi^2} \sum_{k=1}^{2N} \frac{k}{2(2N+1)\pi} \frac{1}{\sin \left( \frac{k}{2(2N+1)} \pi \right)},
\]
(20)
where we used (14) on the last equality. Since
\[
\lim_{x \to 0} \frac{x}{\sin x} = 1,
\]
it follows the function \( \frac{x}{\sin x} \) is continuous on \([0, \pi/2]\), hence uniformly continuous, and hence uniformly bounded on \([0, \pi/2]\). So, there exists a constant \( M > 0 \) such that
\[
\frac{x}{\sin x} \leq M, \quad \forall x \in [0, \pi/2].
\]
It follows that
\[
\frac{k}{2(2N+1)\pi} \frac{1}{\sin \left( \frac{k}{2(2N+1)} \pi \right)} \leq M, \quad k = 1, \ldots, 2N,
\]
so that from (20) we can write
\[ \|D_N\|_1 \leq 1 + \frac{4}{\pi^2} M \sum_{k=1}^{2N} \frac{1}{k}. \]

To see that the above estimate implies at most logarithmic growth, note
\[ \sum_{k=2}^{2N} \frac{1}{k} \leq \sum_{k=2}^{2N} \int_{k-1}^{k} \frac{1}{x} \, dx = \int_{1}^{2N} \frac{1}{x} \, dx = \ln(2N) = \ln 2 + \ln N, \]
and as such,
\[ \|D_N\|_1 \leq 1 + \frac{4}{\pi^2} M (1 + \ln 2) + \frac{4}{\pi^2} M \ln N, \quad \forall N \geq 2. \]  
(21)

Finally, (18) and (21) imply the desired result.

\[ \square \]

3. ([2] #3.1.3) Let \( F_N(x) \) be the Fejér kernel, i.e.
\[ F_N(x) = \frac{1}{N+1} \sum_{n=0}^{N} D_n(x). \]  
(22)

(a) Show that
\[ F_N(x) = \frac{1}{N+1} \left( \frac{\sin((N+1)\pi x)}{\sin(\pi x)} \right)^2, \]  
(23)
in particular \( F_N(x) \geq 0 \).

(b) Show that the family \( \{F_N\} \) is an approximate identity as \( N \to \infty \).

\[ \text{Solution.} \]

\[ \text{Proof of (a).} \] First, recall from (8) that we have
\[ D_n(x) = \frac{\cos((n+1)2\pi x) - \cos(n2\pi x)}{\cos(2\pi x) - 1}, \quad \forall n \in \mathbb{N}, \]
and therefore,
\[ F_N(x) = \frac{1}{N+1} \sum_{n=0}^{N} \frac{\cos((n+1)2\pi x) - \cos(n2\pi x)}{\cos(2\pi x) - 1}, \]
and pleasantly, the sum on the right-hand side telescopes. Consequently,
\[
F_N(x) = \frac{1}{N + 1} \frac{\cos(2(N + 1)\pi x) - 1}{\cos(2\pi x) - 1}
\]
\[
= \frac{1}{N + 1} \frac{\cos^2((N + 1)\pi x) - \sin^2((N + 1)\pi x) - 1}{\cos^2(\pi x) - \sin^2(\pi x) - 1},
\]
and from this last equality, (23) follows immediately from the use of the identity 
\[
\cos^2 x + \sin^2 x = 1.
\]
It is trivial that (23) implies \(F_N(x) \geq 0\) for all \(x \in \mathbb{R}\).

**Proof of (b).** Identify \(T^1\) with the interval \([-1/2, 1/2]\). First observe that for each \(N \in \mathbb{N}\),
\[
\int_{T^1} F_N(x) \, dx = \int_{T^1} \frac{1}{N + 1} \sum_{n=0}^{N} D_n(x) \, dx = \frac{1}{N + 1} \sum_{n=0}^{N} \int_{T^1} D_n(x) \, dx = \frac{1}{N + 1} \sum_{n=0}^{N} 1 = 1,
\]
so that property (i) of the axioms of an Approximate Identity is satisfied. Next, since each \(F_N\) is positive, we note
\[
\|F_N\|_1 = \int_{T^1} |F_N(x)| \, dx = \int_{T^1} F_N(x) \, dx = 1, \quad \forall N \in \mathbb{N}
\]
so that property (ii) of the axioms of an Approximate Identity is satisfied. Finally, fix \(\delta \in (0, 1/2]\). If \(\delta = 1/2\), there is nothing to prove, so suppose \(\delta \in (0, 1/2)\). We observe
\[
\int_{T^1 \setminus B_\delta(0)} |F_N(x)| \, dx = 2^{1/2} \int_{\delta} \frac{1}{N + 1} \left( \frac{\sin\left(\frac{(N + 1)\pi x}{\sin(\pi x)}\right)}{\sin(\pi x)} \right)^2 \, dx, \quad \forall N \in \mathbb{N}. \quad (24)
\]
Since
\[
\pi x \geq \pi \delta, \quad \forall x \in [\delta, 1/2],
\]
then
\[
\frac{1}{\sin \pi x} \leq \frac{1}{\sin \pi \delta}, \quad \forall x \in [\delta, 1/2],
\]
so that (24) implies
\[
\int_{T^1 \setminus B_\delta(0)} |F_N(x)| \, dx \leq \frac{2}{N + 1 \sin^2(\pi \delta)} \int_{\delta} \frac{1}{1/2} \left( \frac{1}{\sin^2((N + 1)\pi x)} dx \right)
\]
\[
\leq \frac{2}{N + 1 \sin^2(\pi \delta)} \left( \frac{1}{2} - \delta \right) \to 0 \quad \text{as} \quad N \to \infty
\]
for fixed $\delta \in (0, 1/2)$. This proves property (iii) of the axioms of an Approximate Identity. It follows the family $\{F_N\}$ is an approximate identity. □

4. ([2] #3.1.4) Let $V_N(x)$ be the de la Valée Poussin kernel, i.e.

$$V_N(x) = 2F_{2N+1}(x) - F_N(x).$$

(a) Show that the family $\{V_N\}$ is an approximate identity as $N \to \infty$.

(b) Find and plot the Fourier coefficients of $V_N$.

**Solution.**

**Proof of (a).** First observe that for any $N \in \mathbb{N}$,

$$\int_{T^1} V_N(x) \, dx = \int_{T^1} (2F_{2N+1}(x) - F_N(x)) \, dx = 2 \int_{T^1} F_{2N+1}(x) \, dx - \int_{T^1} F_N(x) \, dx = 1,$$

so that property (i) is satisfied. Next, for any $N \in \mathbb{N}$,

$$\int_{T^1} |V_N(x)| \, dx \leq 2 \int_{T^1} |F_{2N+1}(x)| \, dx + \int_{T^1} |F_N(x)| \, dx = 3,$$

which shows property (ii) is satisfied. Finally, identify $T^1$ with the interval $[-1/2, 1/2]$, and fix $\delta \in (0, 1/2]$. We note

$$\int_{T^1 \setminus B_\delta(0)} |V_N(x)| \, dx \leq 2 \int_{T^1 \setminus B_\delta(0)} |F_{2N+1}(x)| \, dx + \int_{T^1 \setminus B_\delta(0)} |F_N(x)| \, dx \to 0$$

as $N \to \infty$ since $\{F_N\}$ is an approximate identity as shown in problem 3(b). Thus $\{V_N\}$ is an approximate identity.

**Proof of (b).** First note that

$$F_N(x) = \frac{1}{N+1} \sum_{n=0}^{N} \sum_{k=-n}^{n} e^{2\pi i k x} = \frac{1}{N+1} \sum_{k=-N}^{N} \sum_{n=0}^{N-|k|} e^{2\pi i k x}$$

$$= \frac{1}{N+1} \sum_{k=-N}^{N} (N + 1 - |k|) e^{2\pi i k x} = \sum_{k=-N}^{N} \left(1 - \frac{|k|}{N+1}\right) e^{2\pi i k x}. \quad (25)$$

12
Moreover, using (23) we see

\[
F_{2N+1}(x) = \frac{1}{2(N+1)} \left( \frac{\sin(2(N+1)\pi x)}{\sin(\pi x)} \right)^2
\]

\[
= 2F_N(x) \cos^2 \left( (N + 1)\pi x \right) = \frac{1}{2} F_N(x) \left[ e^{(N+1)i\pi t} + e^{-(N+1)i\pi t} \right]^2
\]

for each \( N \in \mathbb{N} \). Therefore, for each \( N \in \mathbb{N} \) and \( n \in \mathbb{Z} \), we calculate

\[
\hat{V}_N(n) = \int_{T^1} V_N(t) e^{-2\pi i nt} dt = \int_{0}^{1} \left[ 2F_{2N+1}(t) - F_N(t) \right] e^{-2\pi i nt} dt
\]

\[
= \int_{0}^{1} F_N(t) \left[ 1 + e^{2(N+1)i\pi t} + e^{-2(N+1)i\pi t} \right] e^{-2\pi i nt} dt
\]

\[
= \sum_{k=-N}^{N} \left( 1 - \frac{|k|}{N+1} \right) \int_{0}^{1} \left[ e^{2\pi i(k-n)t} + e^{2\pi i(N+1+k-n)t} + e^{2\pi i(-(N+1)+k-n)t} \right] dt. \quad (26)
\]

To calculate the integral in (26), for any \( a, b \in \mathbb{Z} \), consider the function

\[
E_{a,b}(x) := e^{2\pi i(a-b)x}, \quad x \in T^1,
\]

then we see

\[
\int_{0}^{1} E_{a,b}(t) dt = \begin{cases} 
1, & a = b \\
\frac{1}{2\pi i(a-b)}(e^{2\pi i(a-b)} - 1), & a \neq b.
\end{cases}
\]

However, whenever \( a \neq b \), \( a - b \in \mathbb{Z}\setminus\{0\} \) and

\[
e^{2\pi i(a-b)} - 1 = \cos(2\pi(a - b)) + i \sin(2\pi(a - b)) - 1 = 1 - 1 = 0,
\]

so that actually we have the formula

\[
\int_{0}^{1} E_{a,b}(t) dt = \delta_{a,b},
\]

where \( \delta_{a,b} \) is the Kronecker-delta:

\[
\delta_{a,b} := \begin{cases} 
1, & a = b \\
0, & a \neq b.
\end{cases}
\]
Back to the main calculation, we can rewrite (26) as

\[
\hat{V}_N(n) = \sum_{k=-N}^{N} \left(1 - \frac{|k|}{N+1}\right) \left[ \int_0^1 E_{k,n}(t) \, dt + \int_0^1 E_{N+1+k,n}(t) \, dt + \int_0^1 E_{-(N+1)+k,n}(t) \, dt \right]
\]

\[
= \sum_{k=-N}^{N} \left(1 - \frac{|k|}{N+1}\right) \left[ \delta_{k,n} + \delta_{N+1+k,n} + \delta_{-(N+1)+k,n} \right]
\]

\[
= \left(1 - \frac{|n|}{N+1}\right) \mathbf{1}_{[-N,N]}(n) + \left(1 - \frac{|n-(N+1)|}{N+1}\right) \mathbf{1}_{[1,2N+1]}(n) + \\
+ \left(1 - \frac{|n+N+1|}{N+1}\right) \mathbf{1}_{[-2N-1,-1]}(n).
\]  

(27)

So the right-hand side of (27) gives a formula to calculate the Fourier coefficients of \(V_N\), for each \(N \in \mathbb{N}\). More explicitly, note that for fixed \(N \in \mathbb{N}\),

\[
\hat{V}_N(n) = 0, \quad \text{for } |n| \geq 2N+2,
\]  

and this is clear from (27) since in this case all indicator functions vanish. Also clear is

\[
\hat{V}_N(n) = 1, \quad \text{for } |n| \leq N+1,
\]

which is seen after quick computations of the five different cases \(n = 0, N+1, -N-1, n \in (0, N+1), \text{ and } n \in (-N-1, 0)\). It remains to consider the case when \(|n| \in [N+2, 2N+1]\). If \(n > 0\), then write \(n = N+1+j\) for \(j \in \{1, \ldots, N\}\). From (27) we see that

\[
\hat{V}_N(n) = 1 - \frac{j}{N+1}, \quad \text{for } n = N+1+j, \; j \in \{1, \ldots, N\}.
\]

Similarly we get

\[
\hat{V}_N(n) = 1 - \frac{j}{N+1}, \quad \text{for } n = -N-1-j, \; j \in \{1, \ldots, N\},
\]

and thus all Fourier coefficients of \(V_N\) are determined. From our formulas, the plots of the Fourier coefficients describe an isosceles trapezoid whose long side is the line \([-2N-2, 2N+2]\) on \(\mathbb{R}\), and whose short parallel side is the line \([-N-1, N+1]\) at a height of 1. A MATLAB script was written to plot the Fourier coefficients; see the Appendix for the script. The following figure is a plot of the Fourier coefficients of \(V_{50}\) satisfying \(|n| \leq 107\) (recall that from (28), there is no need to plot the rest of the Fourier coefficients):
Figure 1: Plot of Fourier Coefficients of $V_{50}$.

The next figure is a plot of the Fourier coefficients of $V_N$ satisfying $|n| \leq 2N + 7$ for each $N = 0, \ldots, 50$. Each color represents a different $V_N$. The color representing $V_N$ is the color of the point on the coordinate $(2N + 2, 0)$. 
Figure 2: Plot of Fourier Coefficients of $v_N$, $N = 0, \ldots, 50$. 

Fourier Coefficients of $V_N$ for $N = 0, \ldots, 50$.
5. ([2] #3.1.11; Theorem of Fejér and F. Riesz) Let \( P(x) = \sum_{k=-N}^{N} a_k e^{2\pi ikx} \) be a trigonometric polynomial such that \( P(x) > 0 \) for all \( x \in \mathbb{T} \). Prove that there exists a polynomial \( Q(x) \) of the form \( Q(x) = \sum_{k=0}^{N} b_k e^{2\pi ikx} \) such that

\[
P(x) = |Q(x)|^2.
\]

**Solution.** Let

\[
R(z) := \sum_{k=-N}^{N} a_k z^{k+N},
\]

and first note that

\[
R(e^{2\pi ix}) = \left( \sum_{k=-N}^{N} a_k e^{2\pi ikx} \right) e^{2\pi iNx} = P(x) e^{2\pi iNx} = P(x) e^{2\pi iNx},
\]

\[
\implies R(e^{2\pi ix}) e^{-2\pi iNx} = P(x), \quad \forall x \in \mathbb{T}^1. \tag{29}
\]

Since \( P(x) \geq 0 \) for all \( x \in \mathbb{T}^1 \), then the angle of the left-hand side of (29) must be an even multiple of \( \pi \), so that

\[
\theta \left( R(e^{2\pi ix}) \right) = e^{2\pi iNx}.
\]

It follows that

\[
R(e^{2\pi ix}) = |R(e^{2\pi ix})| e^{2\pi iNx} = \left( R(e^{2\pi ix}) \overline{R(e^{2\pi ix})} \right)^{1/2} e^{2\pi iNx} =
\]

\[
\implies R(e^{2\pi ix}) = \overline{R(e^{2\pi ix})} e^{4\pi iNx}, \quad \forall x \in \mathbb{T}^1. \tag{30}
\]

Actually, (30) implies a symmetry of the coefficients of \( R(z) \) (hence, of \( P(x) \)). To see what is meant, notice

\[
R(e^{2\pi ix}) = a_{-N} + a_{-N+1} e^{2\pi ix} + \ldots + a_{N-1} (e^{2\pi ix})^{2N-1} + a_N (e^{2\pi ix})^{2N},
\]

while

\[
\overline{R(e^{2\pi ix})} e^{4\pi iNx} = a_{-N} (e^{-2\pi ix})^{2N} + a_{-N+1} (e^{-2\pi ix})^{2N-1} + \ldots + a_{N-1} e^{-2\pi ix} + a_N.
\]
Subtracting the last two identities, realizing that any element of the unit circle in $\mathbb{C}$ can be written as $e^{2\pi ix}$ for $x \in \mathbb{T}^1$, and using (30) we arrive at
\[ 0 = \sum_{k=1}^{N} (a_k - \overline{a_k})(z^{2k} - 1)z^{N-k}, \quad \forall z \in \mathbb{C} \text{ with } |z| = 1, \tag{31} \]
hence the polynomial on the right-hand side of (31) has uncountably many zeroes on the unit circle while being of finite degree at most $2N$; this implies that this polynomial must be identically the 0-th polynomial, and hence
\[ a_k = \overline{a_k}, \quad \forall k = 1, \ldots, N. \tag{32} \]
This implies that we might as well suppose 0 is not a root of $R(z)$. To see this, first assume that 0 is a root of multiplicity $m$ of $R(z)$. Then it must be the case that
\[ a_{-N+m} = 0, \quad \ell = 0, \ldots, m-1, \]
which by (32) implies
\[ a_{N-m} = 0, \quad \ell = 0, \ldots, m-1, \]
so that $z^{-m}R(z)$ has degree at most $2(N-m)$ (so that in particular $m \leq N$), and we can write
\[ P(x) = \sum_{k=-N+m}^{N-m} a_k e^{2\pi i x}, \]
and $a_{-N+m} = \overline{a_{N-m}} \neq 0$. Moreover, $m < N$, for if $m = N$ then $a_0 e^{2\pi i x} = P(x) > 0$ for each $x \in \mathbb{T}^1$, which is not possible. Thus, if $P(x)$ is such that 0 is a root of $R(z)$ of multiplicity $m < N$, then all this means is that the first and last $N-m$ coefficients of $P(x)$ are 0, and can thus be ignored; the remaining trigonometric polynomial still satisfies all the conditions of the problem.

Therefore, from here on we suppose 0 is not a root of $R(z)$, which implies $a_{-N} = \overline{a_N} \neq 0$. Now let $z \in \mathbb{C} \setminus \{0\}$. On one hand
\[ R(z) = a_{-N} + a_{-N+1}z + \ldots + a_{N-1}z^{2N-1} + a_N z^{2N}, \]
while on the other,
\[ z^{2N} \overline{R(z^{-1})} = \overline{a_{-N}} z^{2N} + \overline{a_{-N+1}} z^{2N-1} + \ldots + \overline{a_{N-1}} z + \overline{a_N} \]
and due to (32), these last two expressions are equal, so we have proved
\[ R(z) = z^{2N} \overline{R(z^{-1})}, \quad \forall z \in \mathbb{C} \setminus \{0\}. \tag{33} \]
Suppose \( z_* \) is a zero of \( R(z) \) (which cannot be 0). Then from (33) it is clear that \( \overline{z_*}^{-1} \) is also a zero of \( R(z) \). By assumption, no zeroes of \( R(z) \) lie on the unit circle, and therefore
\[
z_* \neq \overline{z_*}^{-1},
\]
while both of these are zeroes of \( R(z) \), a polynomial of degree \( 2N \). By the Fundamental Theorem of Algebra and the previous remarks, there must lie \( N \) zeroes of \( R(z) \) in the interior of the unit circle, and the rest lie strictly outside the unit circle. Consequently, we can enumerate the (not necessarily distinct) zeroes of \( R(z) \) inside the unit circle as \( z_1, \ldots, z_N \), and as such the factorization
\[
R(z) = a_N \prod_{k=1}^{N}(z - z_k) \prod_{k=1}^{N}\left(z - \frac{1}{z_k}\right)
\]
holds. It follows by (29) that for each \( x \in \mathbb{T}^1 \),
\[
P(x) = e^{-2\pi iNx}a_N \prod_{k=1}^{N}(e^{2\pi ix} - z_k) \prod_{k=1}^{N}\left(e^{2\pi ix} - \frac{1}{z_k}\right)
\]
\[
= a_N \prod_{k=1}^{N}(e^{2\pi ix} - z_k) \prod_{k=1}^{N}\left[\left(e^{2\pi ix} - \frac{1}{z_k}\right)e^{-2\pi ix}\right]
\]
\[
= a_N \prod_{k=1}^{N}(e^{2\pi ix} - z_k) \prod_{k=1}^{N}\left(1 - \frac{e^{-2\pi ix}}{z_k}\right)
\]
\[
= \left(a_N(-1)^N \prod_{k=1}^{N}\frac{1}{\overline{z_k}}\right) \prod_{k=1}^{N}(e^{2\pi ix} - z_k) \prod_{k=1}^{N}(e^{-2\pi ix} - \overline{z_k})
\]
\[
= \left(a_N(-1)^N \prod_{k=1}^{N}\frac{1}{\overline{z_k}}\right) \left| \prod_{k=1}^{N}(e^{2\pi ix} - z_k) \right|^2. \tag{34}
\]
Finally, since \( P(x) > 0 \), from (34) it follows that the number
\[
c_N := a_N(-1)^N \prod_{k=1}^{N}\frac{1}{\overline{z_k}}
\]
is a positive real number. Hence, if we write
\[
Q(x) = \sqrt{c_N} \prod_{k=1}^{N}(e^{2\pi ix} - z_k)
\]
then clearly $Q(x)$ is of the required form, and by (34),

$$P(x) = |Q(x)|^2,$$

as desired. □

3 Acknowledgements

For problem 5, the author benefited greatly from the extended hint found in [3], problem 3.2.5.

4 Appendix

4.1 MATLAB script for plotting Fourier Coefficients of the de la Vallée Poussin kernel

```
close all
clear all
clc

for N=0:50
for n=-2*N-7:2*N+7
x(N+1,n+2*N+8)=n;
if abs(n)<=N+1
V(N+1,n+2*N+8)=1;
elseif abs(n)>=2*N+2
V(N+1,n+2*N+8)=0;
else
V(N+1,n+2*N+8)=1-(abs(n)-(N+1))/(N+1);
end
end

figure(1)
hold on
plot(x(N+1,:),V(N+1,:),'.','color',rand(1,3))
axis([min(x(N+1,:))-5 max(x(N+1,:))+5 0 1.3]);
title('Fourier Coefficients of V_N for N=0,...,50')
xlabel('n')
ylabel('n-th Fourier Coefficient of V_N')
end
```
figure(2)
plot(x(N+1,:),V(N+1,:),'.','color',rand(1,3))
axis([min(x(N+1,:))-5 max(x(N+1,:))+5 0 1.3]);
title('Fourier Coefficients of V_{50}');
xlabel('n')
ylabel('n-th Fourier Coefficient of V_{50}');

References